Hyperplane sections of convex bodies

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Fourth International Workshop on
Convex Geometry - Analytic Aspects
Cortona, June 4th - 8th, 2007

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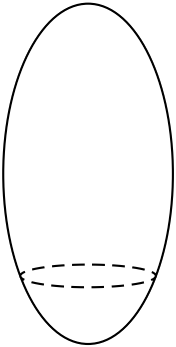

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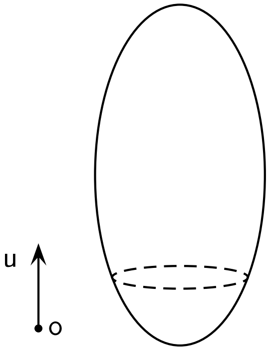

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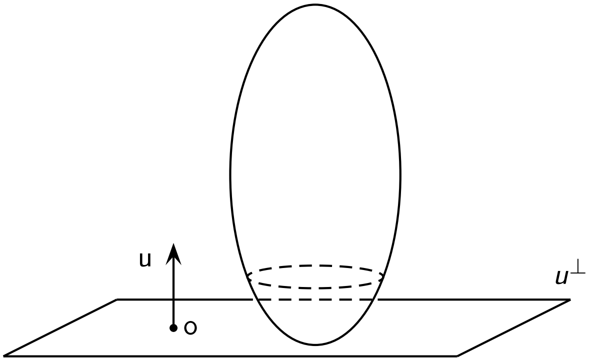

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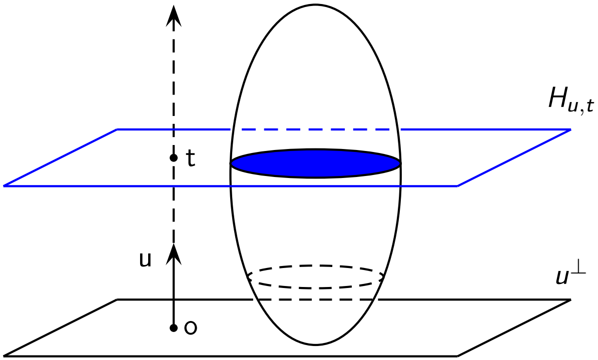

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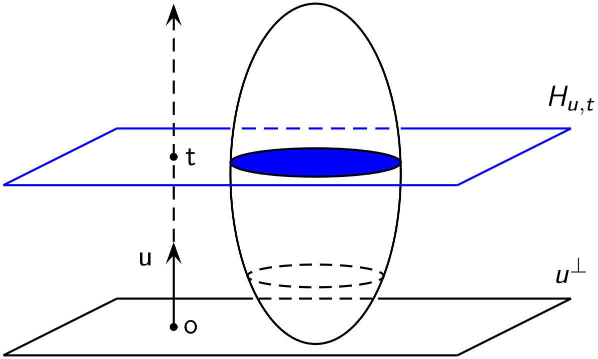
$|A|$: volume of $A \subset \mathbb{R}^n$ (relative to its affine convex hull)



$$H^+_{u,t} := \{x \in \mathbb{R}^n : \langle x, u\rangle \geq t\},$$
$$H^-_{u,t} := \{x \in \mathbb{R}^n : \langle x, u\rangle \leq t\},\ t \in \mathbb{R}$$

Problem. Find the best upper bound for the ratio

$$\rho_{K,u}(t) := \frac{\min\left\{\left|K \cap H^+_{u,t}\right|, \left|K \cap H^-_{u,t}\right|\right\}}{\left|K \cap H_{u,t}\right|}.$$

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$$\rho_{K,u}(t) \leq \frac{2}{\ln 2} \min_{y \in \mathbb{R}^n} \left(\frac{1}{|K|} \int_K |\langle x - y, u\rangle|\, dx \right)$$

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If $K = -K$, then

$$\rho_{K,u}(t) \leq \frac{2}{|K|} \int_K |\langle x, u\rangle|\, dx$$

These type of inequalities are simpler versions of the so-called *relative isoperimetric inequalities,* which, for every sufficiently smooth subset E of a domain G, bound from above the Lebesgue measure either of E or $G \setminus E$ by an appropriate $(n-1)$-dimensional measure $(P(E, G))$ of the boundary $\partial E \cap G$.

