On a problem by Hadwiger

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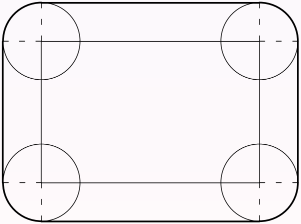
Cortona, June 2007

The outer parallel body

The outer parallel body

$$\left.\begin{array}{l} K \text{ convex body} \\ \rho \geq 0 \end{array}\right\} \rightsquigarrow K + \rho\mathbb{B}^n = \textit{outer parallel body} \text{ of } K \text{ at distance } \rho$$



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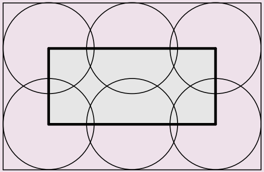
$$\left.\begin{array}{l} K \text{ convex body} \\ 0 \le \rho \le r(K) \end{array}\right\} \rightsquigarrow K \sim \rho\mathbb{B}^n = \left\{x \in \mathbb{R}^n : \rho\mathbb{B}^n + x \subset K\right\}$$

$K \sim \rho\mathbb{B}^n =$ *inner parallel body* of K at distance ρ



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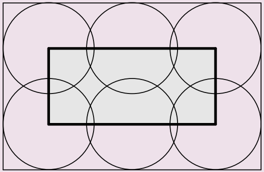

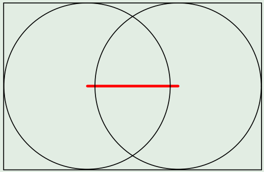

If $\rho = r \rightsquigarrow \ker(K) = \{\text{incenters of } K\}$

The full system of parallel bodies

The full system of parallel bodies of K

$$K_\rho := \begin{cases} K \sim (-\rho)\mathbb{B}^n & \text{for } -r(K) \leq \rho \leq 0 \\ K + \rho\mathbb{B}^n & \text{for } 0 \leq \rho < \infty \end{cases}$$

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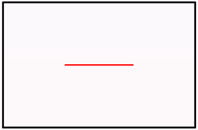

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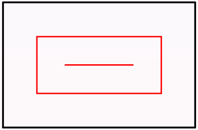

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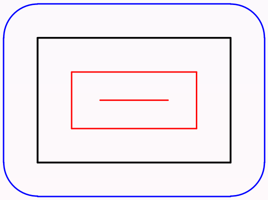

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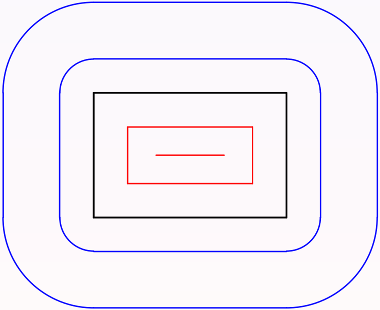

The Steiner polynomial

The volume of the outer parallel body is obtained in the following way:

Theorem (the Steiner formula, 1840)

The volume of the outer parallel body of K at distance $\rho \geq 0$, $K_\rho = K + \rho\mathbb{B}^n$, is expressed as a polynomial of degree the dimension n in the parameter ρ, the so called **Steiner polynomial**, where its coefficients are, up to a constant, the **quermassintegrals** of the body K, $W_i(K)$, for $0 \leq i \leq n$:

$$V(K_\rho) = V(K + \rho\mathbb{B}^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i.$$

The Steiner polynomial

However...

there is **no explicit formula** for the volume of an inner parallel body.

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The alternating Steiner polynomial

The **alternating Steiner polynomial** is obtained replacing ρ by $-\rho$ in the Steiner polynomial:

$$\sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$

The Steiner polynomial in $\mathbb{R}^3$

The Steiner polynomial in $\mathbb{R}^3$

$$V(K+\rho\mathbb{B}^n)=\sum_{i=0}^{3}\binom{3}{i}W_i(K)\rho^i = V(K)+S(K)\rho+M(K)\rho^2+\frac{4}{3}\pi\rho^3$$

- $W_0(K)=V(K)$ is the usual **volume** of K.
- $3W_1(K)=S(K)$ is its **surface area**.
- $3W_2(K)=M(K)$ is its **integral mean curvature**.
- $W_3(K)=\kappa_3$ is the **volume of the unit ball**.

