

# Projection Functions of Convex Bodies

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June 2007

Projection functions:  $K \in \mathcal{K}^n, 1 \leq j \leq n - 1$

$$\pi_j(K) : \mathcal{L}_j^n \rightarrow \mathbb{R}, \quad L \mapsto V_j(K|L)$$

$j$ -th projection function of  $K$

constant width and constant  $j$ -brightness

**Thm** (Schneider) Let  $P \in \mathcal{P}_c^n$  be simplicial,  $K \in \mathcal{K}^n$  and  $1 \leq i \leq n - 2$ . Assume

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- Improvement (Bauer): no central symmetry required.

If

$$S_i(K, \cdot) \leq S_i(P, \cdot) + S_i(P^*, \cdot),$$

then  $K$  and  $P$  or  $P^*$  are homothetic.

**Thm** (Bauer) Let  $2 \leq i \leq n - 2$ . For most  $K_0 \in \mathcal{K}^n$  the following is true: If  $K \in \mathcal{K}^n$  and

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**Thm** (Campi; Gardner, Volčič; Goodey, Schneider, Weil)

There are  $K, K_0 \in \mathcal{K}^n$ , non-congruent, and

$$\pi_i(K) = \pi_i(K_0), \quad \text{for all } 1 \leq i \leq n.$$

## Selected results about determination of convex bodies:

- Non-centrally symmetric bodies are not determined by  $\pi_1$  nor by  $\pi_{n-1}$ .
- A convex body is determined by  $\pi_{n-1}$  iff it is a parallelepiped. [GV]
- Most convex bodies are determined by  $\pi_1$  and  $\pi_{n-1}$ . [GSW]  
Connexion to the covariogram problem!
- Centrally symmetric bodies are determined by any two of their projection functions. [CL]

Overall assumption:  $K_0$  is centrally symmetric.

Still one projection function is **not** enough: constant  $k$ -brightness

- $k = 1$ : classical, convolution
- $k = n - 1$ : Blaschke, convolution
- $2 \leq k \leq n - 2$ : bodies of revolution (Firey)
- $2 \leq k \leq n - 3$ : param. description, smooth case (Goodey/Howard)  
const.  $k$ -brightness  $\neq$  const.  $k$ -girth
- $2 \leq k \leq n - 2$ : perturb. argument, smooth case (Goodey/Howard)

Nakajima's problem (general form):

$(A_{ij})$  Let  $K, K_0 \in \mathcal{K}^n$ ,  $K_0 = K_0^*$ ,  $1 \leq i < j \leq n - 1$ ,  $\alpha, \beta > 0$ .

Assume that

$$\pi_i(K) = \alpha \pi_i(K_0) \quad \text{and} \quad \pi_j(K) = \beta \pi_j(K_0).$$

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YES:

- $n = 3$ :  $K_0$  is  $C^{1,1}$ -smooth (Howard)
- $n \geq 5$ :  $K, K_0$  smooth,  $(i, j) \neq (1, n - 1)$  and  $\neq (n - 2, n - 1)$ ;  
bodies of revolution (Howard, H)

**Thm** (Howard, H) Assume  $A_{ij}$ . Then  $K$  and  $K_0$  are homothetic, if  $i = 1$  and either of the following conditions is satisfied:

(a)  $j < (n + 1)/2$  and  $\det \text{Hess}(K_0) \neq 0$       *or*

(b)  $n = 5, j = 3$ , “some smoothness of  $K_0$ ”.

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**Thm** Assume  $A_{ij}$  for some  $1 \leq i < j \leq n - 2, (i, j) \neq (1, n - 2)$ , and  $\det \text{Hess}(K_0) \neq 0$ . Then  $K$  and  $K_0$  are homothetic.

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### Still open:

- General setting:  $(1, n - 2)$  and  $n \geq 6$ ;  
 $j = n - 1$  and  $n \geq 4$ ;
- Smooth setting:  $(1, n - 1), (n - 2, n - 1)$ .

**Proof.** Let  $\alpha = 1$ . Then

$$V_i(K|U) = V_i(K_0|U) \quad \text{and} \quad V_j(K|L) = \beta V_j(K_0|L)$$

for  $U \in \mathcal{L}_i^n$ ,  $L \in \mathcal{L}_j^n$ . Fix  $L$  and choose  $U \subset L$ .

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Then  $K|L$  and  $K_0|L$  belong to the same  $i$ -th projection class wrt  $L$  and  $K_0|L$  is centrally symmetric.

**Chakerian & Lutwak:**

$$V_j(K_0|L) \geq V_j(K|L)$$

with equality iff  $K_0|L$  is a translate of  $K|L$ .

