Geometric Tomography: Uniqueness, Stability and Consistency under Convexity Assumptions

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CONVEX GOEMETRY, Cortona June 5, 2007

Overview

- Tomographic measurements
- Determination (injectivity)
- stability
- reconstruction
- consistency.

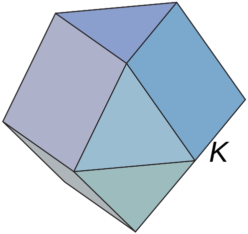
Motivation: the j-th girth function.

S^{n-1} := unit sphere in $\mathbb{R}^n$, $n \geqslant 2$

$K \in \mathcal{K} := \{$convex bodies in $\mathbb{R}^n\}$

endowed with the Hausdorff-metric δ

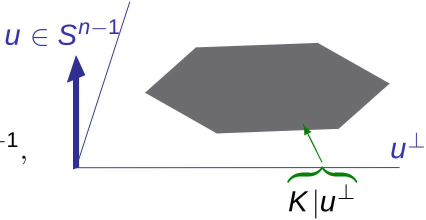
$V_j(K) :=$ j-th intrinsic volume of K.



Tomographic measurements of K:

$$F(K,u) := V_j(K|u^\perp), \quad u \in S^{n-1},$$

($=$ j-th girth function).



Motivation II

Known properties of $K \mapsto F(K,\cdot)$:

- **rotation covariance:**

$$F(\vartheta K, \vartheta(\cdot)) = F(K,\cdot), \qquad \vartheta \in SO_n.$$

- **continuity** from $(\mathcal{K}, \delta)$ to $(L^2, \|\cdot\|)$

  $L^2 =$ Hilbert-space of square integrable functions on S^{n-1},

- **additivity w.r.t. the j-th surface area measure:**

$$S_j(K,\cdot) + S_j(M,\cdot) = S_j(K',\cdot) + S_j(M',\cdot),$$

$$\Rightarrow F(K,\cdot) + F(M,\cdot) = F(K',\cdot) + F(M',\cdot)$$

  for all $K, M, K', M' \in \mathcal{K}$.

General assumptions on tomographic measurements

We consider tomographic measurements $F(K,\cdot)$ assuming

- rotation covariance,
- continuity,
- additivity w.r.t. an **analytic representation** $Q(K,\cdot)$ **of** K**:**

  $$Q(K,\cdot)+Q(M,\cdot)=Q(K',\cdot)+Q(M',\cdot),$$

  $$\Rightarrow F(K,\cdot)+F(M,\cdot)=F(K',\cdot)+F(M',\cdot)$$

  for all $K,M,K',M'\in\mathcal{K}$. For $Q(K,\cdot)$ we allow

  - a power h_K^θ of the support function, $\theta\neq 0$,
  - a power ρ_K^θ of the radial function, $\theta\neq 0$,
  - a surface area measure $S_j(K,\cdot)$.

Additivity w.r.t. an analytic representation

In special cases additivity w.r.t. $Q(K,\cdot)$ means simply

$$F(K \diamond M,\cdot) = F(K,\cdot) + F(M,\cdot).$$

Where $\diamond$ stands for

- Minkowski addition, if $Q(K,\cdot) = h_K$,
- radial addition, if $Q(K,\cdot) = \rho_K$,
- Firey's p-addition, if $Q(K,\cdot) = h_K^p$,
- Blaschke addition, if $Q(K,\cdot) = S_{n-1}(K,\cdot)$.

Multiplier-rotation operators I

For $f \in L^2$, we write $\vartheta f(u) := f(\vartheta^{-1}u)$ and $f \sim \sum_{k=0}^{\infty} f_k$ for its (condensed) **spherical harmonic expansion**. (Similar for measures.)

[K. 07]. Let $K \mapsto F(K,\cdot)$ be continuous, rotation covariant and additive w.r.t. $Q(K,\cdot) \sim \sum_{k=0}^{\infty} Q_k(K,\cdot)$.

Then $F(K,\cdot)$ is a multiplier-rotation operator:

$$F(K,\cdot) \sim \sum_{k=0}^{\infty} a_k \vartheta_k Q_k(K,\cdot),$$

with sequences (a_k) in $\mathbb{R}$ and (ϑ_k) in SO_n.

