Almost spherical sections of a cross-polytope generated by random matrices

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Almost spherical sections of a cross-polytope

Theorem (Kashin)

Let $\delta > 0$*. Let* $E \subset \mathbb{R}^n$ *be a random subspace of dimension* $m = (1-\delta)n$*. Then*

$$\frac{1}{\sqrt{n}} B_2^n \cap E \subset B_1^n \cap E \subset \frac{\varphi(\delta)}{\sqrt{n}} B_2^n \cap E$$

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Diameter of a random section (Garnaev–Gluskin)

$$\varphi(\delta) \asymp \sqrt{\frac{\log(1/\delta)}{\delta}} \quad \text{for } m > c \log n$$

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- A random rotation of $\mathbb{R}^m$.
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Short version of the Khinchin inequality

$$\forall a \in \mathbb{R}^n \quad \frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j=1}^{n} \varepsilon_j a_j \right| \sim \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2} .$$

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How small can be a set $Q \subset \{-1, 1\}^n$ such that

$$\forall a \in \mathbb{R}^n \quad \frac{1}{|Q|} \sum_{\varepsilon \in Q} |\langle \varepsilon, a \rangle| \sim \|a\|_2 \, ?$$

Previous results.

Let $m < n$ and let $\delta = \frac{n-m}{n}$. If $E \subset \mathbb{R}^n$ is the span of m random ± 1 vectors, then

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- Litvak, Pajor, R., Tomczak-Jaegermann, Vershynin (2005)

$$\varphi(\delta) = C^{1+1/\delta} \text{ for } \delta > \frac{c}{\log n} \quad \text{Probability: } 1 - e^{-cn}.$$

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- Artstein-Avidan, Friedman, Milman, Sodin (2006)

$$\varphi(\delta) = c\delta^{-5/2} \log(1/\delta) \text{ for } \delta > cn^{-1/6} \log n \quad \text{Probability: } 1 - \exp(-c\delta n^{1/6}).$$

Random sections.

Proposition

Let $m < n$ and let $\delta = \frac{n-m}{n}$. Let $E \subset \mathbb{R}^n$ be the span of m random ± 1 vectors. Then for any $t \leq c\delta$

$$\frac{1}{\sqrt{n}} B_2^n \cap E \subset B_1^n \cap E \subset \frac{C}{t\delta} \cdot \frac{1}{\sqrt{n}} B_2^n \cap E$$

with probability greater than $1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}$.

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- Probability exponentially close to 1

$$(t/\bar{c}\delta)^{\delta n} \sim e^{-cn} \;\Rightarrow\; t = c^{1+1/\delta}.$$

(LPRTV) $\varphi(\delta) = C^{1+1/\delta}$ for $\delta > \frac{c}{\log n}$.

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- Polynomial bound

$$\varphi(\delta) = C\delta^{-2} \quad \text{with probability } \sim 1 - e^{-m}.$$

(LPRTV) $\varphi(\delta) = C^{1+1/\delta}$ for $\delta > \frac{c}{\log n}$.

- Probability: $1 - e^{-cn}$.

(AFMS) $\varphi(\delta) = c\delta^{-5/2}\log(1/\delta)$ for $\delta > cn^{-1/6}\log n$.

- Probability: $1 - \exp(-c\delta n^{1/6})$.

Equivalence of norms

Let $E \subset \mathbb{R}^n$ be the span of $(1-\delta)n$ random ± 1 vectors. Then for any $t \le c\delta$

$$\forall y \in E, \ \sqrt{n}\, \|y\|_2 \le \frac{c}{t\delta}\, \|y\|_1 \ \text{with probability } 1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}.$$

Form an $n \times m$ matrix from m random ± 1 vectors in $\mathbb{R}^n$. This allows to recast the problem as a question about random matrices with independent entries.

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Let $m < n$ and let $\delta = \frac{n-m}{n}$. Let A be an $n \times m$ random ± 1 matrix. Then for any $t \le c\delta$

$$\forall x \in S^{m-1},\ \|Ax\|_1 \ge t\delta n$$

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Cauchy–Schwartz inequality

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$$\|A\| \leq C\sqrt{n}$$

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Minimization of the norm (convexity)

Theorem

Let $m < n$ *and let* $\delta = \frac{n-m}{n}$. *Let* A *be an* $n \times m$ *random* ± 1 *matrix. Then for any* $t \le c\delta$

$$\mathbb{P}\left(\min_{x \in S^{m-1}} \|Ax\|_1 \ge t\delta n \right) \ge 1 - C\exp(-cn) - (t/\bar{c}\delta)^{\delta n}.$$