Coefficients of $V(K_\rho)\longrightarrow$ Functionals in **Hadwiger's problem**: V, S, M

Mixed volumes and mixed surface area measures

The volume of a linear combination of convex bodies

For convex bodies $K_1, \dots, K_m \subset \mathbb{R}^n$ and real numbers $\rho_1, \dots, \rho_m \geq 0$, the volume of the linear combination $\rho_1 K_1 + \dots + \rho_m K_m$ is expressed as a polynomial of degree the dimension n in the variables $\rho_1, \dots, \rho_m$,

$$V(\rho_1 K_1 + \dots + \rho_m K_m) = \sum_{i_1=1}^{m} \dots \sum_{i_n=1}^{m} V(K_{i_1}, \dots, K_{i_n}) \rho_{i_1} \dots \rho_{i_n}.$$

The coefficients $V(K_{i_1}, \dots, K_{i_n})$ are the **mixed volumes** of $K_1, \dots, K_m$.

The mixed surface area measures

For convex bodies $K_1, \dots, K_{n-1} \subset \mathbb{R}^n$ the **mixed surface area measure** is the finite Borel measure on $\mathbb{S}^{n-1}$ such that

$$V(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)\, dS(K_1, \dots, K_{n-1}, u).$$

A problem by Hadwiger

Problem:

To study the **differentiability of the functionals V, S and M** with respect to the parameter ρ of the full system of parallel bodies.

Let us recall...

The full system of parallel bodies of K:

$$K_\rho := \begin{cases} K \sim (-\rho)\mathbb{B}^n & \text{for } -r(K) \leq \rho \leq 0 \\ K + \rho\mathbb{B}^n & \text{for } 0 \leq \rho < \infty \end{cases}$$

A problem by Hadwiger

Problem:

To study the **differentiability of the functionals** V**,** S **and** M with respect to the parameter ρ of the full system of parallel bodies.

It always holds:		
$'V(\rho) \geq V'(\rho) \geq S(\rho)$		
$'S(\rho) \geq S'(\rho) \geq 2M(\rho)$		
$'M(\rho) \geq M'(\rho) \geq 4\pi$		

Since...

the functionals $V(\rho)^{1/3}$, $S(\rho)^{1/2}$ and $M(\rho)$ are concave in $\left[-r(K), \infty\right)$

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To study the **differentiability of the functionals** V**,** S **and** M with respect to the parameter ρ of the full system of parallel bodies.

It always holds:	If	then K lies in
${}'V(\rho) \geq V'(\rho) \geq S(\rho)$	${}'V(\rho) = S(\rho)$	class $\mathcal{R}_\alpha$
${}'S(\rho) \geq S'(\rho) \geq 2M(\rho)$	${}'S(\rho) = 2M(\rho)$	class $\mathcal{R}_\beta$
${}'M(\rho) \geq M'(\rho) \geq 4\pi$	${}'M(\rho) = 4\pi$	class $\mathcal{R}_\gamma$

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Fixing the problem:

To characterize the convex bodies belonging to the classes $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$, $\mathcal{R}_\gamma$.

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A partial solution (Hadwiger)

A partial solution:

A characterization, not of the convex bodies, but of the **triples of values** $\mathbf{(V, S, M)}$ which can be, respectively, the volume, surface area and integral mean curvature, of some convex body in each class.

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More precisely:

Three positive real numbers V, S and M are the (respective) magnitudes of **some** convex body belonging to the class $\mathcal{R}_\beta$ if, and only if, they satisfy the inequalities

$$V \leq \frac{S^2}{3M}$$

$$V \geq \frac{1}{24\pi^2}\left[6\pi MS - M^3 - (12-\pi^2)\pi\left(\frac{M^2-4\pi S}{\pi^2-8}\right)^{3/2}\right]$$

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Three positive real numbers V, S and M are the (respective) magnitudes of **some** convex body belonging to the class $\mathcal{R}_\gamma$ if, and only if, they satisfy the inequalities

$$V \leq \frac{1}{24\pi^2}\left[6\pi MS - M^3 + (M^2 - 4\pi S)^{3/2}\right]$$

$$V \geq \frac{1}{24\pi^2}\left[6\pi MS - M^3 - (12 - \pi^2)\pi\left(\frac{M^2 - 4\pi S}{\pi^2 - 8}\right)^{3/2}\right]$$

n-dimensional Hadwiger's problem

Hadwiger's problem can be established in $\mathbb{R}^n$.

Lemma

The full system of parallel bodies is a concave family, i.e., they satisfy

$$(1-\lambda)K_\rho + \lambda K_\sigma \subset K_{(1-\lambda)\rho+\lambda\sigma}.$$

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Brunn-Minkowski theorem

The $(n-i)$-th root of the i-th quermassintegral W_i, $i = 0, \dots, n$, is a concave function in $\big[-r(K), \infty\big)$.

