Shadow Boundaries and the Fourier Transform.

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Convex centrally symmetric bodies are uniquely determined by:

- Volumes of projections (Alexandrov's theorem)

- Volumes of central sections (Minkowski's Theorem)

Convex bodies (not necessary centrally symmetric):

- K.J.Falconer, R.J.Gardner: volumes of all hyperplane sections passing through any two fixed points in the interior of the body

- A.Koldobsky, C.Shane: generalization of the previous to derivatives of section functions

- R.Schneider: mean width and Steiner point of projections

- K.Böröczky, R.Schneider: volumes and centroids of hyperplane sections through 0

Let f be an infinitely smooth function on the sphere S^{n-1}, denote

$$f_p(x) = f(x/|x|)|x|^{-n+p}$$

its homogeneous extension to $\mathbf{R^n}$ of degree $-n+p$.

Also define

$$F_\xi(t) = (1-t^2)^{(n-3)/2} \int_{S^{n-1}\cap\xi^\perp} f(t\,\xi+\sqrt{1-t^2}\,\zeta)\,d\zeta.$$

Theorem 1.[GKS] Let f be an even function. The Fourier transform of f_p is given by the following formulas.

i) If $0 < p < 2k+1$, $p \neq n$, p is not an odd integer, then

$$(f_p)^\wedge(\xi) = \cos\frac{p\pi}{2}\Gamma(p)(\int_{-1}^{1} |t|^{-p} \; (F_\xi(t) - F_\xi(0) -$$

$$- \cdots - F_\xi^{(2k-2)}(0)\frac{t^{2k-2}}{(2k-2)!})dt +$$

$$+ \sum_{m=1}^{2k-2} F_\xi^{(m)}(0)\frac{2}{m!(1+m-p)})$$

ii) if $p = 2k-1 \neq n$, then

$$(f_{2k-1})^\wedge(\xi) = \pi(-1)^{k+1}F_\xi^{(2k-2)}(0).$$

Theorem 2. Let f be an odd function.

iii) If $0 < p < 2k+2$, $p \neq n$, p is not an even integer, then

$$(f_p)^\wedge(\xi) = i \sin\frac{p\pi}{2}\Gamma(p)(\int_{-1}^{1} |t|^{-p} \operatorname{sgn} t \, (F_\xi(t) - F'_\xi(0)t -$$

$$- \cdots - F_\xi^{(2k-1)}(0)\frac{t^{2k-1}}{(2k-1)!})dt +$$

$$+ \sum_{m=1}^{2k-1} F_\xi^{(m)}(0)\frac{2}{m!(1+m-p)})$$

iv) if $p = 2k \neq n$, then

$$(f_{2k})^\wedge(\xi) = i\pi(-1)^{k+1} F_\xi^{(2k-1)}(0).$$

Proof.

For $0 < p < 1$ the Fourier transform of (odd) f_p is a homogeneous function of degree $-p$ whose values on the sphere are given by the formula

$$(f_p)^\wedge(\xi) = i \sin \frac{p\pi}{2} \, \Gamma(p) \int_{S^{n-1}} f(\theta) |(\xi,\theta)|^{-p} \, \mathrm{sgn}\,(\xi,\theta) \, d\theta.$$

In general, if $p < 2k+2$,

$$(f_p)^{\wedge}(\xi) = i \sin \frac{p\pi}{2} \Gamma(p) \times \Big(\int_{-1}^{1} |t|^{-p} \operatorname{sgn} t \, (F_\xi(t) - F'_\xi(0)t -$$

$$- \cdots - F_\xi^{(2k-1)}(0) \frac{t^{2k-1}}{(2k-1)!}) dt +$$

$$+ \sum_{m=1}^{2k-1} F_\xi^{(2k-1)}(0) \frac{2}{m!(1+m-p)} \Big)$$

In particular, taking the limit as $p \to 2k$ we get

$$(f_{2k})^{\wedge}(\xi) = i\pi(-1)^{k+1} F_\xi^{(2k-1)}(0).$$

Shadow boundaries

The **support function** of a convex body K in $\mathbf{R^n}$ is defined by

$$h_K(x) = \max_{\xi \in K}(x, \xi), \quad x \in \mathbf{R^n}.$$

Let K be a convex body in $\mathbf{R^n}$. The **shadow boundary** of K under illumination parallel to $\xi \in S^{n-1}$ is defined as the set of all boundary points of K at which there are support lines of K parallel to ξ.

Let K be strictly convex. For $\theta \in \xi^\perp$ let $x(\theta)$ denote the unique point of intersection of K and the support plane perpendicular to θ.

The **average height of the shadow boundary** of K in the direction of ξ is

$$H_K(\xi) = \int_{S^{n-1}\cap\xi^\perp} (x(\theta), \xi) d\theta.$$

Ewald, Larman and Rogers: the average height of the shadow boundary of an arbitrary convex body is defined for almost all $\xi \in S^{n-1}$.

The **average width of the shadow boundary** of K in the direction of ξ is

$$W_K(\xi) = \int_{S^{n-1}\cap\xi^\perp} h(\theta) d\theta.$$

Theorem. Let K be a strictly convex body. K is uniquely determined the average height and average width of all shadow boundaries.

