Convexity, Optimal Mass Transport and Design of Mirrors

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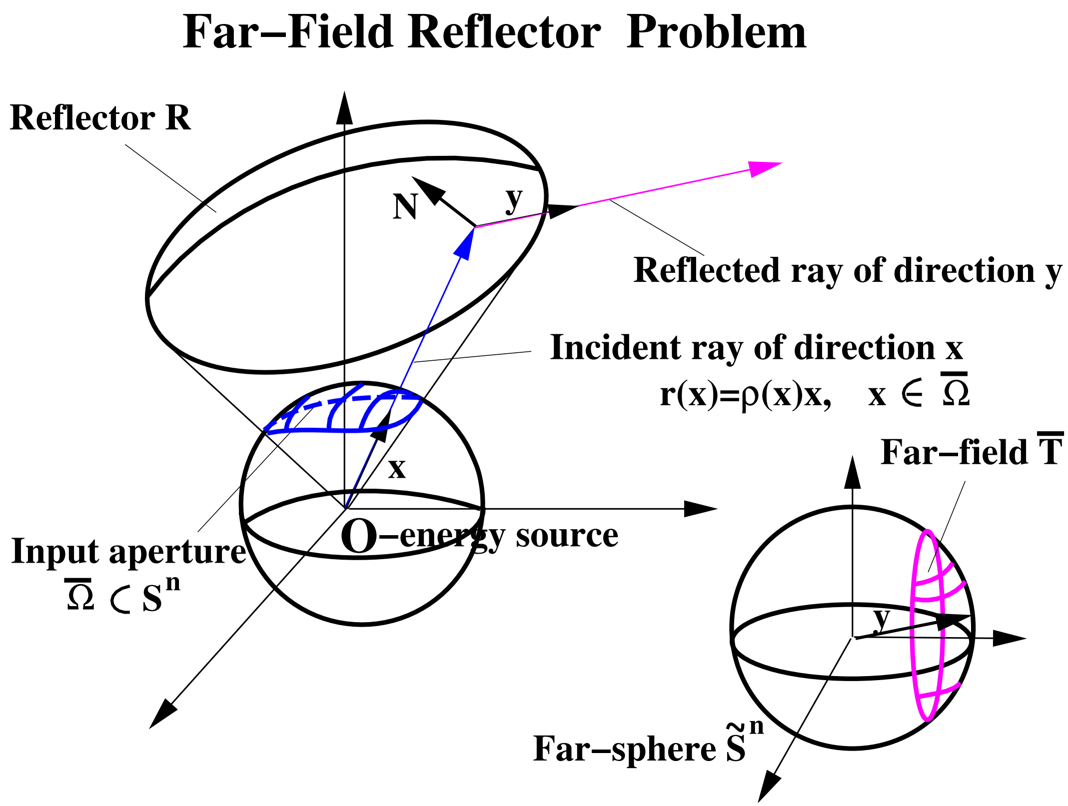


Problem: Given $\bar{\Omega} \subset \mathbf{S}^n,\ g \in L(\Omega),\ g \geq 0$, $\bar{T} \subset \tilde{\mathbf{S}}^n,\ f \in L(T),\ f \geq 0$. **Determine** R such that the rays from $\mathcal{O}$ through the $\bar{\Omega}$ with density distribution $g(\mathbf{x})$ are transformed into reflected rays with directions covering the region $\bar{T}$ with given density distribution f. Blockage is to be avoided!

Part I. From optics to PDE’s

If the reflector R is smooth and star-shaped with respect to $\mathcal{O}$ then we have **the reflection law** and the **reflector map** γ_R

$$y \equiv \gamma_R(x) = x - 2\langle x, N(x)\rangle N(x),\ x \in \bar{\Omega},$$

and **the conservation of energy law along infinitesimal tubes of rays**:

$$f(\gamma_R(x))|J(\gamma_R(x))| = g(x),\ \ x \in \Omega,$$

where J is the Jacobian determinant. Instead of the boundary condition it is required that

$$\gamma_R(\bar{\Omega}) = \bar{T}.$$

A convenient formulation is to describe R using $\gamma_R^{-1} : \bar{T} \to \bar{\Omega}$ so that $\gamma_R^{-1}(\bar{T}) = \bar{\Omega}$ and

$$g(\gamma_R^{-1}(y))|J(\gamma_R^{-1})| = f(y), \quad y \in T. \tag{1}$$

There exists a positive function $p : \bar{T} \to (0, \infty)$ such that

$$g(\gamma_R^{-1}(y)) \left| \frac{\det[Hess(p) + (p - \rho)e]}{\rho^n \det(e)} \right| = f(y), \quad y \in T, \tag{2}$$

where $\rho = \frac{p^2 + |\nabla p|^2}{2p}$. The position vector of the reflector R in terms of p is

$$r(y) = -\nabla_{\mathbf{S}^n} p(y) - p(y)y + \rho(y)y, \quad y \in \bar{T} \text{ and } \gamma_R^{-1} = \frac{r(y)}{\rho(y)} : \bar{T} \to \bar{\Omega}. \tag{3}$$