Multiplier-rotation operators II

$$F(K,\cdot) \sim \sum_{k=0}^{\infty} a_k \vartheta_k Q_k(K,\cdot).$$

- $a_0, a_1, \ldots$ are called **multipliers**,

  $|a_k|$ grows at most polynomially in k,
- if $n \geqslant 3$ we have $\vartheta_k = id$ for all k.

Method of proof I

- By additivity "extend" $K \mapsto F(K,\cdot)$ to a continuous operator T on the space $\mathcal{L}_Q := \operatorname{span}\{Q(K,\cdot) : K \in \mathcal{K}\}$:

$$T\left(\sum_{i=1}^{m} \varepsilon_i Q(K_i,\cdot)\right) := \sum_{i=1}^{m} \varepsilon_i F(K_i,\cdot)$$

  with $K_i \in \mathcal{K}$, $\varepsilon_i \in \{\pm 1\}$, $i = 1, \dots, m$.
- show that $\mathcal{C}^m \subset \mathcal{L}_Q$ (continuous representation),

  m-times differentiable functions on S^{n-1}
  (m appropriately chosen)
- apply spherical harmonic analysis to $T\big|_{\mathcal{C}^m}$ (Schur's Lemma).

Method of proof II

Let B^n be the unit sphere in $\mathbb{R}^n$, $\theta \neq 0$, $j \in \{1, \dots, n-1\}$.

- any $f \in \mathcal{C}^2$ has a representation
$$f = h_K^\theta - h_{rB^n}^\theta$$
with continuous $f \mapsto (K, r)$ ($\theta = 1$: [Schneider '74]).
- similar for ρ_K^θ, $\{f \in \mathcal{C}^m \ : \ f_1 \equiv 0\}$
- any $f \in \mathcal{C}_*^m$ with $m = 2\lceil (n+1)/4 \rceil$ has a representation
$$f = \sum_{i=1}^{j} S_j(K_i, \cdot) - S_j(rB^n, \cdot)$$
with continuous $f \mapsto (K_1, \dots, K_j, r)$
(strengthens [Weil '80] for $j = 1$).

Injectivity

$$F(K,\cdot) \sim \sum_{k=0}^{\infty} a_k \vartheta_k Q_k(K,\cdot).$$

- If $Q_k(K,\cdot) = 0$ whenever $a_k = 0$, then $Q(K,\cdot)$ is uniquely determined by $F(K,\cdot)$
- Hence

$$K \in \mathcal{K}_F := \{K \in \mathcal{K} : Q(K,\cdot) \sim \sum_{a_k \neq 0} Q_k(K,\cdot)\}$$

is uniquely determined by $F(K,\cdot)$.

(When $Q(K,\cdot) = \rho_K^\theta$ include condition $o \in K$,
when $Q(K,\cdot) = S_j(K,\cdot)$ include $(h_K)_1 = o$, dim$K > j$.)

Smoothing properties of $F(K,\cdot)$

$$F(K,\cdot) \sim \sum_{k=0}^{\infty} a_k \vartheta_k Q_k(K,\cdot).$$

- "reconstruction": $Q(K,\cdot) \sim \sum_{a_k \neq 0} a_k^{-1} \vartheta_k^{-1} (F(K,\cdot))_k$.
- We assume $\exists b > 0, \beta \geqslant 0$:

  $\boxed{|a_k|^{-1} \geqslant b k^{\beta}}$ for all k with $a_k \neq 0$.

  and call β smoothing parameter.

Spherical Sobolev spaces and support functions

For $f \in L^2$, $f \sim \sum_{k=0}^{\infty} f_k$, set

$$\|f\|_\eta^2 := \sum_{k=0}^{\infty} (1 \vee k)^\eta \|f_k\|^2, \qquad \eta \geqslant 0.$$

$H^\eta := \{f \in L^2 : \|f\|_\eta < \infty\}$ Sobolev space of derivative order η.

For $0 \leqslant r \leqslant R$ set $\mathcal{K}(r, R) := \{K \in \mathcal{K}(R) : rB^n \subset K \subset RB^n\}$.