Proof. In order to recover the odd part of h_K we will use part (iv) of the previous Theorem.

$$(h_2^-)^\wedge(\xi) = i\pi F'_\xi(0)$$

$$= \frac{d}{dt}\left(\int_{S^{n-1}\cap\xi^\perp} h_K^-(t\ \xi + \sqrt{1-t^2}\ \zeta)\ d\zeta\right)_{t=0}$$

$$= \frac{d}{dt}\left(\int_{S^{n-1}\cap\xi^\perp} h_K(t\ \xi + \sqrt{1-t^2}\ \zeta)\ d\zeta\right)_{t=0}$$

$$= \int_{S^{n-1}\cap\xi^\perp} (\nabla h_K(\zeta), \xi) d\zeta$$

$$= H_K(\xi)$$

Stability

Let $0 < p < n$. Let $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ be the operator defined by

$$I_p(f) = (f_p)^\wedge$$

Schur's Lemma:

$$I_p(H_m) = \lambda_m(n,p) H_m,$$

where H_m is a spherical harmonic of degree m.

Lemma

$$|\lambda_m(n,p)| = \frac{2^p \pi^{n/2} \Gamma((m+p)/2)}{\Gamma((m+n-p)/2)}$$

The supremum norm (or the Hausdorff distance) is defined as

$$\delta_\infty(K, L) = \max_{\theta \in S^{n-1}} |h_K(\theta) - h_L(\theta)|.$$

However it will be more convenient to use the L_2-distance

$$\delta_2(K, L) = \|h_k - h_L\|_2 = \left(\int_{S^{n-1}} (h_K(\theta) - h_L(\theta))^2 \, d\theta \right)^{1/2}$$

Vitale's theorem:

$$c_1(n)\delta_2(K, L) \le \delta_\infty(K, L)$$
$$\le c_2(n) D^{2(n-1)/(n+1)} \delta_2(K, L)^{2/(n+1)},$$

where $D = \operatorname{diam}(K \cup L)$ and c_1, c_2 are constants depending on n only.

Theorem. Let K and L be convex bodies in $\mathbf{R^n}$, contained in a ball of radius R, with infinitely smooth support functions. Let $0 < p < n$. If for some $\epsilon \geq 0$

$$\|I_p(h_K) - I_p(h_L)\|_2 \leq \epsilon,$$

then

$$\delta_\infty(K,L) \leq \begin{cases} C(n,p,R)\epsilon^{\frac{4}{(n-2p+2)(n+1)}}, & \text{if } n > 2p, \\ C(n,p,R)\epsilon^{\frac{2}{(n+1)}}, & \text{if } n \leq 2p. \end{cases}$$

Here $C(n,p,R)$ is a constant that depends only on n, p, R.

Proof.

Let $f = h_K - h_L$ and denote its associated series by

$$\sum_{m=0}^{\infty} Q_m.$$

By Vitale's theorem it is enough to estimate the L_2-norm of f instead of the sup-norm.

Assume first that $n > 2p$.

$$\delta_2(K,L)^2 = \|f\|_2^2 = \sum_{m=0}^{\infty} \|Q_m\|_2^2 =$$

$$= \sum_{m=0}^{\infty} \left(|\lambda_m(n,p)|^{\frac{4}{n-2p+2}} \|Q_m\|_2^{\frac{4}{n-2p+2}} \right) \times$$

$$\times \left(|\lambda_m(n,p)|^{-\frac{4}{n-2p+2}} \|Q_m\|_2^{\frac{2n-4p}{n-2p+2}} \right) \leq$$

$$\leq \left(\sum_{m=0}^{\infty} |\lambda_m(n,p)|^2 \|Q_m\|_2^2 \right)^{\frac{2}{n-2p+2}} \times$$

$$\times \left(\sum_{m=0}^{\infty} |\lambda_m(n,p)|^{-\frac{4}{n-2p}} \|Q_m\|_2^2 \right)^{\frac{n-2p}{n-2p+2}}$$

Parseval's equality:

$$\sum_{m=0}^{\infty} |\lambda_m(n,p)|^2 \|Q_m\|_2^2 = \|I_p f\|_2^2$$

Stirling's formula:

$$|\lambda_m(n,p)|^{-\frac{4}{n-2p}} \approx C(n,p) m^2,$$

as m tends to infinity.

Therefore

$$\|f\|_2^2 \le C(n,p) \left(\|I_p f\|_2^2\right)^{\frac{2}{n-2p+2}} \times$$

$$\times \left(\|Q_0\|_2^2 + \sum_{m=1}^{\infty} m(m+n-2)\|Q_m\|_2^2\right)^{\frac{n-2p}{n-2p+2}}$$

$$\le C(n,p) \epsilon^{\frac{4}{n-2p+2}} \left(\epsilon^2 + \|\nabla_o h_K - \nabla_o h_L\|_2^2\right)^{\frac{n-2p}{n-2p+2}}$$

$$\le C(n,p) \epsilon^{\frac{4}{n-2p+2}} \left(\epsilon^2 + R^2\right)^{\frac{n-2p}{n-2p+2}}$$

If $n \leq 2p$, then $|\lambda_m(n,p)|$ does not approach zero as m tends to infinity.

Therefore $\exists\ C(n,p)$ such that

$$C(n,p)|\lambda_m(n,p)|^2 \geq 1$$

for all m.

$$\|f\|_2^2 = \sum_{m=0}^{\infty} \|Q_m\|_2^2 \leq C(n,p) \sum_{m=0}^{\infty} |\lambda_m(n,p)|^2 \|Q_m\|_2^2 =$$

$$= C(n,p)\|I_p f\|_2^2 \leq C(n,p)\epsilon^2$$

Corollary. Let K and L be convex bodies in $\mathbf{R^n}$ which are contained in a ball of radius R. If their average heights and average widths of shadow boundaries of these two bodies are close in the L_2-norm, then K and L are close with respect to the Hausdorff distance.