On a problem by Hadwiger

E. Saorín Gómez

(joint work with M. A. Hernández Cifre)

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The outer parallel body

\[ \{ K \text{ convex body} \quad \rho \geq 0 \} \mapsto K + \rho B^n = \text{outer parallel body of } K \text{ at distance } \rho \]
The inner parallel body

$K$ convex body

$0 \leq \rho \leq r(K)$

$\sim \to K \sim \rho B^n = \{ x \in \mathbb{R}^n : \rho B^n + x \subset K \}$

$K \sim \rho B^n = \text{inner parallel body of } K \text{ at distance } \rho$
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If $\rho = r \sim \ker(K) = \{ \text{incenters of } K \}$
The full system of parallel bodies of $K$

$$K_{\rho} := \begin{cases} 
K \sim (-\rho)B^n & \text{for } -r(K) \leq \rho \leq 0 \\
K + \rho B^n & \text{for } 0 \leq \rho < \infty 
\end{cases}$$
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The volume of the outer parallel body is obtained in the following way:

**Theorem (the Steiner formula, 1840)**

The volume of the outer parallel body of $K$ at distance $\rho \geq 0$, $K_\rho = K + \rho B^n$, is expressed as a polynomial of degree the dimension $n$ in the parameter $\rho$, the so called **Steiner polynomial**, where its coefficients are, up to a constant, the **quermassintegrals** of the body $K$, $W_i(K)$, for $0 \leq i \leq n$:

$$V(K_\rho) = V(K + \rho B^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) \rho^i.$$
However... there is **no explicit formula** for the volume of an **inner parallel body**.
The Steiner polynomial

However... there is no explicit formula for the volume of an inner parallel body.

The alternating Steiner polynomial

The alternating Steiner polynomial is obtained replacing $\rho$ by $-\rho$ in the Steiner polynomial:

$$\sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$
The Steiner polynomial in $\mathbb{R}^3$

\[ V(K + \rho B^n) = \sum_{i=0}^{3} \binom{3}{i} W_i(K) \rho^i = V(K) + S(K) \rho + M(K) \rho^2 + \frac{4}{3} \pi \rho^3 \]

- $W_0(K) = V(K)$ is the usual volume of $K$.
- $3W_1(K) = S(K)$ is its surface area.
- $3W_2(K) = M(K)$ is its integral mean curvature.
- $W_3(K) = \kappa_3$ is the volume of the unit ball.

Coefficients of $V(K_\rho) \longrightarrow$ Functionals in Hadwiger’s problem: $V, S, M$
Mixed volumes and mixed surface area measures

The volume of a linear combination of convex bodies

For convex bodies $K_1, \ldots, K_m \subset \mathbb{R}^n$ and real numbers $\rho_1, \ldots, \rho_m \geq 0$, the volume of the linear combination $\rho_1 K_1 + \cdots + \rho_m K_m$ is expressed as a polynomial of degree the dimension $n$ in the variables $\rho_1, \ldots, \rho_m$,

$$V(\rho_1 K_1 + \cdots + \rho_m K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} V(K_{i_1}, \ldots, K_{i_n}) \rho_{i_1} \cdots \rho_{i_n}.$$  

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are the mixed volumes of $K_1, \ldots, K_m$.

The mixed surface area measures

For convex bodies $K_1, \ldots, K_{n-1} \subset \mathbb{R}^n$ the mixed surface area measure is the finite Borel measure on $\mathbb{S}^{n-1}$ such that

$$V(K, K_1, \ldots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u).$$
A problem by Hadwiger

Problem:
To study the **differentiability of the functionals** $V$, $S$ and $M$ with respect to the parameter $\rho$ of the full system of parallel bodies.

Let us recall...

**The full system of parallel bodies of $K$:**

$$K_\rho := \begin{cases} 
K \sim (-\rho)B^n & \text{for } -r(K) \leq \rho \leq 0 \\
K + \rho B^n & \text{for } 0 \leq \rho < \infty 
\end{cases}$$
A problem by Hadwiger

Problem:

To study the **differentiability of the functionals** \( V, S \) and \( M \) with respect to the parameter \( \rho \) of the full system of parallel bodies.