For $0 \leqslant \eta < 3/2$, $\{h_K^\theta : K \in \mathcal{K}(r, R)\}$ is bounded in H^η.

- $\theta \in \mathbb{N}$: true for $\mathcal{K}(R) := \{K \in \mathcal{K} : K \subset RB^n\}$.
- follows essentially from [Campi '98].

Stability (Support Function Powers)

Let $F(K,\cdot)$ be additive w.r.t. h_K^θ; $\boxed{|a_k| \geqslant bk^{-\beta}}$ when $a_k \neq 0$.

[K. 07]. For $\gamma > 0$ there is $c = c(n,\theta,b,\beta,\gamma,r,R)$ with

$$\delta_2(K,K') \leqslant c\|F(K,\cdot) - F(K',\cdot)\|^{3/(3+2\beta)-\gamma}$$

for all $K,K' \in \mathcal{K}_F(r,R)$.

Here, $\mathcal{K}_F(r,R) := \mathcal{K}_F \cap \mathcal{K}(r,R)$, $0 < r \leqslant R$,

$\delta_2(K,K') = \|h_K - h_{K'}\|$.

Earlier approaches use $\eta = 1 \Rightarrow$ an exponent $\dfrac{1}{1+\beta}$.

Application I: The first girth function

For $\gamma > 0$ there is $c = c(n, \gamma, R)$ with

$$\delta_2(K, K') \leqslant c \| V_1(K|u^\perp) - V_1(K'|u^\perp) \|^{3/(n+1) - \gamma}$$

for all origin-symmetric $K, K' \in \mathcal{K}(R)$.

Compare the exponent $2/n$ obtained by

[Campi '86] for $n = 3$ and [Goodey & Groemer '90].

Remark: If α_0 is the supremum of all possible Hölder exponents in $\mathbb{R}^3$ then

$$\frac{3}{4} \leqslant \alpha_0 \leqslant \frac{5}{6}.$$

Further applications

- **First girth and Steiner point of hyperplane projections**

  $$\|V_1(K|u^\perp)-V_1(K'|u^\perp)\| \leqslant \varepsilon, \quad \| |s'(K|u^\perp)-s'(K'|u^\perp)| \| \leqslant \varepsilon$$

  then $\delta_2(K,K') \leqslant c\varepsilon^{3/(n+1)-\gamma}$ (cf. [Schneider '06]: $2/n$).

- **directed projection function and averages**

  cf. [Goodey & Weil '06].

The kth projection mean

Let $\mathcal{L}_k^n$, $k = 1, \dots, n-1$, be the Grassmannian of all k-dim subspaces of $\mathbb{R}^n$.

$$F(K,u) := \int_{\mathcal{L}_k^n} h_{K|L}(u)\,dL$$

is the invariant mean of all orth. proj. $K|L$ of K on $L \in \mathcal{L}_k^n$. Then

$$\delta_2(K,K') \leqslant c\|F(K,\cdot) - F(K',\cdot)\|^{3/(3+2(n-k))-\gamma}$$

holds for $K, K' \in \mathcal{K}(R)$,

- if $k = n-1$ (improving [Spriestersbach '98]),
- if k is even and K, K' are origin-symmetric, [K. '04],
- if k is odd and K, K' have constant width, [K. '04].

Stability (Radial Function Powers)

Let $F(K,\cdot)$ be additive w.r.t. ρ_K^θ; $\boxed{|a_k| \geqslant bk^{-\beta}}$ when $a_k \neq 0$.

For $\gamma > 0$ there is $c = c(n, \theta, b, \beta, \gamma, r, R)$ with

$$\rho_2(K,K') \leqslant c\|F(K,\cdot) - F(K',\cdot)\|^{3/(3+2\beta)-\gamma}$$

for all $K, K' \in \mathcal{K}_F(r,R)$.

Here, $\rho_2(K,K') = \|\rho_K - \rho_{K'}\|$.

Applications.

- **Central section volumes:**

$$\rho_2(K,K') \leqslant c \cdot \|V_{n-1}(K \cap u^\perp) - V_{n-1}(K' \cap u^\perp)\|^{3/(n+1)-\gamma}$$

for all origin-symmetric $K, K' \in \mathcal{K}(r,R)$,

cf. [Groemer '96] with $2/n$; [Campi '98].