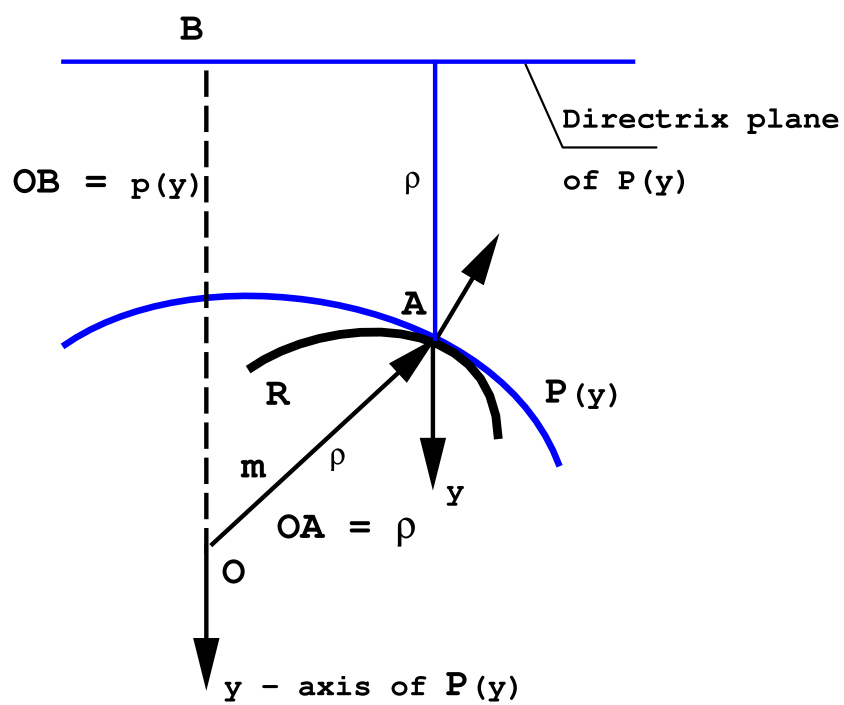
BACK to EUCLID!

FROM PDE's TO GEOMETRY

Part II. Geometry of convex reflectors and weak solutions

First, - an observation,

Suppose we already have a reflector R solving the problem. Then, for each $y \in \bar{T}$ we have a **paraboloid of revolution** $P(y)$ with focus at $\mathcal{O}$ and axis y, tangent to R at the point A. Let $p(y)$ be the focal parameter of $P(y)$.



Geometric description of convex reflectors

Main idea. Since for the reflector R the map γ is supposed to map $\bar{\Omega}$ onto $\bar{T}$, we may try to re-construct R as an envelope of a family of confocal paraboloids of revolution with common focus $\mathcal{O}$ and with axes $\{y \in \bar{T}\}$.

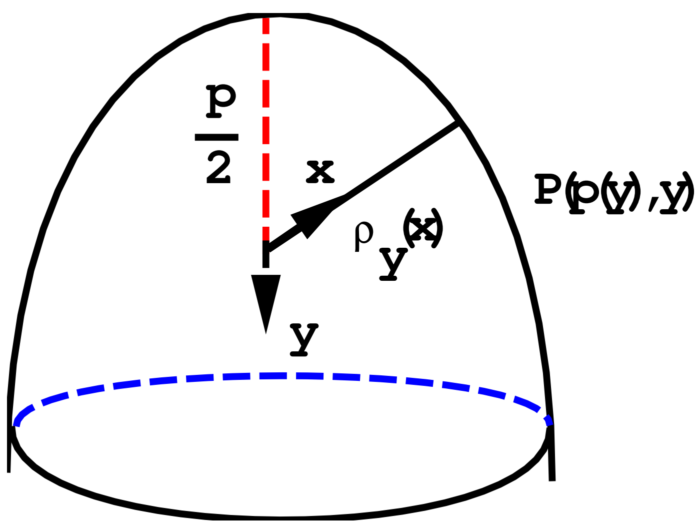
The best way to describe such families of confocal paraboloids of revolution with common focus $\mathcal{O}$ is to use focal functions.

To make things simple, for the time being we forget about $\bar{\Omega}$ and $\bar{T}$ and assume that $\bar{\Omega} = \mathbf{S}^n$ and $\bar{T} = \mathbf{S}^n$. Later we will see that this is not a restriction (in the weak solutions theory) and can be controlled by setting the input density g and output density $f = 0$ ouside Ω and T, respectively!

Genesis

Let $\mathcal{O} \in \mathbf{R}^{n+1},\ n \geq 1$. Let $p : \mathbf{S}^n \to (0, \infty)$, $p \not\equiv \infty$.
Let $P(y, p(y))$ be a paraboloid of revolution with axis $y \in \mathbf{S}^n$, focus $\mathcal{O}$, and *focal parameter* $p(y)$. For a fixed $y \in \mathbf{S}^n$ the polar equation of $P(y, p(y))$ is

$$\rho_y(x) = \frac{p(y)}{1 - \langle x, y \rangle}, x \in \mathbf{S}^n \setminus \{y\}.$$



Let

$$B(y,p(y)) = \{X \in \mathbf{R}^{n+1} \mid |X| - \langle X, y\rangle \leq p(y)\}.$$

This is the solid bounded by $P(y,p(y))$.

Definition. Put

$$B = \bigcap_{y \in \mathbf{S}^n} B(y,p(y)), \quad R = \partial B.$$

R is called a **reflector (defined by p) with light source $\mathcal{O}$**.

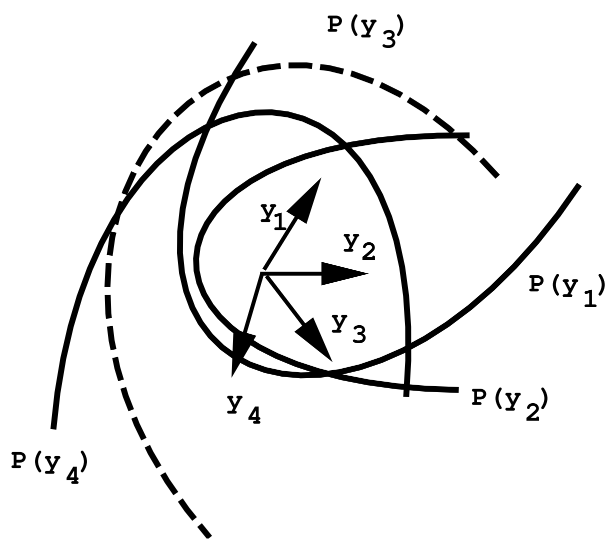
Clearly, R is a convex hypersurface and $\mathcal{O} \in int(B)$.

Notation. **$\mathcal{R}^n$ denotes the set of all compact convex reflectors with light source $\mathcal{O}$ generated by positive functions on S^n.**

Definition. Let $R \in \mathcal{R}^n$. A paraboloid $P(y,a), a > 0, y \in \mathbf{S}^n$, is **supporting to** R if

$$R \subset B(y,a) \quad \text{and} \quad R \cap P(y,a) \neq \emptyset.$$

For a $R \in \mathcal{R}^n$ generated by a function p some $P(y,p(y))$ may not be supporting (as the parabola $P(y_3)$ below).



