Characterizations of duality

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$\mathcal{K}_0$ set of closed convex sets $K \subset \mathbb{R}^d$ with $0 \in K$

$\mathcal{K}_{(0)}$ set of compact convex sets $K \subset \mathbb{R}^d$ with $0 \in \operatorname{int} K$

$\mathcal{C}$ set of closed convex cones in $\mathbb{R}^d$

For $K \in \mathcal{K}_0$, the dual or polar set is defined by

$$K^* := \{x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

The duality mapping * maps each of the spaces $\mathcal{K}_0$, $\mathcal{K}_{(0)}$, $\mathcal{C}$ into itself.

A question of Vitali Milman: Which properties of the duality mapping are sufficient to characterize it?

First we concentrate on $\mathcal{K}_{(0)}$.

Properties of the duality mapping on $\mathcal{K}_{(0)}$:

(D1) $(K^*)^* = K$.

(D2) $K \subset L$ implies $K^* \supset L^*$.

(D3) $(K \cap L)^* = \text{conv}(K^* \cup L^*)$.

(D4) $[\text{conv}(K \cup L)]^* = K^* \cap L^*$.

(D5) $K \cup L$ convex $\Rightarrow (K \cap L)^* = K^* \cup L^*$, $(K \cup L)^* = K^* \cap L^*$.

(D6) Continuity with respect to the Hausdorff metric.

(D7) If $g \in GL(d)$, then $(gK)^* = g^{-t}K^*$.

Notation:

$$K \vee L := \text{conv}(K \cup L)$$

Theorem 1. *Let* $\psi : \mathcal{K}_{(0)} \to \mathcal{K}_{(0)}$ *be a mapping satisfying*

$$\psi \circ \psi = id, \tag{1}$$

$$\psi(K \cap L) = \psi(K) \vee \psi(L) \tag{2}$$

for all $K, L \in \mathcal{K}_{(0)}$. *Then there exists a selfadjoint linear transformation* $g \in GL(d)$ *such that*

$$\psi(K) = gK^* \qquad \text{for } K \in \mathcal{K}_{(0)}.$$

Remark. If (1) is assumed, then each of the following three assumptions implies the other two:

$$\psi(K \cap L) = \psi(K) \vee \psi(L), \tag{3}$$

$$\psi(K \vee L) = \psi(K) \cap \psi(L), \tag{4}$$

$$K \subset L \;\Rightarrow\; \psi(K) \supset \psi(L). \tag{5}$$

If ψ satisfies (1), (2), define

$$\eta(K) := \psi(K)^* \qquad \text{for } K \in \mathcal{K}_{(0)}.$$

Then

$$\eta(K \cap L) = \eta(K) \cap \eta(L), \qquad \eta(K \vee L) = \eta(K) \vee \eta(L).$$

Thus, η is an endomorphism of the lattice $(\mathcal{K}_{(0)}, \cap, \vee)$.

Theorem 1 follows from:

Theorem 2. *Let $\eta : \mathcal{K}_{(0)} \to \mathcal{K}_{(0)}$ be an endomorphism of the lattice $(\mathcal{K}_{(0)}, \cap, \vee)$. Then either η is constant, or there exists a linear transformation $g \in GL(d)$ such that $\eta(K) = gK$ for all $K \in \mathcal{K}_{(0)}$.*

Remark. Observe that η is not assumed to be injective or surjective.

Peter Gruber (1991, 1992) has determined the endomorphisms of the lattices

- $(\mathcal{K}, \cap, \vee)$

where $\mathcal{K}$ is the system of all compact convex sets in $\mathbb{R}^d$ (including $\emptyset$)

- $(\mathcal{B}, \cap, \vee)$

where $\mathcal{B}$ is the system of all unit balls of norms on $\mathbb{R}^d$.

For closed, convex sets K, L, define $K \vee L := \text{cl conv}\,(K \cup L)$. Note that $C \vee D = \text{cl}\,(C + D)$ for $C, D \in \mathcal{C}$.

Theorem 3. *Let $d \geq 3$. Let $\eta : \mathcal{C} \to \mathcal{C}$ be an endomorphism of the lattice $(\mathcal{C}, \cap, \vee)$. Then either η is constant, or there exists a linear transformation $g \in GL(d)$ such that $\eta(C) = gC$ for all $C \in \mathcal{C}$.*

Corollary. *Let $d \geq 3$. Let $\psi : \mathcal{C} \to \mathcal{C}$ be a mapping satisfying*

$$\psi \circ \psi = id,$$

$$C \subset D \;\Rightarrow\; \psi(C) \supset \psi(D) \qquad \text{for } C, D \in \mathcal{C}.$$

Then there exists a selfadjoint linear transformation $g \in GL(d)$ such that $\psi(C) = gC^*$ for all $C \in \mathcal{C}$.

Remark. The assumption $d \geq 3$ in Theorem 3 cannot be deleted.