- **Cone volumes:**

$C_\varphi(u) :=$ cone of revolution with axis pos(u), $u \in S^{n-1}$,
opening angle 2φ, $\varphi \in (0,\pi) \cap \pi\mathbb{Q}$

$$\rho_2(K,K') \leqslant c \cdot \|V_n(K \cap C_\varphi(\cdot)) - V_n(K' \cap C_\varphi(\cdot))\|^{3/(n+5)-\gamma}$$

for all $K, K' \in \mathcal{K}_F(r,R)$.

[Rubin '00]: $\beta = (n+2)/2$ (many cases $\beta = n/2$).

- **Directed section functions** [Goodey & Weil '06].

Stability: $(n-1)$st surface area measure

$F(K,\cdot)$ be add. w.r.t. $S_{n-1}(K,\cdot)$; $\boxed{|a_k| \geqslant bk^{-\beta}}$ when $a_k \neq 0$.

For $\beta > 3/2$, $\gamma > 0$ there is $c = c(n, b, \beta, \gamma, r, R)$ with

$$\delta(K,K') \leqslant c\|F(K,\cdot) - F(K',\cdot)\|^{1/(n(\beta-1/2))-\gamma}$$

for all $K, K' \in \mathcal{K}_{a,S_{n-1}}(r,R)$.

- $\beta \leqslant 3/2$: Hölder exponent $1/n$.
- [Hug & Schneider '02]: Hölder exponent $1/(n(\beta+1)) - \gamma$.
- with $S_{j-1}(K,\cdot)$ replacing $S_{n-1}(K,\cdot)$: same exponent $\times 2^{-j}$.

Application: Brightness Function

For the **brightness function** ($(n-1)$st girth function):

$$\delta(K,K') \leqslant c \cdot \|V_{n-1}(K|u^\perp) - V_{n-1}(K'|u^\perp)\|^{2/(n(n+1))-\gamma}$$

for all origin-symmetric $K, K' \in \mathcal{K}(r,R)$.

- [Campi '88] for $n = 3$ with exponent $1/9 - \gamma$.
- [Bourgain & Lindenstrauss '88]

  with exponent $2/(n(n+4)) - \gamma$.

The proof for general $F(K,\cdot)$ is based on a combination of **Poisson integral estimates** [BL], [HS] and **Sobolev estimates** [C].

Applications II

- **illumination-brightness fct.** $a, b \in \mathbb{R} \setminus \{0\}$

$$F(K,u) := aV_{n-1}(K|u^\perp) + bS_{n-1}(K, S^{n-1} \cap u^+).$$

[Anikonov & Stepanov '94] show injectivity for $a + b \neq 0$.
Hölder exponent $2/(n(n+1)) - \gamma$.

- **directed projection functions** cf. [Goodey & Weil '06]
- **2nd mean section body**

$$F(K, \cdot) := \int_{\mathcal{L}_2^n} \int_{L^\perp} h(K \cap (x+L), \cdot)\, dx dL.$$

[Goodey & Weil '92] show injectivity (up to translations)
Hölder exponent $1/(n(n-1/2)) - \gamma$.

Tomographic transforms and endomorphisms of $\mathcal{K}$

$F(K,\cdot)$ continuous, rotation covariant and additive w.r.t. $h(K,\cdot)$. Assume now: for all $K \in \mathcal{K}$ there is $\overline{K} \in \mathcal{K}$ with $\mathbf{F}(\mathbf{K},\cdot) = \mathbf{h}_{\overline{\mathbf{K}}}$.

Then

$$\Phi : \mathcal{K} \to \mathcal{K}, \quad K \mapsto \overline{K}$$

is a **Minkowski endomorphism** of $\mathcal{K}$: it is continuous and rotation covariant, respects Minkowski addition.

- [Schneider '74a] $n = 2$: complete characterization,
- [Schneider '74b] $n \geqslant 3$: $|a_k| \leqslant a_0$ for all $k \neq 1$.