Definition. Let $R \in \mathcal{R}^n$. A function $p(y), y \in \mathbf{S}^n$, such that $\forall y \in \mathbf{S}^n$ the paraboloid $P(y, p(y))$ is supporting to R is called **the focal function of** R.

The function $\rho(x) = \text{distance}(\mathcal{O}, R)$ in direction $x \in \mathbf{S}^n$ is the **radial function of** R.

Note 1. (Analogue of the Legendre transform/Duality) The functions p and ρ for a reflector $R \in \mathcal{R}^n$ are related by

$$\rho(x) = \inf_{y \in \mathbf{S}^n} \frac{p(y)}{1 - \langle x, y \rangle}, \quad x \in \mathbf{S}^n,$$

$$p(y) = \sup_{x \in \mathbf{S}^n} \rho(x)(1 - \langle x, y \rangle), \quad y \in \mathbf{S}^n.$$

Note $1'$. (Polarity.) Stated differently, for a given reflector R and the convex body B bounded by R we may define the “polar” body B^* as

$$\left\{Y \in \mathbf{R}^{n+1} \mid |X||Y| - \langle X, Y\rangle \leq 1\right\} \quad \forall X \in B.$$

If we consider the boundaries R and R^* as graphs over $\mathbf{S}^n$ and define them by their radial functions $\rho(x),\ x \in \mathbf{S}^n$ and $\rho^*(y),\ y \in \mathbf{S}^n$, then

$$R: \quad \rho(x) = \inf_{y \in \mathbf{S}^n} \frac{1/\rho^*(y)}{1 - \langle x, y\rangle}, \quad x \in \mathbf{S}^n,$$

and

$$R^*: \quad \rho^*(y) = \inf_{x \in \mathbf{S}^n} \frac{1/\rho(x)}{1 - \langle x, y\rangle}, \quad y \in \mathbf{S}^n.$$

In our notation,

$$\rho^*(y) \equiv \frac{1}{p(y)} = \inf_{x \in \mathbf{S}^n} \frac{1/\rho(x)}{1 - \langle x, y\rangle}, \quad y \in \mathbf{S}^n.$$

Note 2. The reflectors in $\mathcal{R}^n$ may not be smooth but the "reflector map" (possibly, multivalued) is defined as

$$\gamma_R(x) = \{y \in \mathbf{S}^n \mid \rho(x) = \frac{p(y)}{1 - \langle x, y \rangle}\}, \quad x \in \mathbf{S}^n.$$

Note 3. The "visibility" set of $y \in \mathbf{S}^n$ is

$$\gamma_R^{-1}(y) = \{x \in \mathbf{S}^n \mid \rho(x) = \frac{p(y)}{1 - \langle x, y \rangle}\}$$

Proposition. $\omega \in \mathcal{B}(\mathbf{S}^n) \implies \gamma_R^{-1}(\omega)$ is Lebesgue measurable. Also, $\sigma_{\mathbf{S}^n}[x \in \mathbf{S}^n \mid x \in (\gamma_R^{-1}(y_1) \cap \gamma_R^{-1}(y_2))] = 0$ if $y_1 \neq y_2$.

Note 4. Another class of (non-convex) reflectors is obtained by

$$\rho(x) = \sup_{y \in \mathbf{S}^n} \frac{p(y)}{1 - \langle x, y \rangle}, \quad x \in \mathbf{S}^n.$$

Theorem. (Characterization of focal functions, V. O.'04) Let $R \in \mathcal{R}^n$ and p its focal function. Then p is a positive bounded function on $\mathbf{S}^n$ and if for $x, y \in \mathbf{S}^n$

$$x - y = \sum_{i=1}^{s} \alpha_i (x - y_i),\ \alpha_i \geq 0, \sum_{i=1}^{s} \alpha_i > 0, \tag{4}$$

where the vectors $\{x - y_1, , ..., x - y_s\}$ are linearly independent, then the inequality

$$p(y) \leq \sum_{i=1}^{s} \alpha_i p(y_i) \tag{5}$$

holds.

Conversely, let p be a positive bounded function on $\mathbf{S}^n$ such that for $x, y \in \mathbf{S}^n$ satisfying (4) the inequality (5) holds. Then there exists a unique $R \in \mathcal{R}^n$ with focal function p.

For smooth functions we have the following

Theorem. Let $p \in C^2(\mathbf{S}^n),\ p > 0$. Suppose that

$$p_{\alpha\alpha} + p - \frac{p^2 + |\nabla p|^2}{2p} > 0 \text{ on } \mathbf{S}^n,$$

where $p_{\alpha\alpha}$ is the second derivative along any great circle of $\mathbf{S}^n$. Then there exists a unique convex reflector with focal function p. Its position vector is given by (formula (3) given earlier)

$$r(y) = -\nabla_{\mathbf{S}^n} p(y) - p(y)y + \rho(y)y : \ \mathbf{S}^n \to \mathbf{R}^{n+1}.$$

Parabolic Polytopes (P-polytopes)

Fix a positive integer $K \geq 2$ and unit vectors $y_1, ..., y_K \in \mathbf{S}^n$ (not all the same). Let $0 < p_i < \infty,\ i = 1, ..., K$. For $y \in \mathbf{S}^n$ put

$$p(y) = \begin{cases} p_i & \text{if } y = y_i,\ i = 1, ..., K \\ \infty & \text{if } y \neq y_i,\ i = 1, ..., K \end{cases}$$

A reflector R with such focal function p is called a **Parabolic Polytope** (P-polytope) with axes $\{y_1, ..., y_K\}$. R may have faces of dimension n only on paraboloids $P(p_1, y_1), ..., P(p_K, y_K)$.

Notation.

$$\mathcal{R}^n_K(y_1, ..., y_K) := \{P\text{−polytopes with axes } y_1, ..., y_K\}.$$

Theorem. Any $R \in \mathcal{R}^n$ can be approximated in $C(\mathbf{S}^n)$ by P−polytopes, that is, $\rho_K(x) \to \rho(x),\ p_K(y) \to p(y)$ on $\mathbf{S}^n$ as $K \to \infty$.