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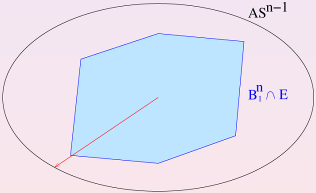

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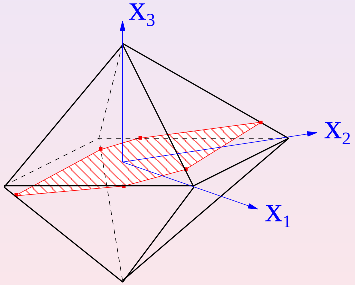

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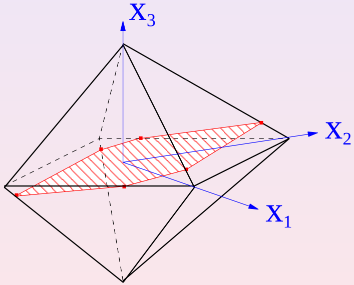
- In other words: if $x \in S^{m-1}$ is a point of minimum, then $A_J x = 0$ for some set $J \subset \{1, \dots, n\}$, $|J| = m - 1$.



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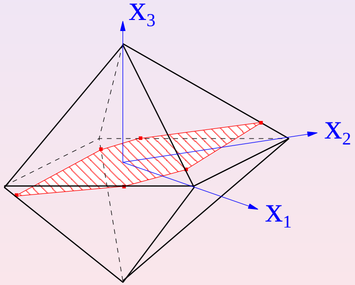
- For a given set J such point $x \in S^{m-1}$ is uniquely defined up to a sign.



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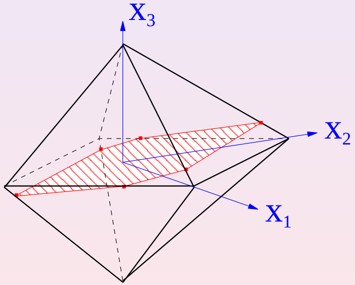
- We have to check at most $\binom{n}{m-1} \sim \left(\frac{n}{n-m}\right)^{n-m} = (1/\delta)^{\delta n}$ points.



Reduction to one dimension (probability)

Theorem

$$\mathbb{P}\left(\min_{x\in S^{m-1}} \|Ax\|_1 \le t\delta n\right) \le (t/\bar{c}\delta)^{\delta n} + C\exp(-cn).$$

- Fix $J \subset \{1,\ldots,n\}$, $|J| = m-1$ and find $x \in S^{m-1}$ such that $A_J x = 0$.

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- Fix $J \subset \{1, \ldots, n\}$, $|J| = m - 1$ and find $x \in S^{m-1}$ such that $A_J x = 0$.
- The matrices A_J and A_{J^c} are independent.
- $\|Ax\|_1 = \|A_{J^c} x\|_1$ is a sum of i.i.d. random variables: $\|Ax\|_1 = \sum_{j \in J^c} |\xi_j|$, where $\xi_j \sim \sum_{k=1}^m \varepsilon_k x_k$.
- This reduces the problem to estimating the small ball probability $p_x(t) = \mathbb{P}\left(\left| \sum_{k=1}^m \varepsilon_k x_k \right| < t \right)$.

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- Ideal case: ξ_j behaves like a normal random variable:

$$\mathbb{P}(|\xi_j| < t) \sim \mathbb{P}(|\gamma| < t) \le Ct.$$

This implies

$$\mathbb{P}\left(\|Ax\|_1 < t \cdot |J^c|\right) \le (C't)^{|J^c|}.$$

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Small ball probability (number theory + probability)

$$p_x(t) = \mathbb{P}\left(\left|\sum_{k=1}^{m} \varepsilon_k x_k\right| < t\right)$$

Examples

- if $x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$, then $p_x(t) = 1/2$.
- if $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, , \dots, \frac{1}{\sqrt{n}})$, then $p_x(t) \sim t + 1/\sqrt{n}$.

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- Littlewood–Offord Problem: How does the small ball probability depend on the arithmetic properties of the sequence $x = (x_1, \dots, x_m)$?
- Random normal theorem: If $x \in S^{m-1}$ satisfies $A_J x = 0$, then with probability exponentially close to 1,

$$p_x(t) \le Ct + e^{-cm} \quad \text{for all } t > 0.$$