Endomorphisms of $\mathcal{K}$

[K. '06]. If $n \geqslant 3$: $\exists$ a spherical zonal distribution G with

$$F(K,\cdot) = h_K * G.$$

Necessary conditions for G are known (e.g. order(G)$\leqslant 2$).

- $*$ denotes convolution on $S^{n-1} = SO_n/\underbrace{SO_{n-1}}_{\{\vartheta \in SO_n \,:\, \vartheta p = p\}}$,

  induced by the usual convolution on SO_n.
- G is **zonal** iff for all test functions φ,

  $$G(\varphi) = G(\vartheta\varphi), \qquad \forall \vartheta \in SO_{n-1}.$$

  G can be identified with a distribution $\tilde{G}$ on $[-1,1]$.
- sufficient conditions for G are unknown.

Weakly monotonic endomorphisms of $\mathcal{K}$

$K \mapsto F(K,\cdot)$ is called **weakly monotonic** iff

$$K \subset K' \;\Rightarrow\; F(K,\cdot) \leqslant F(K',\cdot) \quad (\Leftrightarrow \overline{K} \subset \overline{K'}),$$

whenever $(h_K)_1 = (h_{K'})_1 = o$ (Steiner points at o).

[K. '06]. If $n \geqslant 3$ and $K \mapsto F(K,\cdot)$ is weakly monotonic, there exists a zonal signed measure μ on S^{n-1} with

$$F(K,\cdot) = h_K * \mu.$$

There is a constant c such that $\mu + c\langle p,\cdot\rangle$ is positive.

- The above conditions are necessary and sufficient.
- **Conjecture:** Any Minkowksi endom. is weakly monotonic.

Blaschke Minkowski homomorphisms

Replace additivity w.r.t. h_K by add. w.r.t. $S_{n-1}(K,\cdot)$. Then

$$\Phi : \mathcal{K} \to \mathcal{K}, \quad K \mapsto \overline{K}$$

is a **Blaschke Minkowski homomorphism** of $\mathcal{K}$:

$$\Phi(K\#K') = \Phi(K) + \Phi(K'), \qquad K, K' \in [\mathcal{K}_0].$$

[Schuster '07]. There exists a zonal function f on S^{n-1} with

$$F(K,\cdot) = S_{n-1}(K,\cdot) * f.$$

There is a constant c such that $f + c\langle p,\cdot\rangle$ is positive.

- Necessary and sufficient conditions are unknown.
- If $F(K,\cdot) = F(-K,\cdot)$, f is a support function (Characterization).

Reconstruction from noisy measurements

- **Input:**
  - ∘ a sequence of measurement directions $u_1, u_2, \ldots$ in S^{n-1}
  - ∘ k noisy measurements $f_i := F(K, u_i) + X_i,\ i = 1, \ldots, k$ with i.i.d. errors, $X_1 \sim N(0, \sigma^2)$
- **Task:** Determine $\hat{K}_k \in \mathcal{K}$ close to the unknown $K \in \mathcal{K}$.
- **Action:** Let $\hat{K}_k$ be a solution of the least squares problem

$$(LSQ) \quad \boxed{\begin{array}{c} \min \sum_{i=1}^{k} (f_i - F(M, u_i))^2 \\ \text{subject to } M \in \mathcal{K}. \end{array}}$$

((LSQ) always has a solution!)

Loss-free Discretization I

- If $F(K,\cdot)$ is additive w.r.t. h_K^θ or ρ_K^θ

  very little is known about a loss-free discretization.

- If $F(K,\cdot)$ is additive w.r.t. $S_{n-1}(K,\cdot)$, then *(LSQ)* has a solution
  - which is a polytope with at most $(n+1)\cdot k$ facets,
  - which is a polytope with at most $2\cdot k$ facets,

    if $F(K,\cdot)=F(-K,\cdot)$,

This follows from a **Carathéodory argument** in $\mathbb{R}^k$.

Loss-free discretization for $F(K,u) = V_{n-1}(K|u^\perp)$

[K. 2001] $\Rightarrow$ There is a polytope that solves (LSQ) with facet normals in a fixed, finite set $V \subset S^{n-1}$ (V depends only on $u_1,\dots,u_k$; $m := \#V \leqslant k^{n-1}$).