Transfer of Energy

Let $g \in L(\mathbf{S}^n), g \geq 0 (\not\equiv 0)$. Define the **Energy**

$$G(R, \omega) = \int_{\gamma_R^{-1}(\omega)} g(x) d\sigma(x), \;\; \omega \in \mathcal{B}(\mathbf{S}^n).$$

$G(R, \omega)$ is the total energy delivered by R (via reflection) to the set $\omega \subset \mathbf{S}^n$.

Theorem. $G(R, \cdot)$ is a finite positive Borel measure on $\mathcal{B}(\mathbf{S}^n)$. It is scale invariant w.r.t. rescaling of R. If $R_s, s = 1, 2, ...$ and R are in $\mathcal{R}^n$ and $R_s \to R$ when $s \to \infty$ in $C(\mathbf{S}^n)$ then $G(R_s, \cdot) \stackrel{weakly}{\longrightarrow} G(R, \cdot)$.

The function G has the following important monotonicity property.

Lemma M. Let $R, \bar{R} \in \mathcal{R}_K^n(y_1, ..., y_K)$ with focal functions p and $\bar{p}$. Suppose for some

$$i \in \{1, ..., K\},\ \bar{p}(y_i) > p(y_i),\ \text{ while }\ \bar{p}(y_j) = p(y_j)\ \forall j \neq i.$$

Then

$$G(\bar{R}, y_i) \leq G(R, y_i),\ \ G(\bar{R}, y_j) \geq G(R, y_j),\ \text{ for }\ j \neq i.$$

Furthermore, if $g(x) \geq g_0 > 0$ and $G(\bar{R}, y_i) > 0$ then the above inequalities are strict for y_i and all y_j such that $int(\gamma_R^{-1}(y_i) \cap \gamma_R^{-1}(y_j)) \neq \emptyset$.

Apriori bounds for ρ and p.

Lemma N. Fix $K \geq 2$, $y_1, ..., y_K \in \mathbf{S}^n$ and let $R \in \mathcal{R}^n_K(y_1, ..., y_K)$ be defined by $P(y_1, p_1), P(y_2, p_2), ...P(y_K, p_K)$. Then

$$\gamma_R^{-1}(y_i) = \bigcap_{j=1, j\neq i}^{K} \left\{ x \in \mathbf{S}^n \mid \rho(x) = \frac{p_i}{1 - \langle x, y_i \rangle} \leq \frac{p_j}{1 - \langle x, y_j \rangle} \right\}. \quad (6)$$

In addition, for a pair of indices $i, j \in \{1, ..., K\}, i \neq j$, the set

$$U_{ij} := \left\{ x \in \mathbf{S}^n \mid \frac{p_i}{1 - \langle x, y_i \rangle} \leq \frac{p_j}{1 - \langle x, y_j \rangle} \right\}$$

is a ball in $\mathbf{S}^n$ and ∂U_{ij} is the $(n-1)-$dimensional sphere in $\mathbf{S}^n$ which is the intersection of $\mathbf{S}^n$ with the hyperplane

$$\{X \in \mathbf{R}^{n+1} \mid \langle X, p_i y_j - p_j y_i \rangle = p_i - p_j\}.$$

The centers of the balls U_{ij} and U_{ji} bounded by this sphere are

$$A_{ij} = \frac{p_i y_j - p_j y_i}{q_{ij}}, \tag{7}$$

and $A_{ji} = -A_{ij}$, where $q_{ij} = \sqrt{p_i^2 - 2p_i p_j \langle y_i, y_j \rangle + p_j^2}$. The geodesic radius α_{ij} of U_{ij} in $\mathbf{S}^n$ is given by

$$\cos \alpha_{ij} = \frac{p_i - p_j}{q_{ij}}, \ \alpha_{ij} \in (0, \pi). \tag{8}$$

Combining Lemmas M and N we obtain the estimate,

Theorem. Let $g \in L(\mathbf{S}^n)$, $0 \leq g(x) \leq g_1 < \infty$, $g \not\equiv 0$ and $R \in \mathcal{R}_K^n(y_1, ..., y_K)$. Then for any $i, j = 1, ..., K$

$$-1 < -c_0 \leq \frac{p_i - p_j}{q_{ij}} \leq c_1 < 1, \tag{9}$$

where $c_0, c_1 > 0$ are constants depending on $\tau_k,\ k = i, j$, which is the solution of the equation

$$\text{volume}(V(\tau_k)) = \frac{G(R, y_k)}{\sup_{\mathbf{S}^n} g},$$

where $V(\tau_k)$ is a ball in $\mathbf{S}^n$ with geodesic radius τ_k. In particular, one can rescale R so that $0 < C_0 \leq p(y), \rho(x) \leq C_1 < \infty$, where C_0, C_1 depend only on g and $\min_i G(R, y_i), \max_i G(R, y_i)$, where the $\min$ and $\max$ are over y_i for which $G(R, y_i) \neq 0$, $\int_{\mathbf{S}^n} g(x) d\sigma(x)$. (The set of such y_i is not empty for any $R \in \mathcal{R}_K^n$!)

Proof. Fix a pair i, j such that $G(R, y_i) > 0$ and $G(R, y_j) > 0$ and delete all other paraboloids. Then use monotonicity of G.

