Serrin type overdetermined problems for Hessian equations via Isoperimetric inequality

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Joint work with B. Brandolini, C. Nitsch, C. Trombetti
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\end{align*}
\Rightarrow \quad \Omega \text{ is a ball and } u = \frac{|x|^2 - r^2}{2}
\text{ up to a translation.}
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**One physical interpretation:** Consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of cross section $\Omega$. Fix coordinates with the $z$ axis directed along the pipe (i.e. parallel to the flow), then the flow velocity $v$ is a function of $(x, y)$ only: $\Delta v = -c$. $v = 0$ on $\partial \Omega$ is the adherence condition at pipe wall, while the tangential stress per unit area on the pipe wall is proportional to $\partial v / \partial \nu$.
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Serrin’s result states that **the tangential stress on the pipe wall is the same at all points if an only if the pipe has circular cross section.**
Let $\Omega$ be a $C^2$ domain and $u \in C^2(\overline{\Omega})$

$$\begin{cases}
S_k(D^2u) = \text{constant} & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega \\
\frac{\partial u}{\partial \nu} = \text{constant} & \text{on } \partial\Omega.
\end{cases}$$

$\Omega$ is a ball & $u = \frac{|x|^2 - r^2}{2}$ up to a translation.

$S_k(A)$ is the $k$-th elementary symmetric function of the eigenvalues of the matrix $A$. In particular:

$$S_1(D^2u) = \Delta u \text{ Poisson eq.}, \quad S_n(D^2u) = \det D^2u \text{ Monge-Ampère eq.}$$
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\[ \Rightarrow \]

+ \text{STABILITY: } \frac{\partial u}{\partial \nu} \simeq \text{const. } \Rightarrow \Omega \simeq \text{a ball} \]

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Selected famous papers:


For more references:


Very short proof:

Weinberger’s proof strongly relies on linearity and still maximum principle is involved. Serrin’s proof is much more general and it can be adapted also to nonlinear equations, but it needs that $u_{\nu\nu}$ can be determined in terms of the remaining second partial derivatives.

Very short proof:

- auxiliary function $f = |Du|^2 - 2u = 1$ in $\partial \Omega$
- $f = 1$ on $\partial \Omega$ and $\Delta f \geq 0$ in $\Omega$
- then either $f < 1$ in $\Omega$ either $f \equiv 1$ in $\Omega$, by maximum principle
- $f < 1$ contradicts Pohožaev Identity
- then $f \equiv 1$ in $\Omega \Rightarrow \Delta f = 0$ in $\Omega$
- then $D^2u = I$ in $\Omega \Rightarrow u$ is radial

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Serrin’s problem for Hessian equations

Let $\Omega$ be a $C^2$ domain and $u \in C^2(\Omega)$

\[
\begin{align*}
S_k(D^2 u) &= \left( n^k \right) \\
u &= 0 \\
\partial u / \partial \nu &= 1
\end{align*}
\]

$\Rightarrow u = |x|^2 - \frac{1}{2}$ & $\Omega \equiv B(0,1)$

up to a translation

$S_k(A)$ is the $k$-th elementary symmetric function of the eigenvalues of the matrix $A$, i.e.

\[
S_k(A) = S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}
\]

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $D^2 u$. 

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Overdetermined problems via Isoperimetric Inequality

Cortona, June 2007
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\frac{|x|^2 - 1}{2} & \text{ and } \Omega \equiv B(0, 1)
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$S_k(A)$ is the $k$-th elementary symmetric function of the eigenvalues of the matrix $A$, i.e.

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where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $D^2 u$. 
Features of the proof

No explicit use of the maximum principle

Elementary proof based on the arithmetic-geometric mean inequality

Link with Isoperimetric inequality

The proof involves:

\[ W_k(D) = \frac{1}{n} \left( n - 1 \right) \int_{\partial D} C^{k-1}(\partial D). \]

Notice that \( W_1(D) = |\partial D|^n \), while, for \( k = 0 \), we set \( W_0(D) = |D| \).