It always holds:

<table>
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Since...

the functionals \( V(\rho)^{1/3}, S(\rho)^{1/2} \) and \( M(\rho) \) are concave in \( [-r(K), \infty) \)
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Fixing the problem:
To characterize the convex bodies belonging to the classes \( \mathcal{R}_\alpha, \mathcal{R}_\beta, \mathcal{R}_\gamma \).
A problem by Hadwiger

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Fixing the problem:
To characterize the convex bodies belonging to the classes $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$, $\mathcal{R}_\gamma$. 

If $\,\,\,\,'V(\rho) = S(\rho)$, $K$ lies in $\mathcal{K}^3$. 

If $\,\,\,\,'S(\rho) = 2M(\rho)$, class $\mathcal{R}_\beta$.

If $\,\,\,\,'M(\rho) = 4\pi$, class $\mathcal{R}_\gamma$.
A partial solution:  

A characterization, not of the convex bodies, but of the triples of values \((V, S, M)\) which can be, respectively, the volume, surface area and integral mean curvature, of some convex body in each class.
A partial solution (Hadwiger)

A partial solution:

A characterization, not of the convex bodies, but of the **triples of values** \((V, S, M)\) which can be, respectively, the volume, surface area and integral mean curvature, of some convex body in each class.

More precisely:

Three positive real numbers \(V, S\) and \(M\) are the (respective) magnitudes of some convex body belonging to the class \(R_\beta\) if, and only if, they satisfy the inequalities

\[
V \leq \frac{S^2}{3M}
\]

\[
V \geq \frac{1}{24\pi^2} \left[6\pi MS - M^3 - (12 - \pi^2)\pi \left(\frac{M^2 - 4\pi S}{\pi^2 - 8}\right)^{3/2}\right]
\]
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A characterization, not of the convex bodies, but of the **triples of values** \((V, S, M)\) which can be, respectively, the volume, surface area and integral mean curvature, of some convex body in each class.

More precisely:  

Three positive real numbers \(V, S\) and \(M\) are the (respective) magnitudes of some convex body belonging to the class \(\mathcal{R}_\gamma\) if, and only if, they satisfy the inequalities

\[
V \leq \frac{1}{24\pi^2} \left[ 6\pi MS - M^3 + (M^2 - 4\pi S)^{3/2} \right]
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V \geq \frac{1}{24\pi^2} \left[ 6\pi MS - M^3 - (12 - \pi^2)\pi \left( \frac{M^2 - 4\pi S}{\pi^2 - 8} \right)^{3/2} \right]
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The parallel bodies
The Steiner polynomial
Hadwiger’s problem

Setting the problem.

Hadwiger’s problem in the $n$-dimensional space
Tangential bodies and Hadwiger’s problem
An application: Matheron’s conjecture

*n*-dimensional Hadwiger’s problem

Hadwiger’s problem can be established in $\mathbb{R}^n$.

**Lemma**

The full system of parallel bodies is a concave family, i.e., they satisfy

$$(1 - \lambda)K_\rho + \lambda K_\sigma \subset K_{(1-\lambda)\rho + \lambda\sigma}.$$
n-dimensional Hadwiger’s problem

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**Lemma**

The full system of parallel bodies is a concave family, i.e., they satisfy

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**Brunn-Minkowski theorem**

The $(n - i)$-th root of the $i$-th quermassintegral $W_i$, $i = 0, \ldots, n$, is a concave function in $[-r(K), \infty)$.

Then...
...it always holds:

i) The relations

\[ 'W_i(\rho) \geq W_i'(\rho) \quad \text{and} \quad W_i'(\rho) \geq (n - i)W_{i+1}(\rho) \]

hold for \( i = 0, \ldots, n - 1 \).

ii) If \( i = 0 \), an equality for the derivative of 0-th quermassintegral, i.e., the volume, is obtained:

\[ 'V(\rho) = V'(\rho) = nW_1(\rho) = S(\rho). \]

iii) If \( \rho \geq 0 \), all the quermassintegrals are differentiable and

\[ W_i'(\rho) = (n - i)W_{i+1}(\rho). \]
Setting the n-dimensional Hadwiger’s problem

Definition

A convex body $K \subset \mathbb{R}^n$ belongs to the class $\mathcal{R}_p$, $0 \leq p \leq n - 1$, if

$$'W_i(\rho) = W'_i(\rho) = (n - i)W_{i+1}(\rho)$$

for every $0 \leq i \leq p$, and for $-r(K) \leq \rho \leq \infty$. 
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for every \( 0 \leq i \leq p \), and for \( -r(K) \leq \rho \leq \infty \).