Similar arguments imply that for any reflector R such that

$$\inf_{y\in\mathbf{S}^n} |G(R,y) - \int_{\mathbf{S}^n} g(x)d\sigma| \geq c > 0 \qquad (ND)$$

there exist constants C_0, C_1, depending only on c, such that $R \in \mathcal{R}^n$ and

$$0 < C_0 \leq p(y), \rho(x) \leq C_1 < \infty \ \ \forall x, y \in \mathbf{S}^n.$$

Furthermore, $\exists\varepsilon = \varepsilon(c) > 0$ such that for all R satisfying (ND)

$$|\gamma_R(x) - x| \geq \varepsilon.$$

Weak solutions of the reflector problem

Let F be a positive Borel measure on $\mathbf{S}^n$. A reflector $R \in \mathcal{R}^n$ is a weak solution of the reflector problem if

$$G(R, \omega) = F(\omega) \;\; \forall \omega \in \mathcal{B}(\mathbf{S}^n).$$

Theorem. (L. Caffarelli, V.O., '94) ($\equiv$ CO). Let $g \in L(\mathbf{S}^n), g \geq 0 (\not\equiv 0)$ and F a positive Borel measure on $\mathbf{S}^n$ such that

$$F(\mathbf{S}^n) = \int_{\mathbf{S}^n} g(x) d\sigma \qquad \text{(EC)}$$

$$spt(F) \neq \{y\} \qquad \text{(NDF).}$$

Then the reflector problem has a weak solution $R \in \mathcal{R}^n$. If $g(x) \geq g_0 > 0$ the solution is unique up to a homothety w.r.to $\mathcal{O}$. (In CO uniqueness is proved in $\mathcal{R}^n_K$; in general, P. Guan-X.J. Wang -'98.)

Remark 1. The existence part takes care also of the case when we are given an open Ω and arbitrary $\bar{T}$ on $\mathbf{S}^n$ and one or both are such that their closures are $\neq \mathbf{S}^n$.

Remark 2. The (EC) condition is necessary.

Remark 3. If the (NDF) condition is not satisfied it is easy to construct unbounded soultions. For example, if $spt(F) = \{y\}$ then the paraboloids $P(y, p)$ are solutions with any $p > 0$. If g vanishes on a set of positive $n - dim$ measure, there could also be bounded solutions in $\mathcal{R}^n$.

Remark 4. In fact, if g vanishes on a set of positive $n - dim$ measure, the solution is not unique even if $spt(F) \neq \{y\}$. One can just modify the reflector on the “inactive” part.

Proof. Step 1. Partition $\mathbf{S}^n$ and approximate F by $\{F_K = \sum_{i=1}^K f_i\delta_{y_i}\},\ f_i > 0,\ \sum_{i=1}^K f_i = F(\mathbf{S}^n),\ K = 2, ...$

$$\widehat{\mathcal{R}}_K^n = \{R \in \mathcal{R}_K^n \mid G(R, y_i) \le f_i \ \ \forall i = 2, ..., K;$$

$$P(y_1, p(y_1) = 1) \text{ is supporting;}$$

$$\int_{\mathbf{S}^n} g(x)d\sigma - G(R, y_1) \ge (\min_i f_i)/1000;$$

The set $\widehat{\mathcal{R}}_K^n \ne \emptyset$. Also,

$$\forall R \in \widehat{\mathcal{R}}_K^n \ \ \sum_{i=1}^K G(R, y_i) = \int_{\mathbf{S}^n} g(x)d\sigma = F(\mathbf{S}^n).$$

Fix some sufficiently large $R' \in \widehat{\mathcal{R}}_K^n$ and consider all reflectors in $\widehat{\mathcal{R}}_K^n$ which are also contained in R'. Let $\widehat{\mathcal{R}}_K^n$ denote now the set of these reflectors.

We have the map $R \longrightarrow (1, p(y_2), ..., p(y_K)) \in \mathbb{R}^K$ Denote by Λ the image of $\hat{\mathcal{R}}^n_K$ under this map. We claim that Λ is closed and bounded. Closeness is a consequence of continuity of $G(R, y_i)$. Boundedness - by construction.

Consider the function

$$\phi(R) = \sum_{i=1}^{K} p_R(y_i),\ \Lambda \to (0, C].$$

It is continuous and attains its infimum on some $\bar{R}_K$. We claim that $\bar{R}_K \in \hat{\mathcal{R}}^n_K$. If not, then $\bar{R}_K$ is degenerate. Consider a sequence $p^s \in \Lambda, p^s \to \bar{p}$ as $s \to \infty$. Denote by $\{R^s\}$ the corresponding reflectors. By construction, $G(R^s, y_1) \geq f_1\ \forall s$. Hence, $P(y_1, p^s(y_1))$ supporting for all s. But if $\bar{R}$ is degenerate then there is a supporting paraboloid (may be more than one) with an axis $y_j,\ j \in \{2, ..., K\}$, such that $p^s(y_j) \to 0$. But then, by monotonicity, the amount of energy reflected by $P(y_1, p(y_1) = 1)$ will be $<$ than f_1. A contradiction!

It remains to show that $G(\bar{R}, y_i) = f_i \; \forall i = 1, ..., K$. Suppose $G(\bar{R}, y_j) < f_j$ for some j(which must be > 1). Decrease p_j and that will make $\phi(R)$ strictly smaller without leaving the set $\widehat{\mathcal{R}}_K^n$ (again, by the monotonicity). This is impossible. Hence, $G(\bar{R}_K, y_i) = f_i, \; i = 1, ..., K$.

Step 2. Refine the partition of $\mathbf{S}^n$. The sequence $\{\bar{R}_K\}_{K=2}^{\infty}$ consists of convex hypersurfaces with diameters uniformly bounded away from 0 and ∞. Apply Blaschke's theorem and use weak continuity of $G(R_K, \cdot)$.

Part III. Variational Approach.

Monge-Kantorovich Theory

1. We already saw that each $R \in \mathcal{R}^n$ has two representations:
(a) via radial function $\rho : \mathbf{S}^n \to (0, \infty)$
(b) via focal function $p : \mathbf{S}^n \to (0, \infty)$,
(c) related by the Legendre transform

$$\rho(x) = \inf_{y \in \mathbf{S}^n} \frac{p(y)}{1 - \langle x, y \rangle}, \quad x \in \mathbf{S}^n,$$
$$p(y) = \sup_{x \in \mathbf{S}^n} \rho(x)(1 - \langle x, y \rangle), \quad y \in \mathbf{S}^n.$$

(d) Also,

$$\gamma_R(x) = \{y \in \mathbf{S}^n \mid \rho(x) = \frac{p(y)}{1 - \langle x, y \rangle}\}, \quad x \in \mathbf{S}^n,$$
$$\gamma_R^{-1}(y) = \{x \in \mathbf{S}^n \mid \rho(x) = \frac{p(y)}{1 - \langle x, y \rangle}\}, \quad y \in \mathbf{S}^n.$$

(e) Conversely, a pair $(\rho, p) \in C(\mathbf{S}^n) \times C(\mathbf{S}^n)$, $\rho, p > 0$ satisfying (c) defines a unique reflector in $\mathcal{R}^n$.