Then...

n-dimensional Hadwiger's problem

...it always holds:

i) The relations

$$'W_i(\rho) \geq W_i'(\rho) \quad \text{and} \quad W_i'(\rho) \geq (n-i)W_{i+1}(\rho)$$

hold for $i = 0, \dots, n-1$.

ii) If $i = 0$, an equality for the derivative of 0-th quermassintegral, i.e., the volume, is obtained:

$$'V(\rho) = V'(\rho) = nW_1(\rho) = S(\rho).$$

iii) If $\rho \geq 0$, all the quermassintegrals are differentiable and

$$W_i'(\rho) = (n-i)W_{i+1}(\rho).$$

Setting the n-dimensional Hadwiger's problem

Definition

A convex body $K \subset \mathbb{R}^n$ **belongs to the class $\mathcal{R}_\mathrm{p}$**, $0 \leq p \leq n-1$, if

$$'W_i(\rho) = W_i'(\rho) = (n-i)W_{i+1}(\rho)$$

for every $0 \leq i \leq p$, and for $-r(K) \leq \rho \leq \infty$.

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- $\mathcal{R}_0$ is the family of all convex bodies in $\mathbb{R}^n$.
- $\mathcal{R}_{i+1} \subset \mathcal{R}_i$, $i = 0, \ldots, n-2$.

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Theorem (characterization of $\mathcal{R}_{n-1}$)

The **only sets** in $\mathcal{R}_{n-1}$ are the outer parallel bodies of k-dimensional convex bodies, for $0 \leq k \leq n-1$:

$$\mathcal{R}_{n-1} = \left\{K + \rho\mathbb{B}^n : K \subset \mathbb{R}^k,\ 0 \leq k \leq n-1\right\}.$$

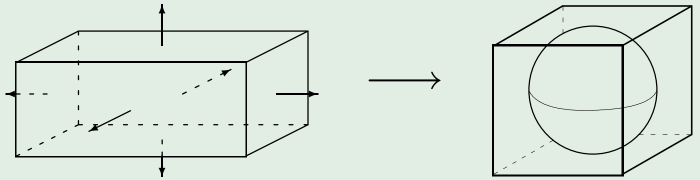
Necessary conditions for a convex body to belong to $\mathcal{R}_p$

Two definitions

- The **form body** of a convex body K, denoted by K^*, is

$$K^* = \bigcap_{u\in\Omega} H^-(\mathbb{B}^n, u),$$

where Ω is the closure of the set of outer normal unit vectors at regular boundary points of K.



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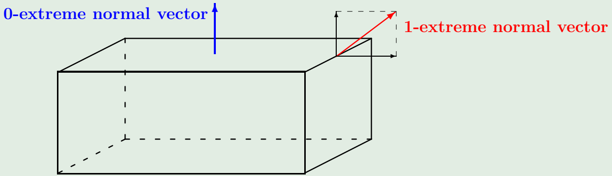
- $u \in \mathbb{R}^n$ is an **r-extreme normal vector** of K if we can not write $u = u_1 + \cdots + u_{r+2}$, with u_i L.I., normal at the same boundary point.



Necessary conditions for a convex body to belong to $\mathcal{R}_p$

Theorem: If $K \in \mathcal{R}_p$ then, for all $\rho \in (-r, 0]$

- $h(K_\rho^*, u) \equiv 1 \quad$ for all $\quad u \in \operatorname{supp} S\left(K_\rho, \overset{(n-p-1)}{\ldots}, K_\rho, \mathbb{B}^n, \overset{(p)}{\ldots}, \mathbb{B}^n, \cdot\right)$

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- $S\left(K_\rho, \overset{(n-p-1)}{\ldots}, K_\rho, \mathbb{B}^n, \overset{(p)}{\ldots}, \mathbb{B}^n, \cdot\right) = S\left(K_\rho^*, K_\rho, \overset{(n-p-1)}{\ldots}, K_\rho, \mathbb{B}^n, \overset{(p-1)}{\ldots}, \mathbb{B}^n, \cdot\right)$

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Consequence

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n-1$.

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- For a polytope P, P_ρ^* is a polytope whose facets are tangent to $\mathbb{B}^n$.

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- For a polytope P, P_ρ^* is a polytope whose facets are tangent to $\mathbb{B}^n$.
- Then $h(P_\rho^*, u) \equiv 1$ if and only if u is a 0-extreme vector of P_ρ^*.

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- For a polytope P, P_ρ^* is a polytope whose facets are tangent to $\mathbb{B}^n$.
- Then $h(P_\rho^*, u) \equiv 1$ if and only if u is a 0-extreme vector of P_ρ^*.
- If $P \in \mathcal{R}_p$ then $h(P_\rho^*, u) \equiv 1$ for all $u \in \operatorname{cl}\{p$-extreme normal vectors of $P_\rho\}$, which leads to a contradiction.