- \( \mathcal{R}_0 \) is the family of all convex bodies in \( \mathbb{R}^n \).
- \( \mathcal{R}_{i+1} \subset \mathcal{R}_i \), \( i = 0, \ldots, n - 2 \).
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Theorem (characterization of \( \mathcal{R}_{n-1} \))

The only sets in \( \mathcal{R}_{n-1} \) are the outer parallel bodies of \( k \)-dimensional convex bodies, for \( 0 \leq k \leq n - 1 \):

\[
\mathcal{R}_{n-1} = \{ K + \rho \mathbb{B}^n : K \subset \mathbb{R}^k, 0 \leq k \leq n - 1 \}.
\]
Necessary conditions for a convex body to belong to $\mathcal{R}_p$

Two definitions

- The **form body** of a convex body $K$, denoted by $K^*$, is

  $$K^* = \bigcap_{u \in \Omega} H^- (B^n, u),$$

  where $\Omega$ is the closure of the set of outer normal unit vectors at regular boundary points of $K$. 

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Two definitions

- The form body of a convex body $K$, denoted by $K^*$, is

  $$K^* = \bigcap_{u \in \Omega} H^-(\mathbb{B}^n, u),$$

  where $\Omega$ is the closure of the set of outer normal unit vectors at regular boundary points of $K$.

- $u \in \mathbb{R}^n$ is an $r$-extreme normal vector of $K$ if we can not write $u = u_1 + \cdots + u_{r+2}$, with $u_i$ L.I., normal at the same boundary point.
Necessary conditions for a convex body to belong to $\mathcal{R}_p$

**Theorem:** If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

- $h(K_\rho^*, u) \equiv 1$ for all $u \in \text{supp } S_{\rho, (n-1)^{-1}}, K_\rho, B^n, (p), B^n, \cdot$
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Necessary conditions for a convex body to belong to \( \mathcal{R}_p \)

**Theorem:** If \( K \in \mathcal{R}_p \) then, for all \( \rho \in (-r, 0] \)

- \( h(K^*_\rho, u) \equiv 1 \) for all \( u \in \text{supp} \ S \left( K_\rho, \left( n-p -1 \right), K_\rho, \mathbb{B}^n, (p), \mathbb{B}^n, \cdot \right) \)

- \( S \left( K_\rho, \left( n-p -1 \right), K_\rho, \mathbb{B}^n, (p), \mathbb{B}^n, \cdot \right) = S \left( K^*_\rho, K_\rho, \left( n-p -1 \right), K_\rho, \mathbb{B}^n, (p-1), \mathbb{B}^n, \cdot \right) \)
Theorem: If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

- $h(K_{\rho}^*, u) \equiv 1$ for all $u \in \mathrm{supp} \ S \left( K_{\rho}, (n-p-1), K_{\rho}, B^n, (p), B^n, \cdot \right)$
- $S \left( K_{\rho}, (n-p-1), K_{\rho}, B^n, (p), B^n, \cdot \right) = S \left( K_{\rho}^*, K_{\rho}, (n-p-1), K_{\rho}, B^n, (p-1), B^n, \cdot \right)$

Consequence

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n - 1$. 
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Necessary conditions for a convex body to belong to $\mathcal{R}_p$

**Theorem**: If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

- $h(K^*, u) \equiv 1$ for all $u \in \text{supp} \ S(K, (n-p-1), K, B^n, (p), B^n, \cdot)$
- $S(K, (n-p-1), K, B^n, (p), B^n, \cdot) = S(K^*, K, (n-p-1), K, B^n, (p-1), B^n, \cdot)$

- For a polytope $P$, $P^*_\rho$ is a polytope whose facets are tangent to $B^n$.