Remarks on the proof of

Theorem 2. *Let* $\eta : \mathcal{K}_{(0)} \to \mathcal{K}_{(0)}$ *be an endomorphism of the lattice* $(\mathcal{K}_{(0)}, \cap, \vee)$*. Then either* η *is constant, or there exists a linear transformation* $g \in GL(d)$ *such that* $\eta(K) = gK$ *for all* $K \in \mathcal{K}_{(0)}$.

Gruber's proof of his second result makes heavy use of the central symmetry and is restricted to that case.

The proof of Gruber's first result can be taken as a model, but requires major new parts.

First we define a new function φ on $\mathcal{K}_0^c$, the system of compact convex sets containing 0.

Definition. For $K \in \mathcal{K}_0^c$, choose a sequence $(K_i)_{i\in\mathbb{N}}$ in $\mathcal{K}_{(0)}$ with $K \in \operatorname{int} K_i$ for all i and with $K_i \downarrow K$. Put

$$\varphi(K) := \bigcap_{i\in\mathbb{N}} \eta(K_i).$$

Then φ is an endomorphism of the lattice $(\mathcal{K}_0^c, \cap, \vee)$.

We have to show:

- Either there exists a convex body $B \in \mathcal{K}_0^c$ with

$$\varphi(K) = B \qquad \text{for all } K \in \mathcal{K}_0^c,$$

- or there exists a linear transformation $g \in GL(d)$ with

$$\varphi(K) = gK \qquad \text{for all } K \in \mathcal{K}_0^c.$$

For $x \in \mathbb{R}^d$, define $\bar{x} := \text{conv}\{0, x\}$.

We follow Gruber (replacing $\emptyset$ by $\bar{0}$ and x by $\bar{x}$) in considering the following 5 cases:

Case 1: $\varphi(\bar{x}) = \varphi(\bar{0})$ for all $x \in \mathbb{R}^d$.

Putting $\varphi(\bar{0}) =: B$, it is easy to see that $\varphi(K) = B$ for all $K \in \mathcal{K}_0^c$.

The aim of the remaining cases is to show the following:

- The φ-image of any segment $\bar{x}$ is a segment $\bar{y}$. This defines a map $g : x \mapsto y$.
- The map g is injective, it maps collinear points to collinear points and 0 to 0. Therefore, $g \in GL(d)$.
- $\varphi(K) = gK$ for $K \in \mathcal{K}_0^c$, therefore $\eta(K) = gK$ for $K \in \mathcal{K}_{(0)}^c$.

Case 2: $\varphi(\bar{x}) = \varphi(\bar{0})$ for some $x \neq 0$, but not for all $x \in \mathbb{R}^d$.

This is impossible.

Proof (as a little example):

Let $A := \{z \in \mathbb{R}^d : \varphi(\bar{z}) = \varphi(\bar{0})\} \quad \Rightarrow \quad A$ is convex

Let $x \in A \setminus \{0\}$, $y := \frac{1}{2}x \quad \Rightarrow \quad y \in A$

Since A is convex and $\neq \mathbb{R}^d$, we can choose $u, v, w \in \mathbb{R}^d \setminus A$ with

$$\bar{u} \cap \bar{v} = \bar{0}, \qquad w \in (x \vee u) \cap (y \vee v).$$

It follows that $\varphi(\bar{x}) = \varphi(\bar{0}) \subset \varphi(\bar{u})$, $\varphi(\bar{y}) \subset \varphi(\bar{v})$, and

$$\varphi(\bar{0}) \subset \varphi(\bar{w}) \subset [\varphi(\bar{x}) \vee \varphi(\bar{u})] \cap [\varphi(\bar{y}) \vee \varphi(\bar{v})] = \varphi(\bar{u}) \cap \varphi(\bar{v}) = \varphi(\bar{0})$$

$\Rightarrow \quad \varphi(\bar{w}) = \varphi(\bar{0})$, a contradiction

The proof in the first two (easy) cases is analogous to Gruber's.

Case 3: $\varphi(\bar{x}) \neq \varphi(\bar{0})$ for all $x \in \mathbb{R}^d \setminus \{0\}$, $\varphi(\bar{0}) \neq \bar{0}$.

Case 4: $\varphi(\bar{x}) \neq \varphi(\bar{0})$ for all $x \in \mathbb{R}^d \setminus \{0\}$, $\varphi(\bar{0}) = \bar{0}$; there exists a point $p \in \mathbb{R}^d$ with $\dim \varphi(\bar{p}) \geq 2$.

Both are impossible.

For these two cases we gave new proofs.

Case 5: $\varphi(\bar{0}) = \bar{0}$, and $\dim \varphi(\bar{x}) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

With a new proof, we show: Each segment $\bar{x}$ is mapped under φ to a segment $\bar{y}$, this defines a map $g : x \mapsto y$. This map g is injective and maps collinear points to collinear points.

As in Gruber's proof, the map g is linear.

Then the rest is easy.

In the proof of Theorem 3 (for cones), we replace segments $\bar{x}$ by rays $\{\lambda x : \lambda \geq 0\}$ and in the end apply the second fundamental theorem of projective geometry.