Modified Shephard's problem on projections of convex bodies.

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The Busemann-Petty problem

Given two convex origin-symmetric bodies K and L in $\mathbf{R}^n$ such that

$$\text{vol}_{n-1}(K \cap H) \leq \text{vol}_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbf{R}^n$, does it follow that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

Answer:

YES if $n \leq 4$

NO if $n \geq 5$

Larman, Rogers, Ball, Giannopoulos, Bourgain, Gardner, Papadimitrakis, Zhang, Koldobsky, Schlumprecht

Question: What information about central sections guarantees affirmative answer in all dimensions?

Define

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^\perp), \qquad \xi \in S^{n-1},$$

where $\xi^\perp$ is the central hyperplane in $\mathbf{R}^n$ orthogonal to ξ.

Extend S_K from the sphere to $\mathbf{R}^n$ as a homogeneous function of degree -1.

Fractional powers of the Laplace operator in $\mathbf{R}^n$ can be defined as

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} (|x|^\alpha \hat{f}(x))^\wedge$$

Theorem. (A.Koldobsky, V.Y., M.Yaskina)

Let $\alpha \geq n-4$, K and L be origin-symmetric infinitely smooth convex bodies in $\mathbf{R}^n$, $n \geq 4$, so that for every $\xi \in S^{n-1}$

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi), \qquad \forall \xi \in S^{n-1}$$

Then

$$\text{vol}_n(K) \leq \text{vol}_n(L).$$

On the other hand, if $\alpha < n-4$ this is no longer true.

Idea of proof:

Formula [K]:

$$S_K(\xi) = \frac{1}{\pi(n-1)}(\|x\|_K^{-n+1})^\wedge(\xi),$$

so the condition of Theorem can be written as

$$(|x|_2^\alpha \|x\|_K^{-n+1})^\wedge \le (|x|_2^\alpha \|x\|_L^{-n+1})^\wedge.$$

Suppose $(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge \ge 0$, then

$$(|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(|x|_2^\alpha \|x\|_K^{-n+1})^\wedge$$
$$\le (|x|_2^{-\alpha}\|x\|_K^{-1})^\wedge(|x|_2^\alpha \|x\|_L^{-n+1})^\wedge.$$

Therefore, integrating over S^{n-1} and using Parseval's formula

$$\int_{S^{n-1}} \|x\|_K^{-n}\ dx \le \int_{S^{n-1}} \|x\|_K^{-1}\|x\|_L^{-n+1}\ dx$$

By Hölder's inequality,

$$n \ \mathrm{vol}_n(K) \leq n(\mathrm{vol}_n(K))^{1/n}(\mathrm{vol}_n(L))^{(n-1)/n}$$

So the main question is whether $|x|_2^{-\alpha}\|x\|_K^{-1}$ is a positive definite distribution.

In [KYY] it was shown that $|x|_2^{-\alpha}\|x\|_K^{-1}$ is a positive definite distribution if $\alpha \geq n-4$, and not necessarily positive definite if $\alpha < n-4$.

Shephard's problem

Given two convex origin-symmetric bodies K and L in $\mathbf{R}^n$ such that

$$\mathrm{vol}_{n-1}(K|H) \leq \mathrm{vol}_{n-1}(L|H)$$

for every hyperplane H in $\mathbf{R}^n$, does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

Answer:

YES if $n = 2$

NO if $n \geq 3$

Petty, Schneider

Define the projection function

$$P_K(\theta) = \mathrm{vol}_{n-1}(K|\theta^{\perp}), \qquad \theta \in S^{n-1}$$

and extend it to $\mathbf{R}^n$ as a homogeneous function of degree 1.

Question: Does the condition

$$(-\Delta)^{\alpha/2} P_K(\theta) \le (-\Delta)^{\alpha/2} P_L(\theta), \qquad \forall \theta \in S^{n-1}$$

imply the inequality for the volumes of K and L?

Conjecture (A.Koldobsky): should be similar to sections with $\alpha = n-2$ being the borderline case.

Reasons to believe:

i) $\|x\|_K^{-n+3}$ is a positive definite distribution [K]

ii) $\|x\|_K^{-1}|x|_2^{-n+4}$ is also positive definite [KYY] (In general $\|x\|_K^{-n+p+3} \cdot |x|_2^{-p}$ is positive definite for a certain range of p)

Seems plausible that $\|x\|_K|x|_2^{-n+2}$ should also be positive definite.

However this is **wrong**. We show that the borderline case is actually $\alpha = n$.

Koldobsky, Ryabogin, Zvavitch:

$$P_K(\theta) = \mathrm{vol}_{n-1}(K|\theta^\perp) = -\frac{1}{\pi}\widehat{f_K}(\theta), \quad \forall\theta \in S^{n-1},$$

where f_K is the curvature function of K extended to $\mathbf{R}^n$ as a homogeneous function of degree $-n-1$.

Therefore,

$$(-\Delta)^{\alpha/2} P_K(\theta) = -\frac{1}{\pi}(|x|_2^\alpha f_K(x))^\wedge(\theta)$$

Positive connection

Let K, $L \subset \mathbf{R}^n$, be infinitely smooth origin-symmetric convex bodies with positive curvature and $1 < \alpha < n+1$.

Suppose $(|x|_2^{-\alpha} \|x\|_{L^*})^\wedge \geq 0$.

