A Classification of $\text{SL}(n)$-invariant Valuations

joint work with Monika Ludwig

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Valuations

\( \Phi : S \rightarrow IR \)

\[ \Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L) \]

for \( K, L, K \cup L, K \cap L \in S \)
Valuations

\[ \Phi : \mathcal{K}^n \to \mathbb{R} \]

\[ \Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L) \]

for \( K, L, K \cup L, K \cap L \in \mathcal{K}^n \)

\( \mathcal{K}^n \) ... convex sets containing origin in their interior
Valuations

\[ \Phi : \mathcal{K}_0^n \rightarrow IR \]

\[ \Phi(K) + \Phi(L) = \Phi(K \cup L) + \Phi(K \cap L) \]

for \( K, L, K \cup L, K \cap L \in \mathcal{K}_0^n \)

\( \mathcal{K}^n \) ... convex sets
\( \mathcal{K}_0^n \) ... convex sets containing origin in their interior
Hadwiger’s characterization theorem

\[ \Phi : \mathcal{K}^n \to IR \] is a continuous, rigid motion invariant valuation.

\[ \Phi(K) = c_0 V_0(K) + \cdots + c_n V_n(K) \]

\( V_i(K) \) is the \( i \)-th intrinsic volume of \( K \).

Hadwiger, Klain
Hadwiger’s characterization theorem

\[ \Phi : \mathcal{K}^n \to \mathbb{IR} \text{ continuous, rigid motion invariant valuation} \]

\[ \Phi(K) = c_0 \, V_0(K) + \cdots + c_n \, V_n(K) \]

\[ V_i(K) \ldots \text{i-th intrinsic volume of } K \]

McMullen, Alesker, Fu, Bernig
Hadwiger’s characterization theorem

\[
\Phi : \mathcal{K}^n \to IR \text{ continuous, rigid motion invariant valuation}
\]

\[
\Phi(K) = c_0 V_0(K) + \cdots + c_n V_n(K)
\]

\(V_i(K)\) \ldots \(i\)-th intrinsic volume of \(K\)
Semicontinuous valuations

\[ K_n \to K \implies \Phi(K) \geq \limsup_{n \to \infty} \Phi(K_n) \]
Semicontinuous valuations

Φ upper semicontinuous:

\[ K_n \to K \implies \Phi(K) \geq \limsup_{n \to \infty} \Phi(K_n) \]

\[ \Phi : \mathcal{K}^n \to \mathbb{R} \] upper semicontinuous, \( SL(n) \) and translation invariant valuation,

\[ \Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K) \]

Ludwig, Reitzner
Semicontinuous valuations

\[ \Phi : \mathcal{K}^n \to \mathbb{R} \] upper semicontinuous \(,\) \(\text{SL}(n)\) and translation invariant valuation,

\[ \uparrow \]

\[ \Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K) \]

\(\Omega(K)\) \ldots affine surface area of \(K\)

\[ \Omega(K) = \int_{\partial K} \kappa(x) \frac{1}{n+1} \, dx, \]

Blaschke, Dolzmann, Hug, Leichtweiß, Lutwak, Schütt, Werner, \ldots
Semicontinuous valuations

\[ \Phi : K^n \rightarrow IR \] upper semicontinuous, \( SL(n) \) and translation invariant valuation, vanishing on polytopes

\[ \Phi(K) = c_2 \Omega(K) \]

\( \Omega(K) \) ... affine surface area of \( K \)

\[ \Omega(K) = \int_{\partial K} \kappa(x) \frac{1}{n+1} \, dx, \]
Semicontinuous valuations

\[ \Phi : \mathcal{K}^n \to IR \text{ upper semicontinuous , } \text{SL}(n) \text{ and translation invariant valuation,} \]

\[ \Phi(K) = c_0 V_0(K) + c_1 V_n(K) + c_2 \Omega(K) \]

\[ \Omega(K) \ldots \text{affine surface area of } K \]

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Semicontinuous valuations

\[ \Phi : \mathcal{K}^n \to \mathbb{R} \] upper semicontinuous, \( \text{SL}(n) \) and translation invariant valuation,

\[ \Phi(K) = c_0 \, V_0(K) + c_1 \, V_n(K) + c_2 \, \Omega(K) \]

\( \Omega(K) \) ... affine surface area of \( K \)