2. Splitting: For each $R \in \mathcal{R}^n$ we have

$$\log \rho(x) - \log p(y) \leq -\log(1 - \langle x, y \rangle) \ \forall x, y \in \mathbf{S}^n$$

and for each x equality is achieved for some y and for each y equality is achieved for some x.

3. The functional: Let $g \in L(\mathbf{S}^n), g \geq 0, \ g \not\equiv 0$ and F, as before, a finite positive Borel measure on $\mathbf{S}^n$ such that

$$F(\mathbf{S}^n) = \int_{\mathbf{S}^n} g, \ \ spt(F) \neq \{y\}.$$

Put

$$Q[\rho, p] := \int_{\mathbf{S}^n} \log \rho(x) g(x) d\sigma(x) - \int_{\mathbf{S}^n} \log p(y) dF(y)$$

and consider it on the set

$$Adm := \{(\rho, p) \in C(\mathbf{S}^n) \times C(\mathbf{S}^n),\ \rho, p > 0, |$$
$$\log \rho(x) - \log p(y) \leq -\log(1 - \langle x, y \rangle)\ \forall x, y \in \mathbf{S}^n\}.$$

Problem P1: $Q[\rho, p] \longmapsto \max$ over Adm.

Theorem. (Existence of maximizers of Q.) Let g and F be as above. Then $\exists$ a maximizing pair $(\rho_{\max}, p_{\max})$ of Q in Adm. If $g(x) \geq g_0 > 0$ then the maximizer is unique up to a homothety w.r. to $\mathcal{O}$. Otherwise, two maximizers coincide (possibly, after a homothety) on $spt(g)$ and $spt(F)$, respectively.

Proof. It takes several steps. First the theorem is proved for P-polytopes with F approximated by finite sums of atomic measures.

- The maximizer can be sought among reflectors in $\hat{\mathcal{R}}^n_K(y_1, ..., y_K)$.

- Using scale invariance one can restrict the search to reflectors for which $\sup_{\mathbf{S}^n} p(y) = 1$.

- The functional Q is continuous and the maximizer exists.

- It is shown that if the maximizer is a reflector which passes through the origin then $Q[\rho, p] = -\infty$.

Last step is to pass to a limit in K.

Prior to the above theorem T. Glimm and V.O. ('03) proved

Theorem. Let $R \in \mathcal{R}^n$ and ρ, p are its radial and focal functions. Then (i) and (ii) below are equivalent:
(i) (ρ, p) maximizes Q on Adm,
(ii) R is a weak solution of the reflector problem.

Proof. Use a perturbation argument.

It follows from the this theorem that the CO theorem implies existence of maximizers in Problem P1. Conversely, the theorem on existence of maximizers and this theorem imply the CO theorem.

Important Remark. The theorem by T. Glimm and V.O. shows that the reflector problem can be treated as an infinite-dimensional linear programming problem and its numerical solution may be approached via a sequence of finite-dimensional linear programming problems.

Remark. The above variational approach is motivated by the optimal mass transport problem considered by L. V. Kantorovich in 1939 and later by him and many other authors. It consists in finding an optimal "plan" for redistributing a given measure in a prescribed manner, with optimality measured against a given cost function. The functional Q corresponds to the "dual" of the "primal" mass transport problem.

In fact, the above theorem on existence of maximizers of Q is the dual of a theorem by W. Gangbo and V.O.('04) concerning existence of minimizers to the primal problem, proved by a very different method for less general F but slightly more general G and more general costs c.

Prior to this result, T.Glimm and V.O. '03 and, independently, X.J. Wang, '04, (under more restrictive assumptions) showed existence of maximizers under the additional conditions:

$$F(\omega) := \int_{\omega\subset \mathbf{S}^n} f d\sigma,\ f \in L(\mathbf{S}^n),\ f \not\equiv 0, \text{ and } spt(g) \cap spt(F) = \emptyset.$$

In the same paper by T.Glimm and V.O. the following problem was considered:

Problem P2. Let $\mathcal{S}$ be the the set of measurable maps $s : \mathbf{S}^n \to \mathbf{S}^n$ satisfying

$$\int_{\mathbf{S}^n} \phi(s(x))g(x)d\sigma(x) = \int_{\mathbf{S}^n} \phi(y)dF(y),\ \forall \phi \in C(\mathbf{S}^n).$$

Determine $s_0 \in \mathcal{S}$ such that

$$\inf_{\mathcal{S}} \left\{ -\int_{\mathbf{S}^n} \log(1 - \langle x, s(x) \rangle) g(x) d\sigma(x) \right\} = -\int_{\mathbf{S}^n} \log(1 - \langle x, s_0(x) \rangle) g(x) d\sigma(x).$$

Theorem. Let R be a weak solution of the reflector problem with F as in the theorem by CO. Then the Problem $P2$ has a minimizer s_0 and $s_0 = \gamma_{R_{\max}}$ $a.e.$ on $spt(g)$. Such R is unique up to a homothety on the "active part" $(= spt(g))$. In addition,

$$Q[\rho_{\max}, p_{\max}] = -\int_{\mathbf{S}^n} \log(1 - \langle x, \gamma_{R_{\max}}(x)\rangle) g(x) d\sigma(x).$$

The problem P2 is an analogue of the mass transportation problem considered for Euclidean domains by Monge in 1781 with the Euclidean distance as the transportation cost. In our case, the cost function is $c(x,y) := -\log(1 - \langle x, y\rangle),\ x, y \in \mathbf{S}^n$.