Then

$$(-\Delta)^{\alpha/2} P_K(\theta) \geq (-\Delta)^{\alpha/2} P_L(\theta), \quad \forall \theta \in S^{n-1}$$

implies

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

Proof. From the hypothesis,

$$(|x|_2^{\alpha} f_K(x))^\wedge \leq (|x|_2^{\alpha} f_L(x))^\wedge$$

$$(|x|_2^{-\alpha}\|x\|_{L^*})^\wedge(|x|_2^{\alpha}f_K(x))^\wedge$$
$$\leq (|x|_2^{-\alpha}\|x\|_{L^*})^\wedge(|x|_2^{\alpha}f_L(x))^\wedge$$

Integrating over S^{n-1} and using Parseval's formula

$$\int_{S^{n-1}} \|\theta\|_{L^*} f_K(\theta)d\theta \leq \int_{S^{n-1}} \|\theta\|_{L^*} f_L(\theta)d\theta$$

$$V_1(K,L) \leq \text{vol}_n(L)$$

By Minkowski's first inequality,

$$\text{vol}_n(L)^{\frac{1}{n}}\text{vol}_n(K)^{\frac{n-1}{n}} \leq V_1(K,L) \leq \text{vol}_n(L).$$

So the main question is whether $|x|_2^{-\alpha}\|x\|_K$ is a positive definite distribution for all convex symmetric bodies.

Easy to show: If $n \le \alpha < n+1$, then $(|x|_2^{-\alpha}\|x\|_K)^\wedge \ge 0$ for all bodies K in $\mathbf{R}^n$.

Therefore the answer to the modified Shephard problem is YES for $n \le \alpha < n+1$.

Lemma.

Let $n-2 \le \alpha < n$, $\alpha \ne 1$. Then there exists an origin-symmetric convex body L in $\mathbf{R}^n$, $n \ge 3$, such that $|x|_2^{-\alpha}\|x\|_L$ is not a positive definite distribution.

Proof

1) $n-2 < \alpha < n$. For a large $N > 0$ let L be an ellipsoid with the norm:

$$\|x\|_L = (x_1^2 + \cdots + x_{n-1}^2 + Nx_n^2)^{1/2}.$$

Define a star body $K \subset \mathbf{R}^n$ by the formula:

$$\rho_K(\theta) = \rho_L^{\frac{1}{1-\alpha}}(\theta), \qquad \theta \in S^{n-1}.$$

Then

$$|x|_2^{-\alpha}\|x\|_L = \|x\|_K^{-\alpha+1}, \qquad \forall x \in \mathbf{R}^n \setminus \{0\}.$$

By [GKS] we get

$$(\|x\|_K^{-\alpha+1})^\wedge(\xi) = \frac{\pi(\alpha-1)}{\Gamma(\alpha-n)\cos\frac{\pi(n-\alpha)}{2}}\times$$

$$\times \int_0^\infty t^{-n+\alpha-1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt,$$

One can show

$$(\|x\|_K^{-\alpha+1})^\wedge(\xi) < 0$$

if $N > 0$ is large enough.

2) $\alpha = n-2$. In this case [GKS] gives

$$(\|x\|_K^{-\alpha+1})^\wedge(\xi) = \pi(1-\alpha)A''_{K,\xi}(0) < 0.$$

Theorem. Let $n-2 \le \alpha < n$. There are convex origin-symmetric bodies $K, L \subset \mathbf{R}^n$, $n \ge 3$ such that $\forall \theta \in S^{n-1}$

$$(-\Delta)^{\alpha/2} P_L(\theta) \le (-\Delta)^{\alpha/2} P_K(\theta),$$

but

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$

Proof (for $\alpha \ne 1$).

As shown, there exists a body K, such that $(|x|_2^{-\alpha}\|x\|_{K^*})^\wedge(\xi) < 0$ for some ξ.

Let $\Omega = \{\theta \in S^{n-1} : (|x|_2^{-\alpha}\|x\|_{K^*})^\wedge(\theta) < 0\}$.
Let $v \in C^\infty(S^{n-1})$, $v \geq 0$, even function, supp$v \subset \Omega$.

$$\left(|y|_2^{1-\alpha} v(y/|y|_2)\right)^\wedge(x) = |x|_2^{-n-1+\alpha} g(x/|x|_2)$$

for some $g \in C^\infty(S^{n-1})$.

Choose an $\varepsilon > 0$ small enough and define

$$f_L(x) = f_K(x) + \varepsilon |x|_2^{-n-1} g(x/|x|_2) > 0.$$

L exists by Minkowski's existence theorem.

Multiply both sides by $|x|_2^\alpha$ and apply the FT

$$-\pi(-\Delta)^{\alpha/2} P_L(\theta) = -\pi(-\Delta)^{\alpha/2} P_K(\theta) + (2\pi)^n \varepsilon v(\theta)$$
$$\geq -\pi(-\Delta)^{\alpha/2} P_K(\theta).$$

On the other hand,

$$-\pi \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_{K^*})^\wedge(\theta) (-\Delta)^{\alpha/2} P_L(\theta) d\theta =$$

$$= -\pi \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_{K^*})^\wedge(\theta) (-\Delta)^{\alpha/2} P_K(\theta) d\theta +$$

$$+ (2\pi)^n \varepsilon \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_{K^*})^\wedge(\theta) v(\theta) d\theta <$$

$$-\pi \int_{S^{n-1}} (|x|_2^{-\alpha} \|x\|_{K^*})^\wedge(\theta) (-\Delta)^{\alpha/2} P_K(\theta) d\theta,$$

since v is supported in the set, where $(|x|_2^{-\alpha} \|x\|_{K^*})^\wedge < 0$.

Using the argument with Minkowski's first inequality,

$$\mathrm{vol}_n(L) < \mathrm{vol}_n(K).$$