Bivaluations on Convex Bodies

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Cortona 2007

Definition

$\mu : \mathcal{K}^n \to \mathbb{R}$ is a valuation ($\mathcal{K}^n$ set of convex bodies in $\mathbb{R}^n$)

$$\Longleftrightarrow$$

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$

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Example

$V_n(\,\cdot\,)$ volume

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$V_n(\,\cdot\, + L)$	$L \in \mathcal{K}^n$
$\Omega(\,\cdot\,)$	affine surface area

Hadwiger's Classification Theorem (1951)

A functional $\mu : \mathcal{K}^n \to \mathbb{R}$ is a continuous and rigid motion invariant valuation

$$\Longleftrightarrow$$

$\exists \; c_0, c_1, \ldots, c_n \in \mathbb{R}$:

$$\mu(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K)$$

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Klain 1995
Alesker, Bernig, Betke, Chen, Fu, Goodey, Groemer, Haberl, Hadwiger, Hug, Kiderlen, Klain, Kneser, Lawrence, McMullen, Peri, Reitzner, Sallée, Schneider, Schuster, Weil, ...

Definition (Klain & Rota)

$\mu : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}$ is a bivaluation

$$\iff$$

$$\mu(K, \cdot\,) : \mathcal{K}^n \to \mathbb{R}$$

are valuations for all $K, L \in \mathcal{K}^n$

$$\mu(\,\cdot\,, L) : \mathcal{K}^n \to \mathbb{R}$$

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Example

$$\mu(K,L) \quad = \quad V_i(K)\,V_j(L)$$

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Example

$$\begin{aligned} \mu(K, L) &= V_i(K)\, V_j(L) \\ \mu(K, L) &= \int_{K \times L} f(x, y)\, dx\, dy \end{aligned}$$

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\mu(K,L) &= V(K,\dots,K,L) \;=\; \lim_{\varepsilon\to 0} \frac{V(K+\varepsilon L)-V(K)}{\varepsilon}
\end{aligned}$$

Theorem (L. 2007)

A functional $\mu : \mathcal{K}^n \times \mathcal{K}_0^n \to \mathbb{R}$ *is a bivaluation and a Minkowski surface area*

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$\exists c \geq 0$:

$$\mu(K, L) = c\, V(K, \dots, K, \Pi L^*)$$

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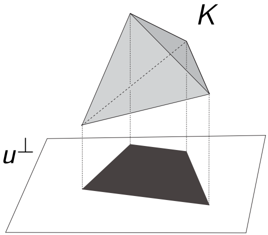
- $\Pi\, K$ projection body of K

Projection Bodies

Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^\perp), \;\; u \in S^{n-1}$$



- $h(K, u) = \max\{u \cdot x : x \in K\}$
  support function of K
- ΠK projection body of K
- $V_{n-1}(\cdot)$ $(n-1)$-dimensional volume
- $u^\perp$ hyperplane orthogonal to u
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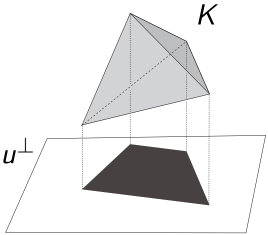

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Note (Petty 1967)

$\Pi(\phi K) = |\det \phi| \, \phi^{-t} \, \Pi K$ for all $\phi \in \mathrm{GL}(n), K \in \mathcal{K}_0^n$

Surface Area in Minkowski Space

$(\mathbb{R}^n, B)$ normed space with unit ball B

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$\mu : \mathcal{K}^n \times \mathcal{K}^n_c \to [0, \infty)$ is a Minkowski surface area

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- μ is continuous
- $\mu(\,\cdot\,, B)$ is a measure on $\mathcal{B}(H)$ for every hyperplane H
- $P \in \mathcal{K}^n$ polytope with facets $F_1, \ldots, F_m \;\Rightarrow$

$$\mu(F_1, B) \leq \sum_{i=2}^{m} \mu(F_i, B)$$

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Definition

Let $K \in \mathcal{K}^n$ and $L \in \mathcal{K}^n_c$. Then

$$\mu(K, L) = V(K, \ldots, K, \Pi L^*)$$

is called Holmes-Thompson surface area of K in the normed space with unit ball L.

Outline of the Proof.

$\mu(\,\cdot\,, B)$ Minkowski surface area

$$\overset{\textit{Busemann}}{\Longrightarrow}$$

$\exists\ \mathbb{I}\,B \in \mathcal{K}^n$ (isoperimetrix):

$$\mu(K, B) = V(K, \ldots, K, \mathbb{I}\,B)$$

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$$\overset{\textit{bivaluation}}{\Longrightarrow}$$

$\mathbb{I}$ is a Minkowski valuation:

$$\mathbb{I}\,B + \mathbb{I}\,C = \mathbb{I}(B \cup C) + \mathbb{I}(B \cap C)$$

for all $B, C \in \mathcal{K}_0^n$ such that $B \cup C \in \mathcal{K}_0^n$

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$$\overset{\textit{invariance}}{\Longrightarrow}$$

$$\mathbb{I}(\phi B) = |\det\phi|^{-1}\phi\,\mathbb{I}\,B$$

for all $\phi \in \mathrm{GL}(n)$

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Definition

$\mathrm{Z} : \mathcal{K}_0^n \to \mathbb{R}$ is GL(n) covariant of weight q

$$\Leftrightarrow$$

$$\mathrm{Z}(\phi K) = |\det \phi|^{q} \phi\, \mathrm{Z}\, K$$

for all $K \in \mathcal{K}_0^n$, $\phi \in \mathrm{GL}(n)$

Classification of Minkowski Valuations

Theorem (L. 2007)

$\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ *is a non-trivial valuation such that*

$$\mathrm{Z}(\phi P) = |\det \phi|^{-1} \phi \, \mathrm{Z}\, P$$

for $\phi \in \mathrm{GL}(n)$, $P \in \mathcal{P}_0^n$

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$\mathcal{P}_0^n$ set of convex polytopes P in $\mathbb{R}^n$ with $0 \in \operatorname{int} P$

Outline of the Proof.

μ Minkowski surface area

$$\Longrightarrow$$

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Classification of Minkowski Valuations

Theorem (L. 2007)

$\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}^n, + \rangle$ *is a non-trivial valuation which is* $\mathrm{GL}(n)$ *covariant of weight* q

$$\Longleftrightarrow$$

$\exists\, c_0 \in \mathbb{R}, c_1 \geq 0$:

$$\mathrm{Z}\,P = \begin{cases} c_0\, m(P) + c_1\, \mathrm{M}\, P & \text{for} \quad q = 1 \\ c_1\, \Pi\, P^* & \text{for} \quad q = -1 \end{cases}$$

for all $P \in \mathcal{P}_0^n$

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$$m(K) = \int_K x\, dx \quad \text{moment vector}$$

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for all $P \in \mathcal{P}_0^n$

$$h(\mathrm{M}\, K, u) = \int_K |u \cdot x|\, dx \quad \text{moment body}$$