**Consequence**

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n-1$. 
Necessary conditions for a convex body to belong to $\mathcal{R}_p$

**Theorem:** If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

- $h(K_\rho^*, u) \equiv 1$ for all $u \in \text{supp } S\left(K_\rho, (n-p-1), K_\rho, \mathbb{B}^n, (p), \mathbb{B}^n, \cdot \right)$
- $S\left(K_\rho, (n-p-1), K_\rho, \mathbb{B}^n, (p), \mathbb{B}^n, \cdot \right) = S\left(K_\rho^*, K_\rho, (n-p-1), K_\rho, \mathbb{B}^n, (p-1), \mathbb{B}^n, \cdot \right)$

- For a polytope $P$, $P_\rho^*$ is a polytope whose facets are tangent to $\mathbb{B}^n$.
- Then $h(P_\rho^*, u) \equiv 1$ if and only if $u$ is a 0-extreme vector of $P_\rho^*$.

**Consequence**

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n - 1$. 
Necessary conditions for a convex body to belong to $\mathcal{R}_p$

**Theorem:** If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

1. $h(K^*_\rho, u) \equiv 1$ for all $u \in \text{supp} \ S\left(K_\rho, (n-p-1), K_\rho, B^n, (p), B^n, \cdot \right)$
2. $S\left(K_\rho, (n-p-1), K_\rho, B^n, (p), B^n, \cdot \right) = S\left(K^*_\rho, K^*_\rho, (n-p-1), K^*_\rho, B^n, (p-1), B^n, \cdot \right)$

- For a polytope $P$, $P^*_\rho$ is a polytope whose facets are tangent to $B^n$.
- Then $h(P^*_\rho, u) \equiv 1$ if and only if $u$ is a 0-extreme vector of $P^*_\rho$.
- If $P \in \mathcal{R}_p$ then $h(P^*_\rho, u) \equiv 1$ for all $u \in \text{cl}\{p\text{-extreme normal vectors of } P_\rho\}$, which leads to a contradiction.

**Consequence**

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n - 1$. 
Definition:

\( K \) is called a **p-tangential body** of \( \mathbb{B}^n \) if each \((n - p - 1)\)-extreme support plane of \( K \) supports \( \mathbb{B}^n \), \( p = 0, \ldots, n - 1 \).

- A supporting hyperplane is said to be **p-extreme** if its outer normal vector is a \( p \)-extreme direction.
Tangential bodies and the Hadwiger problem

**Definition:**

$K$ is called a **p-tangential body** of $B^n$ if each $(n - p - 1)$-extreme support plane of $K$ supports $B^n$, $p = 0, \ldots, n - 1$.

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**Some properties**

- A 0-tangential body of $B^n$ is $B^n$ itself.
**Definition:**

A body $K$ is called a **$p$-tangential body** of $B^n$ if each $(n - p - 1)$-extreme support plane of $K$ supports $B^n$, $p = 0, \ldots, n - 1$.

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- A 0-tangential body of $B^n$ is $B^n$ itself.
- The 1-tangential bodies are the *cap-bodies*. 
Tangential bodies and the Hadwiger problem

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*K* is called a **p-tangential body** of \( B^n \) if each \((n - p - 1)\)-extreme support plane of *K* supports \( B^n \), \( p = 0, \ldots, n - 1 \).

- A supporting hyperplane is said to be **p-extreme** if its outer normal vector is a \( p \)-extreme direction.

Some properties

- A 0-tangential body of \( B^n \) is \( B^n \) itself.
- The 1-tangential bodies are the cap-bodies.
- **Tangential body** \( = (n - 1)\)-tangential body.
Theorem (R. Schneider)

Let \( K \subset \mathbb{R}^n \) be a convex body with interior points, and \( 0 < \rho < r(K) \). Then \( K \sim \rho B^n \) is homothetic to \( K \) if, and only if, \( K \) is homothetic to a tangential body of \( B^n \).
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Theorem: tangential bodies lying in $\mathcal{R}_p$

A tangential body $K \subset \mathbb{R}^n$ lies in the class $\mathcal{R}_p$ if and only if, $K$ is a $(n - p - 1)$-tangential body.
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Theorem: tangential bodies lying in $\mathcal{R}_p$

A tangential body $K \subset \mathbb{R}^n$ lies in the class $\mathcal{R}_p$ if and only if, $K$ is a $(n - p - 1)$-tangential body.