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Semicontinuous valuations

\[ \Phi : \mathcal{K}^n \rightarrow \mathbb{IR} \text{ upper semicontinuous, } \text{SL}(n) \text{ and translation invariant valuation,} \]

\[ \Phi(K) = c_0 \ V_0(K) + c_1 \ V_n(K) + c_2 \ \Omega(K) \]

\[ \Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \ dx, \]

\( \Omega(K) \) ... affine surface area of \( K \)
GL($n$) invariant semicontinuous valuations

$\Phi : K_0^n \rightarrow IR$ upper semicontinuous, GL($n$) invariant valuation

$\Phi(K) = c_0 V_0(K) + c_1 \Omega_c(K)$

Ludwig, Reitzner
GL(n) invariant semicontinuous valuations

\[ \Phi : \mathcal{K}_0^n \rightarrow IR \] upper semicontinuous, GL(n) invariant valuation

\[ \Phi(K) = c_0 \, V_0(K) + c_1 \, \Omega_c(K) \]

\[ \Omega_c(K) \ldots \text{centro-affine surface area of } K \]

\[ \Omega_c(K) = \int_{\partial K} \kappa_0(x)^{\frac{1}{2}} \, h_K(x) \, dx \]

\[ \kappa_0(x) = \frac{\kappa(x)}{h_K(x)^{n+1}}. \]

Lutwak, Titeica
\[ \Phi : K_{0}^{n} \rightarrow IR \text{ upper semicontinuous, } \text{SL}(n) \text{ invariant valuation, homogeneous of degree } q \]

\[
\Phi(K) = \begin{cases} 
  c_0 \, V_0(K) + c_1 \, \Omega_c(K) & \text{for } q = 0 \\
  c_1 \, \Omega_p(K) & \text{for } -n < q < n, \, q \neq 0, \\
  c_0 \, V_n(K) & \text{for } q = n \\
  c_0 \, V_n(K^*) & \text{for } q = -n \\
  0 & \text{for } q < -n \text{ or } q > n 
\end{cases}
\]

where \( p = \frac{n(n-q)}{n+q} \).

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$\Omega_p(K)$ ... $L_p$ - affine surface area of $K$

$$\Omega_p(K) = \int_{\partial K} \kappa_0(x)^{\frac{p}{n+p}} h_K(x) \, dx$$

Hug, Lutwak, Werner, ...
$\Omega_p(K) \ldots L_p$ - affine surface area of $K$

$$\Omega_p(K) = \int_{\partial K} \kappa_0(x)^{\frac{p}{n+p}} h_K(x) \, dx$$

$\Omega(K) = \Omega_1(K)$
$\Omega_p(K)$ ... $L_p$ - affine surface area of $K$

$$\Omega_p(K) = \int_{\partial K} \kappa_0(x)^{\frac{p}{n+p}} h_K(x) \, dx$$

$\Omega(K) = \Omega_1(K)$

$\Omega_c(K) = \Omega_n(K)$
$\Phi: \mathcal{K}_0^n \rightarrow IR$ upper semicontinuous, $SL(n)$ invariant valuation, vanishing on polytopes

\[ \Phi(K) = \int_{\partial K} \phi(\kappa_0(x)) \, h_K(x) \, dx \]

with $\phi: [0, \infty) \rightarrow [0, \infty)$ concave, $\lim_{t \to 0} \phi(t) = \lim_{t \to \infty} \frac{\phi(t)}{t} = 0$. 

Ludwig, Reitzner
Proof

\[ + \ 0 \]
Proof
Proof

\[\begin{align*}
\text{Diagram:} \\
\text{Line segment with label } 0
\end{align*}\]
Proof

\[ \Phi(K \cap H, a, -a) \rightarrow 0 \text{ as } a \rightarrow 0 \]
Proof

\[ \Phi([K \cap H, a, -a]) \rightarrow 0 \text{ as } a \rightarrow 0 \]
Proof
Proof
Proof
Proof

\[
E \rightarrow K = \implies \Phi(K) \preceq \Phi(E)
\]
Proof

\[ E_m \rightarrow K \]
Proof

\[ E_m \rightarrow K \quad \implies \quad \Phi(K) \gtrapprox \Phi(E_m) \]
Proof
Proof
Proof

\[ E_m \rightarrow K \implies \Phi(K) \simeq \Phi(E_m) \]
Proof

\[ \Phi(E_m) \simeq \Phi(K) \simeq \Phi(E_m) \]
Proof

$\Phi(K) \approx \Phi(E_m)$