On the roots of Steiner polynomials

María A. Hernández Cifre

(joint work with Martin Henk)

Universität Magdeburg, Universidad de Murcia

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The relative Steiner polynomial

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The volume of the Minkowski sum $K + \lambda E$ is a polynomial of degree n in the parameter λ, the so called **relative Steiner polynomial** of K,

$$V(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E)\lambda^i.$$

The coefficients $W_i(K; E) = V\left(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\right)$ are called the **relative quermassintegrals** of K.

The (classical) Steiner polynomial, 1840

$$V(K + \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\lambda^i.$$

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- $W_0(K;E)=V(K)$, $W_n(K;E)=V(E)$ and
- $W_i(K;E)=W_{n-i}(E;K)$.

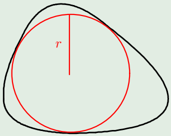
Inradius and circumradius

- The **relative inradius** $r(K;E)$ of K with respect to E:

$$r(K;E) = \max\{r \geq 0 : \text{some translate of } rE \subset K\}.$$

The classical inradius and circumradius

If $E = B_n$: $r(K)$



Inradius and circumradius

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- The **relative circumradius** $R(K;E)$ of K with respect to E:

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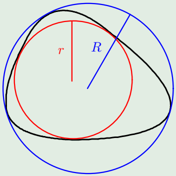

Inradius and circumradius

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- The **relative circumradius** $R(K;E)$ of K with respect to E:

$$R(K;E) = \min\{R > 0 : \text{some translate of } K \subset RE\} = \frac{1}{r(E;K)}.$$

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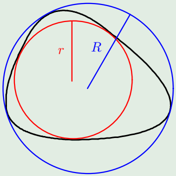

Teissier's problem

B. Teissier: Bonnesen-type inequalities in algebraic geometry I. Introduction to the problem. *Seminar on Differential Geometry*, Princeton Univ. Press, Princeton, N. J., 1982, 85–105.

Relation of the Steiner polynomial to Algebraic Geometry.

$$\sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^i$$

- **Problem:** to look for the relations between the zeros of the Steiner polynomial and the relative inradius $r(K;E)$ and circumradius $R(K;E)$ of the body K.

Teissier's problem and Bonnesen's inequality

Bonnesen (Blaschke)'s inequality

$$W_1(K;E)^2 - A(K)A(E) \geq \frac{A(E)^2}{4}\Big(R(K;E) - r(K;E)\Big)^2$$

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Classical Bonnesen's inequality

When $E = B_2$ the classical Bonnesen inequality is obtained:

$$p(K)^2 - 4\pi A(K) \geq \pi^2\big(R(K) - r(K)\big)^2,$$

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$$\Uparrow$$

$$A(K) + 2W_1(K;E)\lambda + A(E)\lambda^2 \leq 0 \quad \text{if} \quad -R(K;E) \leq \lambda \leq -r(K;E)$$

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- **Steiner polynomial:** $A(K) + 2W_1(K;E)\lambda + A(E)\lambda^2$
- Bonnesen's inequality $\Longrightarrow \lambda_1 \leq -R(K;E) \leq -r(K;E) \leq \lambda_2 \leq 0$

Teissier's problem and Sangwine-Yager's conjecture

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Teissier raised the problem of determining when an extension of this fact can be stated in arbitrary dimension:

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Teissier raised the problem of determining when an extension of this fact can be stated in arbitrary dimension:

- For which convex bodies do the real parts of the roots of the relative Steiner polynomial $\sum_{i=0}^{n}\binom{n}{i}W_i(K;E)\lambda^i$, say $a_1 \leq \cdots \leq a_n$, verify

$$a_1 \leq \cdots \leq a_n \leq 0 \quad \text{and} \quad -r(K;E) \leq a_n?$$

Teissier's problem and Sangwine-Yager's conjecture

J. R. Sangwine-Yager: Bonnesen-style inequalities for Minkowski relative geometry. *Trans. Amer. Math. Soc.* **307** (1) (1988), 373–382.

Conjecture

Let $K \in \mathcal{K}^n$. If $a_1 \leq \cdots \leq a_n$ are the real parts of the roots of the relative Steiner polynomial $\sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^i$, then

$$a_1 \leq -R(K;E) \leq -r(K;E) \leq a_n \leq 0.$$

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- Bonnesen's inequality $\Longrightarrow$ Conjecture is true if $n = 2$.

Negativity of the real parts of the roots

Definition

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- Dimensions 3, 4, 5 $\longrightarrow$ Hurwitz criterion **YES**

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Is the Steiner polynomial a Hurwitz polynomial? **NO**

- Dimensions 3, 4, 5 $\longrightarrow$ Hurwitz criterion **YES**
- Arbitrary dimension $\longrightarrow$ Counterexamples

Tangential bodies

Definition:

K is a **p-tangential body** of E, $0 \leq p \leq n-1$, if each support plane of K that is not a support plane of E contains only $(p-1)$-singular points of K.

- x is an **r-singular point** of K if the dimension of the normal cone in x is not smaller than $n - r$.

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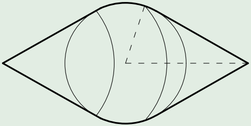
- The 1-tangential bodies are the cap-bodies.

Cap-body of E

Convex hull of E and countably many points exterior to it such that the line segment joining any two of these points intersects E.