Proof. It is easy to check that $\gamma_R \in \mathcal{S}$. Also, since

$$\rho(x) = \inf_{y \in \mathbf{S}^n} \frac{p(y)}{1 - \langle x, y\rangle}, x \in \mathbf{S}^n,$$

we have $\forall s \in \mathcal{S}$

$$\int_{\mathbf{S}^n} c(x,s(x))g(x)d\sigma(x) \geq \int_{\mathbf{S}^n} \log\rho(x)g(x)d\sigma(x)$$
$$-\int_{\mathbf{S}^n} \log p(s(x))g(x)d\sigma(x) = \int_{\mathbf{S}^n} \log\rho(x)g(x)d\sigma(x)$$
$$-\int_{\mathbf{S}^n} \log p(y)dF(y) = \int_{\mathbf{S}^n} \log\rho(x)g(x)d\sigma(x)$$
$$-\int_{\mathbf{S}^n} \log p(\gamma_R(x))dF(y) = \int_{\mathbf{S}^n} c(x,\gamma_R(x))g(x)d\sigma(x).$$

Thus, γ_R is a minimizer. If s_0 is another minimizer then we must have equalities in all of the above inequalities. Then

$$\log\rho(x) - \log p(s_o(y) = c(x,s_0(x)) \; a.e. \text{ on } spt(g)$$

and then $s_0(x) = \gamma_R(x)$ $a.e.$ on $spt(g)$.

Uniqueness of R (up to constant multiple): Follows from uniqueness of γ_R.

Part IV. Regularity

- **’96** X.J. Wang studied regularity assuming: $n = 2$, smooth data, $\Omega \subset S^2_{\frac{1}{2}+}$, $T \subset S^2_{\frac{1}{2}-}$, $\bar{\Omega} \cap \bar{T} = \emptyset$, plus other conditions on Ω and T.

- **’98** P.F. Guan and X.J. Wang proved regularity for smooth data on $\mathbf{S}^n$.

- **’03** O. Schnürer introduced a flow equation for hypersurfaces converging to solutions of the reflector problem both for $\mathbf{S}^n$ and for subdomains of $\mathbf{S}^n$.

- **'04-05** L. Caffarelli, C. Gutierrez, Q. Huang proved C^1 regularity assuming: $0 < c_0 \leq \rho(x) \leq c_1 < \infty$ and
$$0 < \lambda \leq \frac{\gamma_R^{-1}(E)}{|E|} \leq \Lambda < \infty \ \ \forall E \in \mathcal{B}(\mathbf{S}^n).$$
see also G. Loeper, **'06**.

- **'04** V. Oliker, The support function of a reflector is always $C^1(\mathbf{S}^n)$.

Part V. Stability of solutions

Theorem. (V.O.) Let $n \geq 2$. Assume that all $R \in \mathcal{R}^n$ are rescaled so that $p(\bar{y}) = 1$ and $0 < g_0 \leq g(x) \leq g_1 < \infty \ \ \forall x \in \mathbf{S}^n$. Let

$$G(R^i, \omega) = F^i(\omega), \ \forall \omega \in \mathcal{B}(\mathbf{S}^n), \ i = 0, 1,$$

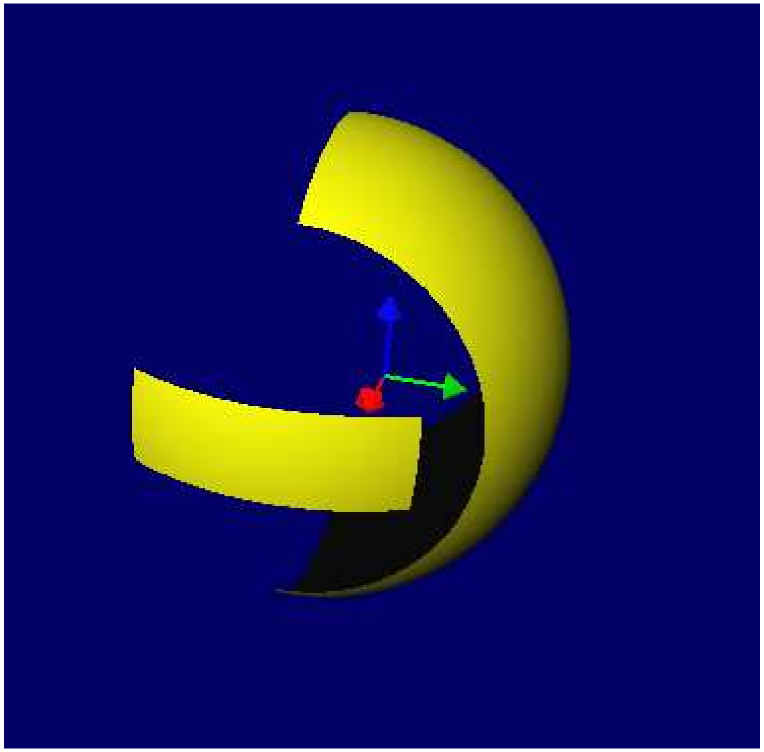
where $G(R^i, \cdot)$ is constructed with the same $g \in L(\mathbf{S}^n)$ and F^i as in the CO theorem. Then

$$\log \frac{p^1(y)}{p^0(y)} \leq C \Delta^{\frac{1}{n-1}},$$

where $\Delta = sup_{\omega \subset \mathbf{S}^n} \{F^0(\omega) - F^1(\omega)\}$,
$C = C(n, r^0_{\text{min}}, r^0_{\text{max}}, r^1_{\text{min}}, r^1_{\text{max}}, g_0, g_1)$ with $r^i_{\text{min}}, r^i_{\text{max}}$ being the radii of the minimal outer and maximal inner balls of R^i with center $\mathcal{O}$. Same bound holds also for $\rho^0(x)/\rho^1(x)$. When $n = 1$ the exponent is $= 1$.

Example

A point source at the origin $\mathcal{O}$ irradiates at the reflector through an aperture $\bar{\Omega}$. On the figure the input aperture is shown on the right.



The input density

$$g(\theta) = \sin\theta,$$

where θ is the angle in the meridional direction (toward the viewer).