Consequence

There are no polytopes in $\mathcal{R}_p$, for all $1 \leq p \leq n-1$.

Tangential bodies and the Hadwiger problem

Definition:

K is called a **p-tangential body** of $\mathbb{B}^n$ if each $(n-p-1)$-extreme support plane of K supports $\mathbb{B}^n$, $p = 0, \ldots, n-1$.

- A supporting hyperplane is said to be **p-extreme** if its outer normal vector is a p-extreme direction.

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Some properties

- A 0-tangential body of $\mathbb{B}^n$ is $\mathbb{B}^n$ itself.

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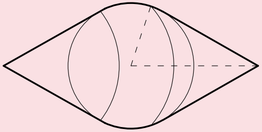

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- The 1-tangential bodies are the cap-bodies.

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- A 0-tangential body of $\mathbb{B}^n$ is $\mathbb{B}^n$ itself.
- The 1-tangential bodies are the cap-bodies.
- **Tangential body** $= (n-1)$-tangential body.

Tangential bodies and the Hadwiger problem

Theorem (R. Schneider)

Let $K \subset \mathbb{R}^n$ be a convex body with interior points, and $0 < \rho < r(K)$. Then $K \sim \rho\mathbb{B}^n$ is homothetic to K if, and only if, K is homothetic to a tangential body of $\mathbb{B}^n$.

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Theorem: tangential bodies lying in $\mathcal{R}_p$

A tangential body $K \subset \mathbb{R}^n$ lies in the class $\mathcal{R}_p$ if and only if, K is a $(n - p - 1)$-tangential body.

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Let $K \subset \mathbb{R}^n$ be a convex body with interior points, and $0 < \rho < r(K)$. Then $K \sim \rho\mathbb{B}^n$ is homothetic to K if, and only if, K is homothetic to a tangential body of $\mathbb{B}^n$.

Theorem: tangential bodies lying in $\mathcal{R}_p$

A tangential body $K \subset \mathbb{R}^n$ lies in the class $\mathcal{R}_p$ if and only if, K is a $(n-p-1)$-tangential body.

In particular:

- The only tangential bodies in $\mathcal{R}_{n-2}$ are the cap-bodies.
- There are no tangential bodies in $\mathcal{R}_{n-1}$; just the ball.

The original Hadwiger problem

Characterization of $\mathcal{R}_\gamma$:

$$\mathcal{R}_\gamma = \left\{K + \rho\mathbb{B}^3 : K \text{ planar convex body}\right\}.$$

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Using an additional condition...

If $K \in \mathcal{R}_\beta$ and $M(\rho)$ is a linear function when $-r \leq \rho \leq 0$, then K is a cap-body of a set $K_0 + \rho\mathbb{B}^3 \in \mathcal{R}_\gamma$, with $\ker(K) = K_0$.

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Conjecture

The **only sets** in $\mathcal{R}_\beta$ are cap-bodies of outer parallel bodies of planar convex bodies $K_0 \subset \mathbb{R}^2$ with kernel K_0.

The volume of inner parallel bodies

Steiner formulae: quermassintegrals of outer parallel bodies

For $K \subset \mathbb{R}^n$, $\rho \geq 0$ and $i = 0, \dots, n$ it holds

$$W_k(K + \rho\mathbb{B}^n) = \sum_{i=0}^{n-k} \binom{n-k}{i} W_{k+i}(K)\rho^i.$$

Steiner formulae for inner parallel bodies? (Matheron)

Let $K \subset \mathbb{R}^n$. Then

$$W_k(K \sim \rho\mathbb{B}^n) = \sum_{i=0}^{n-k} \binom{n-k}{i} W_{k+i}(K)(-\rho)^i$$

for $0 < \rho < r(K)$ and $k = 0, \dots, n$ if and only if $K = L + \lambda\mathbb{B}^n$, $\lambda \geq \rho$.

The Matheron conjecture

Conjecture (Matheron)

Let $K \subset \mathbb{R}^n$. Then for $0 < \rho < r(K)$

$$V(K \sim \rho\mathbb{B}^n) \geq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$

The equality holds if and only if $K = L + \lambda\mathbb{B}^n$, $\lambda \geq \rho$.

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Result

Let $K \subset \mathbb{R}^n$, n odd, be a convex body lying in $\mathcal{R}_{n-2}$. Then

$$V(K \sim \rho\mathbb{B}^n) \leq \sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i.$$

The equality holds if and only if $K \in \mathcal{R}_{n-1}$.

On a problem by Hadwiger

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Cortona, June 2007