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R. Kannan, L. Lovász e M. Simonovits (1995): $K \subset \mathbf{R}^n$ with centroid at the origin

$$\frac{\min(|E|, |K \setminus E|)}{P(E, K)} \leq \frac{2}{\ln 2} \left(\frac{1}{|K|} \int_K |x| dx \right)$$

Proposition. The maximum of the ratio $\rho_{K,u}(t)$, over the set of parallel hyperplane sections of K, is attained when the section bisects the volume of K:

$$\rho_{K,u}(t) = \frac{\min\left\{\left|K \cap H^+_{u,t}\right|, \left|K \cap H^-_{u,t}\right|\right\}}{|K \cap H_{u,t}|} \leq \frac{|K|}{2\,|K \cap H_u|},$$

(H_u: hyperplane orthogonal to u which cuts K into two parts with equal volumes.)

D. Hensley (1980): $K = -K, \quad \forall u \in \mathcal{S}^{n-1}$

$$\frac{|K|}{2|K \cap u^{\perp}|} \leq \sqrt{3} \left(\frac{1}{|K|} \int_K |\langle x, u \rangle|^2 \, dx \right)^{\frac{1}{2}}$$

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$(c_2(p,n) \to \Gamma(p+1)^{1/p}$ when $n \to \infty)$

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$$\int_{-1}^{1}\phi(\alpha t)dt \leq \frac{2}{|K|}\int_K \phi(\langle x,u\rangle)dx \leq \int_{-n}^{n}\phi(\alpha t)\left(1-\frac{|t|}{n}\right)^{n-1}dt \tag{2}$$

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$$(2) \Rightarrow (1)$$

with $\phi(t) := |t|^p$, $p \geq 1$.

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$c_3(p,n) \to \frac{1}{2}\left(\int_0^{+\infty}|t-1|^p e^{-t}dt\right)^{-\frac{1}{p}}$ when $n\to\infty$

Extension: $\phi : \mathbb{R} \to \mathbb{R}$ even convex function,

$$\gamma = \frac{|K|}{2\max_{t\in\mathbb{R}}|K\cap H_{u,t}|}$$

$$\begin{aligned} \int_{-1}^{1}\phi(\gamma t)dt &\leq \frac{2}{|K|}\int_K \phi(\langle x,u\rangle)dx \leq \\ &\leq \frac{2}{\left(1+\frac{1}{n}\right)^n}\int_{-1}^{n}\phi\left(\frac{2n\gamma t}{n+1}\right)\left(1-\frac{t}{n}\right)^{n-1}dt \end{aligned}$$

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C. Schütt (1997): Applications to the floating body and the illumination body.

M. Meyer e E. Werner (1998): Applications to Santaló regions.

Problem. Find an upper bound for the relative isoperimetric ratio $\rho_{K,u}(t)$ in terms of the p-th moments of inertia of K.

Theorem 1. Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Let $u \in S^{n-1}$ and $\alpha = |K| / (2 |K \cap H_u|)$, where H_u denotes the hyperplane orthogonal to u which bisects the volume of K. Then

$$\frac{1}{2} \int_{-1}^{1} \phi(\alpha t) dt \leq \frac{1}{|K|} \int_K \phi(\langle x, u \rangle) dx \qquad (3)$$

The inequality is sharp: if ϕ is strictly convex, there is equality if and only if K is a cylinder in the direction u.

Corollary 1.. Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $u \in S^{n-1}$, and $p \geq 1$. Denote by H_u the hyperplane orthogonal to u which bisects the volume of K. Then

$$\frac{|K|}{2\,|K \cap H_u|} \leq \left(\frac{p+1}{|K|} \int_K |\langle x, u \rangle|^p \, dx \right)^{\frac{1}{p}}.$$

There is equality if and only if K is a cylinder in the direction u

Theorem 2. Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin, and $p \geq 1$. Let $H_{u,t}$ be a hyperplane orthogonal to $u \in S^{n-1}$, which cuts K into two parts, say $K \cap H^+_{u,t}$ and $K \cap H^-_{u,t}$. Then

$$\frac{\min\left\{\left|K \cap H^+_{u,t}\right|, \left|K \cap H^-_{u,t}\right|\right\}}{|H_{u,t} \cap K|} \leq \left(\frac{p+1}{|K|} \int_K |\langle x, u\rangle|^p \, dx\right)^{\frac{1}{p}}.$$

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Corollary 2. Let K be in isotropic position, and $|K| = 1$. Then

$$\frac{\min\left\{\left|K \cap H^+_{u,t}\right|, \left|K \cap H^-_{u,t}\right|\right\}}{|H_{u,t} \cap K|} \leq \sqrt{3}L_K < cn^{\frac{1}{4}}.$$

(L_K: isotropic constant of K.)

Outline of the proof of Theorem 1.

