Hausdorff Approximation of 3D convex Polytopes

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Let C be a convex body in in $\mathbb{R}^d$, and assume for simplicity that $C \subset B_2^d$. It has been proved by Bronshtein and Ivanov and by Dudley that for all $k \geq c_0^{(d-1)/2}$ there exists a convex polytope $Q \subset C$ with no more than k vertices, such that

(1) $$\boxed{d_H(Q, C) \leq \frac{c_0}{k^{2/(d-1)}}}$$

where d_H is the Hausdorff distance. A corollary of this and a theorem of Macbeath is a similar result with the symmetric distance (volume difference) replacing the Hausdorff distance, namely: There exists a polytope $Q \subset C$, with at most k vertices, such that

(2) $$\boxed{\frac{|C \setminus Q|}{|C|} \leq \frac{c_1 d}{k^{2/(d-1)}}}.$$

The fact that (2) is best possible (up to the constant involved) has been proved By Gordon-Reisner-Schütt and exact asymptotic estimate of the constant involved has been achieved by Mankiewicz-Schütt. This shows that (1) is of best possible order as well.

In a previous work the authors have shown how to apply a result of Reisner-Schütt-Werner, together with Euler's relation $V - E + F = 2$ for polytopes in $\mathbb{R}^3$, to construct an algorithm, which is linear in time (with respect to n), that "deletes" $n-k$ vertices of a given polytope P in $\mathbb{R}^3$ (or $\mathbb{R}^2$) with n vertices. So that the resulting k-polytope

$$Q = \text{conv(the remaining } k \text{ vertices)}$$

approximates P in the sense of (2):

$$\boxed{\frac{|P \setminus Q|}{|P|} \leq \frac{c_1}{k} \qquad (\frac{c_1}{k^2} \text{ in the 2D case})}.$$

In the present work we construct a linear time (in n) algorithm that, for a polytope P in $\mathbb{R}^3$ with n vertices such that $P \subset B_2^3$, and for $k < n$ (k big enough as before), selects $n-k$ vertices to be "erased", so that

$$Q = \text{conv(the remaining } k \text{ vertices)}$$

satisfies

$$\boxed{d_H(Q,P) \leq \frac{c}{k} \qquad (\frac{c}{k^2} \text{ in the 2D case})}.$$

In the following we describe the main tools of the algorithm and the principal difference between this algorithm and the one for the symmetric distance. This principal difference prevented us from extending the algorithm to dimensions higher than 3, even with the price of increasing the complexity (this was possible in the case of the symmetric distance algorithm).

The first main tool is a version of a result of Reisner-Schütt-Werner, the proof of this version constitutes a part of the proof of that result).

Lemma 1 *There exist constants* $c_0, c_1 > 0$ *such that for any* $\varepsilon > 0$ *and any positive integers* d, n, *such that* $n > c_0^d/\varepsilon$ *the following holds: Let* P *be a convex polytope with* n *vertices in* $\mathbb{R}^d$, *which is contained in the Euclidean unit ball* B_2^d *of* $\mathbb{R}^d$. *For a vertex* v *of* P *we denote by* h_v *the distance from* v *to the convex hull of all the vertices of* P *other than* v. *Then the set*

(3)
$$A_\varepsilon = \left\{ v \,\middle|\, v \text{ is a vertex of } P \text{ and } \boxed{h_v \leq \frac{c_1}{(\varepsilon n)^{\frac{2}{d-1}}}} \right\}$$

(this is $\frac{c_1}{\varepsilon n}$ *if* $d = 3$*) has at least* $(1 - \varepsilon)n$ *elements.*

(In the symmetric-distance version of this lemma the inequality defining A_ε is

$$(4) \qquad \boxed{\frac{|P| - |\text{conv}(\text{vert}(P) \setminus \{v\}|}{|P|} \leq \frac{c_1}{(\varepsilon n)^{\frac{d+1}{d-1}}}}$$

(this is $\frac{c_1}{\varepsilon^2 n^2}$ if $d = 3$)).

The second tool is Euler's relation, which easily implies the following:

Lemma 2 *Let P be a polytope in $\mathbb{R}^3$ with n vertices. For any $0 < \beta < 1$ there are at least βn vertices of P with degree less than $\frac{3(2-\beta)}{(1-\beta)}$.*

Using the two lemmas, we select β and ε appropriately, as to have sufficiently many vertices v of P that have at the same time a small degree and a small h_v. Small degree would mean that sufficiently many of these vertices are not neighbors of each other. This guarantees that for some $0 < \gamma < 1$ we can remove γn of these vertices, let

$$Q_1 = \text{conv}(\text{vert}(P) \text{ except the } \gamma n \text{ removed})$$

and get (due to the fact that the removed vertices are not neighbors):

$$d_H(Q_1, P) \leq \frac{c}{n}.$$

We then apply the same operation, with the same γ to Q_1 (that has $(1-\gamma)n$ vertices) and get a $Q_2 \subset P$ with $(1-\gamma)^2 n$ vertices and

$$d_H(Q_2, P) \leq \frac{c}{(1-\gamma)n}.$$

Proceeding in this way about $i_0 = \frac{\log(n/k)}{\log(1/(1-\gamma))}$ steps, we remove $n-k$ vertices and let $Q = Q_{i_0}$. The time needed for this is

$$\leq \text{constant} \cdot n \sum_{i=0}^{\infty} (1-\gamma)^i = \frac{\text{constant} \cdot n}{\gamma}.$$

Another simple calculation shows that

$$d_H(Q, P) \leq \frac{c}{(1-\gamma)^{i_0 - 1} n} \approx \frac{c}{k}.$$

An important difference between the symmetric-distance algorithm and the Hausdorff-distance algorithm is that in the former the Euler's relation was used only in order to improve the time-cost of the algorithm. The estimate (4) could be added up to get $\frac{c}{n}$ without worrying about the non-neighborliness of the removed vertices. In the Hausdorff-distance algorithm, the estimate (3) can not be added up. Therefore we need to find non-neighbor vertices v for the very correctness of the algorithm.

This difference raises a theoretical question. It is well known that the best approximation of an n-polytope by a k-polytope ($k < n$) in the symmetric-distance sense is attained by removing certain $n-k$ vertices of P. This is not the case in the Hausdorff distance case, even in dimension 2. Yet, can the estimate $\frac{c_0}{k^{\frac{2}{d-1}}}$ be achieved by removing vertices of P?

The answer to this is yes. The proof is very similar to the proof of Lemma 1 above:

Theorem 3 *There exist constants* $c_1, c_2 > 0$ *such that for any positive integer* d*, any integers* $n > k > c_2^d$ *and any convex polytope* P *in* $\mathbb{R}^d$ *with* n *vertices, there exists a polytope* Q*, which is the convex hull of* k *of the vertices of* P *and satisfies*

$$\boxed{d_H(P,Q) < \frac{c_1 R}{k^{2/(d-1)}}}.$$

R above is the minimal radius of a Euclidean ball containing P.

Remark An analogue of the algorithm (and also of Theorem 3) is true where the role of vertices is taken by facets.