In particular:

- The only tangential bodies in $\mathcal{R}_{n-2}$ are the cap-bodies.
- There are no tangential bodies in $\mathcal{R}_{n-1}$; just the ball.
Characterization of $\mathcal{R}_\gamma$:

$$\mathcal{R}_\gamma = \{ K + \rho B^3 : K \text{ planar convex body} \}.$$
The original Hadwiger problem

Characterization of $\mathcal{R}_\gamma$:

$$\mathcal{R}_\gamma = \{ K + \rho \mathbb{B}^3 : K \text{ planar convex body} \}.$$

- There are no polytopes in $\mathcal{R}_\beta$. 
The original Hadwiger problem

Characterization of $\mathcal{R}_\gamma$:

$$\mathcal{R}_\gamma = \{ K + \rho \mathbb{B}^3 : K \text{ planar convex body} \}.$$ 

- There are no polytopes in $\mathcal{R}_\beta$.
- The only tangential bodies in $\mathcal{R}_\beta$ are the cap-bodies.
Characterization of $\mathcal{R}_\gamma$:

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- There are no polytopes in $\mathcal{R}_\beta$.
- The only tangential bodies in $\mathcal{R}_\beta$ are the cap-bodies.

Using an additional condition...

If $K \in \mathcal{R}_\beta$ and $M(\rho)$ is a linear function when $-r \leq \rho \leq 0$, then $K$ is a cap-body of a set $K_0 + \rho \mathbb{B}^3 \in \mathcal{R}_\gamma$, with $\ker(K) = K_0$. 
The original Hadwiger problem

Characterization of $\mathcal{R}_\gamma$:

$$\mathcal{R}_\gamma = \{ K + \rho \mathbb{B}^3 : K \text{ planar convex body} \}.$$ 

- There are no polytopes in $\mathcal{R}_\beta$.
- The only tangential bodies in $\mathcal{R}_\beta$ are the cap-bodies.

Using an additional condition...

If $K \in \mathcal{R}_\beta$ and $M(\rho)$ is a linear function when $-r \leq \rho \leq 0$, then $K$ is a cap-body of a set $K_0 + \rho \mathbb{B}^3 \in \mathcal{R}_\gamma$, with $\ker(K) = K_0$.

Conjecture

The only sets in $\mathcal{R}_\beta$ are cap-bodies of outer parallel bodies of planar convex bodies $K_0 \subset \mathbb{R}^2$ with kernel $K_0$. 

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On a problem by Hadwiger
The volume of inner parallel bodies

Steiner formulae: quermassintegrals of outer parallel bodies

For $K \subset \mathbb{R}^n$, $\rho \geq 0$ and $i = 0, \ldots, n$ it holds

$$W_k(K + \rho \mathbb{B}^n) = \sum_{i=0}^{n-k} \binom{n-k}{i} W_{k+i}(K) \rho^i.$$ 

Steiner formulae for inner parallel bodies? (Matheron)

Let $K \subset \mathbb{R}^n$. Then

$$W_k(K \sim \rho \mathbb{B}^n) = \sum_{i=0}^{n-k} \binom{n-k}{i} W_{k+i}(K)(-\rho)^i$$

for $0 < \rho < r(K)$ and $k = 0, \ldots, n$ if and only if $K = L + \lambda \mathbb{B}^n$, $\lambda \geq \rho$. 
The Matheron conjecture

Conjecture (Matheron)

Let $K \subset \mathbb{R}^n$. Then for $0 < \rho < r(K)$

$$V(K \sim \rho B^n) \geq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$ 

The equality holds if and only if $K = L + \lambda B^n$, $\lambda \geq \rho$. 

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On a problem by Hadwiger
The Matheron conjecture

**Conjecture (Matheron)**

Let \( K \subset \mathbb{R}^n \). Then for \( 0 < \rho < r(K) \)

\[
V(K \sim \rho \mathbb{B}^n) \geq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.
\]

The equality holds if and only if \( K = L + \lambda \mathbb{B}^n, \lambda \geq \rho \).

**Result**

Let \( K \subset \mathbb{R}^n, n \text{ odd} \), be a convex body lying in \( \mathcal{R}_{n-2} \). Then

\[
V(K \sim \rho \mathbb{B}^n) \leq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.
\]

The equality holds if and only if \( K \in \mathcal{R}_{n-1} \).
On a problem by Hadwiger

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(joint work with M. A. Hernández Cifre)

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