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- The 1-tangential bodies are the cap-bodies.
- Each p-tangential body is also a q-tangential body, $p \leq q \leq n-1$.

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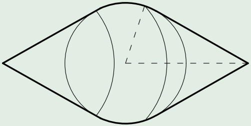

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J. Favard: Sur les corps convexes, *J. Math. Pures Appl.* **12** (9) (1933), 219–282.

Theorem (Favard, 1933)

Let $K, E \in \mathcal{K}^n$ with non-empty interior. Then K is a p-tangential body of E, for all $p = 1, \ldots, n-1$, if and only if $V(K) = W_{n-p}(K; E)$.

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- K p-tangential body of $E \Longrightarrow K$ q-tangential body, $p \leq q \leq n-1$,

$$\Longrightarrow V(K) = W_0(K;E) = W_1(K;E) = \cdots = W_{n-p}(K;E).$$

A 2-tangential body with not Hurwitz Steiner polynomial

Theorem

There exist 2-tangential bodies of B_{15} in $\mathbb{R}^{15}$ for which their Steiner polynomial has complex roots with strictly positive real part.

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- $$\sum_{i=0}^{n} \binom{n}{i} W_i(K)\lambda^i = V(K)\left[\sum_{i=0}^{n-2} \binom{n}{i} \lambda^i + n\beta(K)\lambda^{n-1} + \alpha(K)\lambda^n\right]$$

with $\alpha(K) = \frac{\kappa_n}{V(K)}$ and $\beta(K) = \frac{W_{n-1}(K)}{V(K)}$.

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- Then $\sum_{i=2}^{n}\binom{n}{i}\mu^i$ has roots with positive real part when $n = 15$.
- We construct a 2-tangential body in $\mathbb{R}^{15}$ with $\beta(K), \alpha(K) \to 0$.

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- We construct a 2-tangential body in $\mathbb{R}^{15}$ with $\beta(K), \alpha(K) \to 0$.
- Zeros of polynomials are continuous functions of the coefficients.

In- and circumradius bounds in the conjecture

Recall (Sangwine-Yager's conjecture):

Let $K \in \mathcal{K}^n$. If $a_1 \leq \cdots \leq a_n$ are the real parts of the roots of the Steiner polynomial $\sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^i$, then

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The circumradius bound: (now $K \subset \mathbb{R}^3$ and $E = B_3$)

Lemma

Let $K \subset \mathbb{R}^3$ be a planar convex body, with area A and perimeter p. Then, all the roots of its Steiner polynomial have real part greater than $-R(K)$ if and only if $p(K)^2 < 128A(K)/(3\pi)$ and $p(K) < 16R(K)/3$.

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- Many symmetric lenses verify the above lemma.

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The circumradius bound: (now $K \subset \mathbb{R}^3$ and $E = B_3$)

- A **proper** 3-dimensional set $K \subset \mathbb{R}^3$ as a counterexample:

  The outer parallel body $L_\rho = L + \rho B_3$ of one of the previous lenses L has Steiner polynomial

$$\left(2A(L)\rho + \frac{\pi}{2}p(L)\rho^2 + \frac{4}{3}\pi\rho^3\right) + \left(2A(L) + \pi p(L)\rho + 4\pi\rho^2\right)\lambda + \left(\frac{\pi}{2}p(L) + 4\pi\rho\right)\lambda^2 + \frac{4}{3}\pi\lambda^3.$$

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- Its roots have real part $> -R(L_\rho) = -R(L) - \rho$, for all $\rho > 0$.

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- Its roots have real part smaller than $-r(B_3, L_\rho) = -\dfrac{1}{R(L)+\rho}$, for many values of ρ.

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If K is a p-tangential body of E, $p = 0, \dots, n-1$, there always exists a root of the Steiner polynomial with real part greater than $-1 = -r(K; E)$.

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Solutions for Teissier's problem

- All the sets in $\mathbb{R}^2$ provide a solution for Teissier's problem.
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...but they are not solutions: the negativity part does not hold in general.

- 1-tangential bodies (cap-bodies) provide a solution:

Theorem

Let K be a 1-tangential body of E. If $a_1 \leq \cdots \leq a_n$ are the real parts of the roots of the Steiner polynomial, $a_1 \leq -R(K; E) \leq -r(K; E) \leq a_n \leq 0$.

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Bounds for the roots of the Steiner polynomial in terms of $r(K;E)$ and $R(K;E)$:

Theorem

Let γ_i, $i = 1, \dots, n$, denote the roots of the Steiner polynomial of a convex body $K \in \mathcal{K}^n$. The following properties hold:

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1. They lie in the ring $\dfrac{1}{n} r(K;E) \le |\gamma_i| \le nR(K;E)$.
2. $\big|\mathrm{Re}(\gamma_1)\big| + \cdots + \big|\mathrm{Re}(\gamma_n)\big| \ge nr(K;E)$.
3. $\big|\mathrm{Re}(\gamma_1)\big| + \cdots + \big|\mathrm{Re}(\gamma_n)\big| \le nR(K;E)$, if $\mathrm{Re}(\gamma_i) \le 0$ for all i.

On the roots of Steiner polynomials

María A. Hernández Cifre

(joint work with Martin Henk)

Universität Magdeburg, Universidad de Murcia

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