[Gardner & Milanfar 2003] observed that this also follows from [Campi, Colesanti & Gronchi '95].

If α_j is the area of the facet with normal v_j, $j = 1,\dots,m$, $\Rightarrow$

$$(LSQ) \rightsquigarrow (LSQ') \qquad \boxed{\begin{array}{c} \min \sum_{i=1}^{k} \Big(f_i - \sum_{j=1}^{m} \alpha_j |\langle u_i, v_j\rangle|\Big)^2 \\ \text{subject to } \alpha_1,\dots,\alpha_m \geqslant 0. \end{array}}$$

which is a quadratic program.

Convergence: Is the estimator consistent?

Let $\hat{K}_k$ be a solution of (LSQ). Does it follow that

$$\hat{K}_k \to K, \quad \text{as} \quad k \to \infty,$$

(almost surely) if the test directions $u_1, u_2, \ldots$ are suitably chosen?

We require that $F(K, \cdot)$ is add. w.r.t. $S_{n-1}(K, \cdot)$ and the convexity assumption:

$$\forall K \in \mathcal{K} : \ F(K, \cdot) = h_{\overline{K}} \quad \text{for some} \quad \overline{K} \in \mathcal{K}.$$

Recall examples:

- brightness function $F(K, u) = V_{n-1}(K|u^\perp) = h(\Pi_{n-1}K, u)$,
- 2nd mean section body $F(K, u) = h(M_2(K), u), \ldots$.

Consistency

Let $u_1, u_2, \ldots$ be **uniformly spread** ($u_1, u_2, \ldots, u_m$ forms an ε_m-net in S^{n-1} with $\varepsilon_m = O\left(m^{-1/(n-1)}\right)$ as $m \to \infty$).

$F(K, \cdot) = h_{\overline{K}}$ add. w.r.t. $S_{n-1}(K, \cdot)$, $\beta \geqslant 3/2$, $K \in \mathcal{K}_F(r, R)$, $\hat{K}_k$ a solution of (LSQ) and $\gamma > 0$, then, almost surely,

$$\boxed{\delta(\hat{K}_k, K) \leqslant \begin{cases} C \cdot k^{-\frac{1}{5(\beta - 1/2)}}, & \text{if } n = 2 \\ C \cdot k^{-\frac{1}{4n(\beta - 1/2)} + \gamma}, & \text{if } n = 3, 4, 5, \\ C \cdot k^{-\frac{1}{n(n-1)(\beta - 1/2)} + \gamma}, & \text{if } n \geqslant 6. \end{cases} \quad \text{for all } k \geqslant k_0.}$$

cf. [Gardner, K. 07+]

Key ingredients for the proof

- stability results for $F(K,\cdot)$,
- ε-entropy H_ε (in the Hausdorff metric) for

$$\mathcal{I}_F := \{\overline{K} \subset B^n \ : \ \exists K \in \mathcal{K} \text{ with } F(K,\cdot) = h_{\overline{K}}\} \subset \mathcal{K}(1).$$

[Bronshtein '76]: $H_\varepsilon \leqslant c_n \varepsilon^{-(n-1)/2}$.

When $F(K,u) = V_{n-1}(K|u^\perp)$, then
$\mathcal{I}_F$ is the family of centered zonoids in B^n, and

$$H_\varepsilon \leqslant c_n' \varepsilon^{-2(n-1)/(n+2)}$$

[Gardner, K., Milanfar 2006] based on [Matoušek 96].

Consistency for the brightness function

$$\delta(\hat{K}_k, K) \leqslant \begin{cases} C \cdot k^{-\frac{4}{15}}, & \text{if } n = 2 \\ C \cdot k^{-\frac{n+2}{(2n+1)(n+1)n} + \gamma}, & \text{if } n = 3, 4, \\ C \cdot k^{-\frac{2}{(n+1)n(n-1)} + \gamma}, & \text{if } n \geqslant 5. \end{cases}$$

for all $k \geqslant k_0$.

Example:

For $n = 3$ the exponent is $-5/84 \approx -0.060$,

cf. $-5/147 \approx -0.034$ in [Garnder, K. Milanfar 06].