Functional version of (3):

Let $f(t) := |K \cap H_{u,t}|$ and $f(\tilde{t}) := |K \cap H_u| = 1$, then

$$|K| = \int_{-\infty}^{+\infty} f(t)dt \quad \alpha = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(\tilde{t})} = \int_{-\infty}^{\tilde{t}} f(t)dt$$

$$\int_K \phi(\langle x, u\rangle)dx = \int_{-\infty}^{+\infty} \phi(t)f(t)dt$$

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$$\int_K \phi(\langle x, u\rangle)dx = \int_{-\infty}^{+\infty} \phi(t)f(t)dt$$

We have to prove

$$\int_{-\alpha}^{\alpha} \phi(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt$$

for every $\frac{1}{n-1}$-concave integrable function

$$\int_{-\infty}^{+\infty} tf(t)dt = 0, \quad \int_{-\infty}^{\tilde{t}} f(t)dt = \int_{\tilde{t}}^{+\infty} f(t)dt$$

Step 1. Reduction to the log-affine case

There exists a log-affine function

$$g(t) = e^{a(t-\tilde{t})}\chi_{[-c,b]}(t),$$

where $[-c, b] \subseteq \mathrm{support}\,(f)$, such that

$$\int_{-\infty}^{\tilde{t}} g(t)dt = \int_{-\infty}^{\tilde{t}} f(t)dt, \int_{\tilde{t}}^{+\infty} g(t)dt = \int_{\tilde{t}}^{+\infty} f(t)dt,$$

$$\int_{-\infty}^{+\infty} tg(t)dt = 0$$

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \qquad (4)$$

for every convex function $\phi : \mathbb{R} \to \mathbb{R}$.

Lemma. (Fradelizi, 1999) Let $v : \mathbb{R} \to \mathbb{R}$ be an integrable compactly supported function such that $\int_{-\infty}^{+\infty} v(t)dt = 0$ and $\int_{-\infty}^{+\infty} tv(t)dt = 0$. Set $V(t) = \int_{-\infty}^{t} v(s)ds$. Let ϕ be a convex function and μ be the positive Borel measure on $\mathbb{R}$ such that $\phi'' = \mu$. Then the function $W(t) = \int_{-\infty}^{t} V(s)ds$ is compactly supported and

$$\int_{-\infty}^{+\infty} \phi(t)v(t)dt = \int_{-\infty}^{+\infty} W(t)d\mu(t).$$

$(v := f - g)$

Inequality (4) follows from $W \geq 0$.

Step 2. Reduction to the constant case.

Among all log-affine functions the quantity

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt$$

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Set $d := \frac{1}{2}\int_{-\infty}^{+\infty} g(t)dt$. Then we have to prove that

$$\int_{-d}^{d} \phi(t)dt \leq \int_{-c}^{b} \phi(t)g(t)dt, \tag{5}$$

where $g(t) = e^{(t-\tilde{t})}\chi_{[-c,b]}(t)$

If we set $x := (b+c)/2$, then b, c, d are $\mathcal{C}^\infty$ positive functions of $x > 0$.
Inequality (5) becomes

$$I(x) := \int_{-c(x)}^{b(x)} \phi(t) e^{(t-\tilde{t})} dt - \int_{-d(x)}^{d(x)} \phi(t) dt \geq 0$$

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We have $I(0) = 0$, and by differentiating

$$\int_{-c(x)}^{b(x)} t e^{(t-\tilde{t})} dt = 0 \quad d(x) = \frac{1}{2} \int_{-c(x)}^{b(x)} e^{(t-\tilde{t})} dt$$

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we get

$$I' = b' e^{(b-\tilde{t})} \left(\frac{b+c}{c} \right) \left(\frac{c}{b+c} \phi(b) + \frac{b}{b+c} \phi(c) - \phi(d) \right) \geq$$
$$\geq b' e^{(b-\tilde{t})} \left(\frac{b+c}{c} \right) \left(\phi\left(\frac{2bc}{b+c} \right) - \phi(d) \right) \geq 0$$

$$\Downarrow$$

$$I(x) \geq 0$$

The equality case.

There is equality if and only if there is equality in the corresponding functional form, that is

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt = \int_{-\infty}^{+\infty} \phi(t)f(t)dt$$

where $g(t) = f(\tilde{t})\chi_{[-\alpha,\alpha]}(t)$.

The equality case.

There is equality if and only if there is equality in the corresponding functional form, that is

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt = \int_{-\infty}^{+\infty} \phi(t)f(t)dt$$

where $g(t) = f(\tilde{t})\chi_{[-\alpha,\alpha]}(t)$.
Then by the previous lemma the second primitive W of $v = f - g$ satisfies

$$\int_{-\infty}^{+\infty} W(t)d\mu(t) = 0$$

where $W \geq 0$.

ϕ strictly convex $\Rightarrow$ μ positive $\Rightarrow$ $W = 0 \Rightarrow f = g$