The required reflector should convert the input into an output bundle of rays covering the far-field region $\bar{T}$ with density

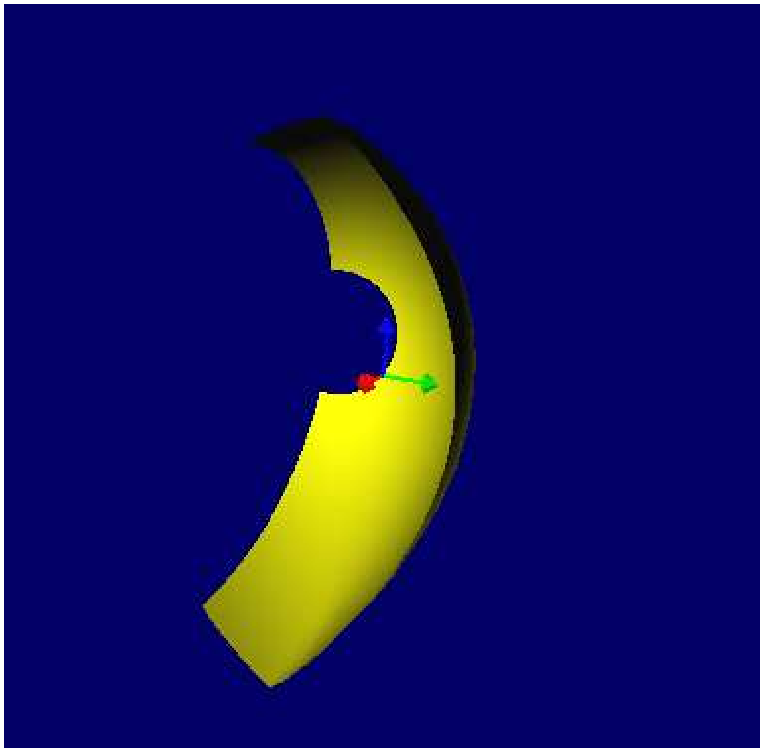
$$\frac{k}{\cos^3\alpha\sin^3\beta},$$

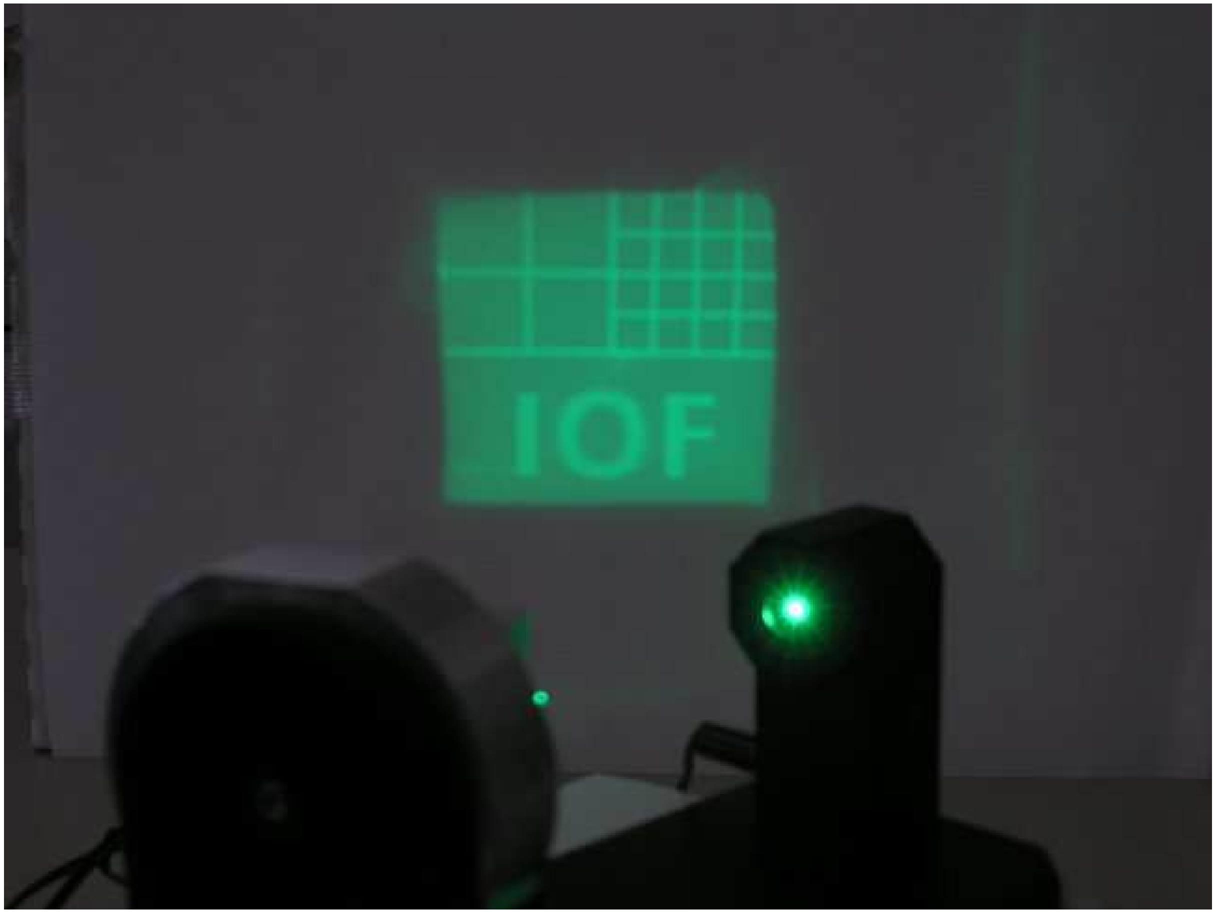
where k is the balancing constant and α, β are spherical coordinates.

The solution was found numerically on $\bar{\Omega}$ and $\bar{T}$ each represented by grids with 66K nodes.

The data for this example was kindly suggested by Pablo Benitez and Juan C. Miñano, Madrid Polytechnical U.

Reflector redirecting and redistributing the input intensity over a given far-field region and with specified output intensity





Logo of the Fraunhofer Optics Institute created with a mirror designed by solving numerically the near-field reflector problem.