A Pohožaev type identity for Hessian equation (see also K. Tso, 1990)

Many integrations by part

Newton's inequalities (the key ingredient)

Notice that we don't assume \( \Omega \) or \( u \) to be \( k \)-convex: hence we are not, by assumption, in the elliptic realm and we make no use of Maximum principle.
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The proof involves:
- Quermassintegrals (or intrinsic volumes)

\[ W_k(D) = \frac{1}{n^{(n-1)\choose k-1}} \int_{\partial D} C_{k-1}(\partial D). \]

Notice that \( W_1(D) = \frac{\lvert \partial D \rvert}{n}, \) while, for \( k = 0, \) we set \( W_0(D) = \lvert D \rvert. \)
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The proof involves:

- Quermassintegrals (or intrinsic volumes)

\[ W_k(D) = \frac{1}{n \binom{n-1}{k-1}} \int_{\partial D} C_{k-1}(\partial D). \]

Notice that \( W_1(D) = \frac{|\partial D|}{n} \), while, for \( k = 0 \), we set \( W_0(D) = |D| \).

- A Pohožaev type identity for Hessian equation (see also K. Tso, 1990)
- Many integrations by part
- Newton’s inequalities (the key ingredient)

Notice that we don’t assume \( \Omega \) or \( u \) to be \( k \)-convex: hence we are not, by assumption, in the elliptic realm and we make no use of Maximum principle.
Monge-Ampère ($k = n$)

Let $n \geq 2$. Assume that $\Omega$ is a $C^2$ domain such that there exists a function $u \in C^2(\Omega)$ which satisfies

\[
\begin{align*}
\det D^2 u &= 1 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega \\
\frac{\partial u}{\partial \nu} &= 1 \quad \text{on } \partial \Omega
\end{align*}
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then $\Omega$ is the unitary ball and $u = \frac{|x|^2 - 1}{2}$ (up to a translation).
Let $n \geq 2$. Assume that $\Omega$ is a $C^2$ domain such that there exists a function $u \in C^2(\overline{\Omega})$ which satisfies

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then $\Omega$ is the unitary ball and $u = \frac{|x|^2 - 1}{2}$ (up to a translation).

Notice that we don’t assume $\Omega$ and $u$ to be convex: the first step of our proof consists exactly in proving this.

Once proved that $u$ is convex, the overdetermined condition about $\partial u / \partial \nu$ can be read as

\[Du(\Omega) = B\]
Proof for Monge-Ampère

Integrating the equation

$$|\Omega| = \int_{\Omega} \det D^2 u \, dx =$$
Proof for Monge-Ampère

Integrating the equation

\[ |Ω| = \int_Ω \det D^2 u \, dx = \int_{Du(Ω)} 1 \, dy = \]

\[ \text{change } y = Du(x) \]

Let \( v \) be the conjugate function of \( u \).
Then \( v \) solves the problem

\[
\begin{cases}
\det D^2 v = 1 \\
v(y) = \langle y, Dv(y) \rangle 
\end{cases}
\]
on \( \partial B \).

Notice that \( Dv(y) \) is the point on \( \partial Ω \) where the outer normal is \( y \).
Hence the boundary condition for \( v \) can be rewritten also as

\[ v(y) = h_{Ω}(y), \quad y \in \partial B, \]

where \( h_{Ω} \) is the support function of \( Ω \).
Proof for Monge-Ampère

Integrating the equation

\[ |\Omega| = \int_{\Omega} \det D^2 u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n. \]
Proof for Monge-Ampère

Integrating the equation

\[ |\Omega| = \int_{\Omega} \det D^2 u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n. \]

Next we will prove that \( W_{n-1}(\Omega) = \omega_n \), hence \( \Omega \) realizes the equality in the isoperimetric inequality between \( W_0 \) and \( W_{n-1} \) and it is forced to be a ball.
Proof for Monge-Ampère

Integrating the equation

\[ |\Omega| = \int_{\Omega} \det D^2 u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n. \]

Let \( v \) be the conjugate function of \( u \).

\[ v(y) = \max \{ \langle y, x \rangle - u(x) : x \in \Omega \} \quad \text{for } y \in \mathbb{R}^n \]
Proof for Monge-Ampère

Integrating the equation

\[ |\Omega| = \int_\Omega \det D^2 u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n. \]

Let \( v \) be the conjugate function of \( u \).