Hence, for  $L \in \mathcal{L}_j^n$ ,

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i.e.  $\beta \leq 1$  with equality iff  $K_0|L$  is a translate of  $K|L$  for all  $L \in \mathcal{L}_j^n$ .

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$$\mathbf{P} := \{u \in \mathbb{S}^{n-1} : h_K, h_{K_0} \text{ sod at } \pm u \text{ and } \det \text{Hess}(K_0)(u) \neq 0\}$$

Fix a vector  $u \in \mathbf{P}$ .

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Fix a vector  $u \in \mathbf{P}$ .

Choose  $W \in \mathcal{L}_{j+1}^n$  with  $u \in W$ . Then  $U := u^\perp \cap W \in \mathcal{L}_j^n$ . From

$$V_j((K|W)|U) = \beta V_j((K_0|W)|U)$$

we get

$$\begin{aligned} & \int |\langle x, u \rangle| S_j^W(K|W; dx) + \int |\langle x, u \rangle| S_j^W(K^*|W; dx) \\ &= 2\beta \int |\langle x, u \rangle| S_j^W(K_0|W; dx). \end{aligned}$$

The injectivity of the cosine transform on even measures yields

$$S_j^W(K|W; \cdot) + S_j^W(K^*|W; \cdot) = 2\beta S_j^W(K_0|W; \cdot).$$

Since  $u \in \mathbf{P}$ ,

$$\begin{aligned} \det(d^2 h_{K|W}(u)|u^\perp \cap W) + \det(d^2 h_{K^*|W}(u)|u^\perp \cap W) \\ = 2\beta \det(d^2 h_{K_0|W}(u)|u^\perp \cap W). \end{aligned}$$

This leads to

$$\wedge^j L(h)(u) + \wedge^j L(h)(-u) = 2\beta \wedge^j L(h_0)(u)$$

where

$$L(h)(u) := d^2 h_K(u)|u^\perp : u^\perp \rightarrow u^\perp.$$

Since  $L(h_0)(u)$  is an isomorphism,

$$L_{h_0}(h)(u) := L(h_0)(u)^{-1/2} \circ L(h)(u) \circ L(h_0)(u)^{-1/2} .$$

Thus the assumptions yield

$$\begin{aligned} \wedge^i L_{h_0}(h)(u) + \wedge^i L_{h_0}(h)(-u) &= 2 \wedge^i \text{id} , \\ \wedge^j L_{h_0}(h)(u) + \wedge^j L_{h_0}(h)(-u) &= 2\beta \wedge^j \text{id} . \end{aligned}$$

The linear maps  $L_{h_0}(h)(u)$  and  $L_{h_0}(h)(-u)$  have a common ONB of eigenvectors  $e_1, \dots, e_{n-1}$  with corresponding, nonnegative eigenvalues  $x_1, \dots, x_{n-1}$  and  $y_1, \dots, y_{n-1}$  satisfying

$$(*) \begin{cases} x_I + y_I = 2, & |I| = i, \\ x_J + y_J = 2\beta, & |J| = j, \end{cases}$$

where

$$x_I := \prod_{i \in I} x_i, \quad x_\emptyset := 1 \quad \text{for } I \subset \{1, \dots, n-1\}.$$

We may assume that  $0 \leq x_1 \leq \dots \leq x_{n-1}$ , hence  $0 \leq y_{n-1} \leq \dots \leq y_1$ .

- If  $x_1 = 0$ , then  $x_2 = \dots = x_{n-1} =: x$  and  $y_2 = \dots = y_{n-1} =: y$ .

Hence

$$x^i + y^i = 2, \quad x^j + y^j = 2\beta,$$

which implies that

$$1 = \frac{x^i + y^i}{2} \leq \frac{(x^i)^{j/i} + (y^i)^{j/i}}{2} = \beta,$$

i.e.  $\beta \geq 1$ .

- The case  $y_{n-1} = 0$  is checked similarly.

- In the remaining case, all numbers are positive.

**Lemma** Let  $2 \leq k \leq n - 3$ ,  $x_1, \dots, x_{n-1} > 0$  and  $y_1, \dots, y_{n-1} > 0$ .  
Assume that

$$x_I + y_I = 2\beta \quad \text{whenever } |I| = k.$$

Then there is a subset  $J \subset \{1, \dots, n-1\}$  with  $|J| = n-2$  and  $x, y > 0$  such that  $x_\iota = x, y_\iota = y$  for  $\iota \in J$ .

The lemma and again Jensen's inequality imply that  $\beta \geq 1$ , and thus the result follows.