\[ v(y) = \max\{\langle y, x \rangle - u(x) : x \in \overline{\Omega}\} \quad \text{for} \quad y \in \mathbb{R}^n \]

Then

\[ Dv = (Du)^{-1} \quad \text{in} \quad Du(\Omega), \]

and

\[ \begin{cases} D^2 v(y) = D^2 u(x)^{-1} \\ v(y) + u(x) = \langle y, x \rangle \end{cases} \quad \text{where} \quad x = Dv(y) \quad \text{and} \quad y = Du(x) \]
Integrating the equation

\[ |\Omega| = \int_{\Omega} \det D^2 u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n. \]

Let \( \nu \) be the conjugate function of \( u \). Then \( \nu \) solves the problem

\[
\begin{cases} 
\det D^2 \nu = 1 & \text{in } B \\
\nu(y) = \langle y, D\nu(y) \rangle & \text{on } \partial B.
\end{cases}
\]
Proof for Monge-Ampère

Integrating the equation

$$|\Omega| = \int_\Omega \det D^2u \, dx = \int_{Du(\Omega)} 1 \, dy = \int_B 1 \, dy = |B| = \omega_n.$$  

Let $v$ be the conjugate function of $u$. Then $v$ solves the problem

$$\begin{cases} 
\det D^2v = 1 & \text{in } B \\
v(y) = \langle y, Dv(y) \rangle & \text{on } \partial B.
\end{cases}$$

Notice that

$$Dv(y) = the \ point \ on \ \partial \Omega \ where \ the \ outer \ normal \ is \ y.$$  

Hence the boundary condition for $v$ can be rewritten also as

$$v(y) = h_\Omega(y) \quad y \in \partial B,$$

where $h_\Omega$ is the support function of $\Omega$. 
Proof for Monge-Ampère

Then we have

\[ W_{n-1}(\Omega) = \frac{1}{n} \int_{\partial B} h_{\Omega}(y) = \frac{1}{n} \int_{\partial B} v(y) = \]

integration by parts

+ geometric-arithmetic mean ineq.

\[ \geq \int_{B} \Delta v \, dy \]

\[ = \int_{B} \left( \det D^2 v \right)^{1/n} \, dy \]

\[ = \int_{B} 1 \, dy = \omega_n \]

Moreover, with some other integrations by parts, we obtain

\[ (n+2) \int_{\Omega} (-u) \, dx = \frac{1}{n} \int_{\partial B} v = W_{n-1}(\Omega). \]

Finally

\[ \omega_n \leq W_{n-1}(\Omega) = (n+2) \int_{\Omega} (-u). \]

\[ = (n+2) \int_{\Omega} \left( -u \right) \left( \det D^2 u \right)^{1/n} \leq n+2 \int_{B} |Dv|^2 \]

\[ = n+2 \int_{B} |y|^2 = \omega_n. \]
Then we have

\[ W_{n-1}(\Omega) = \frac{1}{n} \int_{\partial B} h_{\Omega}(y) = \frac{1}{n} \int_{\partial B} v(y) = \int_{\partial B} < Dv(y), y > \]
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Finally

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\omega_n \leq W_{n-1}(\Omega) = (n + 2) \int_\Omega (-u) = (n + 2) \int_\Omega (-u)(\det D^2 u)^{1/n}
\leq \frac{n+2}{n} \int_\Omega (-u)\Delta u = \frac{n+2}{n} \int_\Omega |Du|^2
\]
Then we have

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Finally

\[ \omega_n \leq W_{n-1}(\Omega) = (n + 2) \int_\Omega (-u) = (n + 2) \int_\Omega (-u)(\det D^2 u)^{1/n} \]

\[ \leq \frac{n + 2}{n} \int_\Omega (-u) \Delta u = \frac{n + 2}{n} \int_\Omega |Du|^2 = \frac{n + 2}{n} \int_B |y|^2 = \omega_n. \]
Let $\Omega$ be a $C^2$ domain and $u \in C^2(\overline{\Omega})$

\[
\begin{align*}
S_k(D^2 u) &= \text{constant} \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

Assume that $\partial u / \partial \nu$ is close to a constant in some norm on $\partial \Omega$.

**Question:** is it $\Omega$ close in some sense to a ball and is it possible to control the distance of $\Omega$ from a ball, in some suitable norm, in terms of the distance of $\partial u / \partial \nu$ from the constant?
Let $\Omega$ be a $C^2$ domain and $u \in C^2(\overline{\Omega})$

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\begin{cases}
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The question of stability, also for the original Serrin problem, is almost unknown. As far as the authors know, the only work about it is


Stability for the Laplacian \((k = 1)\)

**Theorem** (Brandolini-Nitsch-Salani-Trombetti 2007)

Let \(\Omega\) be a bounded, connected, open set of class \(C^{2,\alpha}\) in \(\mathbb{R}^n\). Then there exist positive constant \(C\) and \(\delta_0\), depending only on the \(C^{2,\alpha}\)-regularity of \(\Omega\) and dimension \(n\), such that the following holds: if \(u \in C^2(\overline{\Omega})\) is a solution to the problem

\[
\begin{cases}
\Delta u = n & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]

satisfying the condition

\[
||Du| - 1|| \leq \delta \leq \delta_0 \quad \text{on } \partial\Omega,
\]

then there exist two concentric balls \(B_r\) and \(B_R\) such that

\[
B_r \subset \Omega \subset B_R \quad \text{and} \quad R - r \leq C\delta^{\frac{1}{4n+9}}.
\]
Stability for the Laplacian ($k = 1$)

To prove it, we show that the *isoperimetric deficit* $D(\Omega)$ of $\Omega$ is small, precisely

$$0 \leq D(\Omega) = \frac{|\partial \Omega|}{n \omega_n^{1/n} |\Omega|^{(n-1)/n}} - 1 \leq C \frac{\delta^{1/(4n+9)}}{1}$$

where $C$ is a constant depending only on $n$ and the regularity of $\Omega$. Then we can use recent results about stability for isoperimetric inequality.
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where \(C\) is a constant depending only on \(n\) and the regularity of \(\Omega\).

Then we can use recent results about stability for isoperimetric inequality.

Stability of the isoperimetric inequality


They prove that there exist a ball \(B\), with the same volume of \(\Omega\), such that the measure of the symmetric difference between \(\Omega\) and \(B\) is controlled by the square root of the deficit:

\[
|\Omega \triangle B| \leq C(n) \sqrt{D(\Omega)},
\]

where \(\Omega \triangle B = (\Omega \cup B) \setminus (\Omega \cap B)\).
Stability for Monge-Ampère ($k = n$)

**Theorem** (Brandolini-Nitsch-Salani-Trombetti 2007)

Let $n \geq 2$ and let $\epsilon$ and $\delta$ be two positive (suitably small) real numbers. Assume that $\Omega$ is a $C^2$ convex domain and $u \in C^2(\overline{\Omega})$ satisfies

\[
\begin{aligned}
(1 - \epsilon) \leq \det D^2 u & \leq (1 + \epsilon) & \text{in } \Omega \\
u = 0 & \quad \text{on } \partial\Omega \\
|\partial u/\partial \nu - 1| & \leq \delta & \text{on } \partial\Omega.
\end{aligned}
\]

Then $\Omega$ is a $C^2_+$ domain, $u$ is strictly convex in $\overline{\Omega}$ and

\[
R - r \leq C_1 (\delta^2 + \epsilon^2)^{\frac{1}{n+3}},
\]

\[
\|u(x) - \frac{|x|^2 - R^2}{2}\|_{L^\infty(\Omega)} \leq C_2 (\delta^2 + \epsilon^2)^{\frac{1}{n+3}} \quad \text{(up to a translation)},
\]

where $R$ and $r$ are the circumradius and the inradius of $\Omega$, respectively, and $C_1$, $C_2$ are constants depending only on the dimension $n$. 
Arguing as before, we can prove that both $W_0(\Omega) = |\Omega|$ and $W_{n-1}(\Omega)$ are close to $\omega_n$. Then, we can use stability result for the isoperimetric inequalities for quermassintegrals to get the conclusion.
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**Stability for isoperimetric inequality in the class of convex bodies**

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Stability for isoperimetric inequality in the class of convex bodies