Analytic and discrete aspects of the covariogram problem

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Outline

- Introduction to the covariogram problems.
- Detection of central symmetry.
- Reconstruction of lattice-convex sets.
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- We are given an (unknown) object $A$ located in space.
- The diffraction information of $A$ is available.
- How do we reconstruct $A$?
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A general analytic formulation

Let $f$ be a distribution on $\mathbb{R}^d$ (with compact support).

How do we reconstruct $f$ from $|\hat{f}|$?

This is not possible in general, since the phase information can be prescribed ‘arbitrarily’.

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Let $K \subseteq \mathbb{R}^d$ be nonempty and compact with $K = \text{cl}(\text{int}(K))$. Then the function $x \mapsto g_K(x)$ on $\mathbb{R}^d$ defined by

$$g_K(x) = \text{vol}(K \cap (K + x))$$

is said to be the covariogram of $K$.

The function $g_K$ provides the same data as $|\hat{1}_K|$.

Thus, reconstruction of $K$ from $g_K$ is a special case of the phase retrieval problem.

The reconstruction is not unique, since $g_K(x)$ does not change
- with respect to translations of $K$ and
- with respect to reflections of $K$ in a point.

These are the trivial ambiguities.

In general, there are other reasons of non-uniqueness.

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The function $g_A$ provides the same data as $|\hat{\delta}_A|$, where $\delta_A := \sum_{a \in A} \delta_a$.

Again, $g_A$ does not change with respect to translations and point reflections of $A$.

There are other reasons for non-uniqueness.

E.g., considers finite sets $S, T \subseteq \mathbb{R}^d$ such that the sum of $S$ and $T$ is direct. Then the sum of $S$ and $-T$ is also direct and...

the sets $S \oplus T, S \oplus (-T)$ have the same covariogram.

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Results on uniqueness

- Within centrally symmetric objects the reconstruction is unique up to translations (no additional assumptions are required).
- Within planar convex bodies $K \subseteq \mathbb{R}^2$ the reconstruction of $K$ from $g_K$ is unique, up to translations and reflections (A. & Bianchi, 2009).
- Within three-dimensional convex polytopes the reconstruction from the covariogram is unique, up to translations and reflections (Bianchi, 2009).
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Detecting central symmetry

- Can we detect from the diffraction data that the underlying object is centrally symmetric?
  - In certain cases, yes.
  - E.g., if $K, H$ are convex bodies in $\mathbb{R}^d$, $K$ is centrally symmetric and $g_K = g_H$. Then $H$ is a translate of $K$. (Consequence of the Brunn-Minkowski inequality).
- Other cases?
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- Other cases?
Theorem 1 (A. 2009)

Let \( A, B \subseteq \mathbb{R}^d \) be finite, let \( A \) be centrally symmetric and \( g_A = g_B \). Then \( B \) is a translate of \( A \).

Proof idea:
- The case \( d = 1 \) is settled by induction.
- The case of general \( d \) is reduced to the case \( d = 1 \) by inductive argument...
- using some folklore results due to Renyi, Heppes et al.
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Corollary 2

Let $K = A + [0, 1]^d$ and $H = B + [0, 1]^d$ where $A, B \subseteq \mathbb{Z}^d$ are finite. Let $K$ be centrally symmetric and $g_K = g_H$. Then $H$ is a translate of $K$.

- Proof idea (borrowed from Gardner, Gronchi and Zong):
  - $1_K = \delta_A * 1_{-[0,1]^d}$.
  - Fourier transforms of distributions with compact support are analytic functions.
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Detecting central symmetry in further cases

- Assume $K \subseteq \mathbb{R}^d$ is nonempty, compact and $K = \text{cl}(\text{int}(K))$.
- Can the central symmetry of $K$ be detected from $g_K$?
Detecting central symmetry in further cases

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A finite subset $K$ of $\mathbb{Z}^d$ is said to be *lattice-convex* if $K$ is the intersection of $\mathbb{Z}^d$ with a convex set.

Problem: reconstruction of $K$ from $g_K$ in the class of lattice-convex sets.

The problem was posed by Daurat, Gérard, Nivat (2005) and Gardner, Gronchi, Zong (2005).
Covariogram problem for lattice convex sets

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Reconstruction is not unique

- One cannot hope for a unique reconstruction, up to translations and reflections. Examples were given by Daurat, Gérard, Nivat (2005) and Gardner, Gronchi, Zong (2005).

- Covariograms are the same.
Reconstruction is not unique

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- This is the reason!
An infinite family of counterexamples
Direct sums are rarely lattice-convex

Theorem 3 (A. & Langfeld, 2011)

Let $k, \ell$ be integers with $k > \ell \geq 0$. We define

- $T := (\{0, \ldots, k\} \times \{0\}) \cup (\{0, \ldots, \ell\} \times \{1\})$ (a set of lattice width one),
- $w_1 := (-k - 1, 1), w_2 := (\ell + 1, 1)$,
- the lattice $\mathbb{L} := \mathbb{Z}w_1 + \mathbb{Z}w_2$.

Let $S$ be a set with $o \in S \subseteq \mathbb{Z}^2$. Then the following conditions are equivalent:

(i) The sum of $S$ and $T$ is direct and lattice-convex.

(ii) $S$ is lattice-convex with respect to $\mathbb{L}$ and $\text{conv} S$ is a polygon in $\mathbb{R}^2$ such that

- every edge of $\text{conv} S$ is parallel to $w_1$ or $w_2$ (in the case $k > \ell + 1$),
- every edge of $\text{conv} S$ is parallel to $w_1, w_2, \text{ or } w_1 + w_2$ (in the case $k = \ell + 1$).
Direct lattice-convex summands of lattice-convex sets

- The situation that a lattice-convex set has a direct lattice-convex summand is very uncommon (work in progress).
Notation for the discrete uniqueness result

Let $K$ be a finite lattice-convex set in $\mathbb{R}^2$ such that $\text{conv} K$ is two-dimensional.

The support set of $K$ in direction $u \in \mathbb{R}^d$ is defined by

$$F(K, u) := \{ x \in K : \langle x, u \rangle = h(K, u) \}.$$

The set of outer edge normals:

$$U(K) := \{ u \in \mathbb{Z}^2 \setminus \{0\} : u \text{ is an outer normal to an edge of } \text{conv} K \text{ and } \gcd(u)=1 \}.$$

To measure the number of lattice points on the edges and the difference of parallel edges of $K$ we introduce

$$m'(K) := \min \{ \#F(K, u) : u \in U(K) \} ,$$
$$m''(K) := \min \{ \#F(K, u) - \#F(K, -u) + 1 : u \in \mathbb{Z}^2 \setminus \{0\} \wedge \#F(K, u) > \#F(K, -u) > 1 \}$$
$$m(K) := \min \{ m'(K), m''(K) \}.$$
Notation for the discrete uniqueness result

- Let $K$ be a finite lattice-convex set in $\mathbb{R}^2$ such that $\text{conv } K$ is two-dimensional.
- The support set of $K$ in direction $u \in \mathbb{R}^d$ is defined by
  \[ F(K, u) := \{ x \in K : \langle x, u \rangle = h(K, u) \} . \]
- The set of outer edge normals:
  \[ U(K) := \{ u \in \mathbb{Z}^2 \setminus \{0\} : u \text{ is an outer normal to an edge of } \text{conv} K \text{ and } \gcd(u)=1 \}. \]
- To measure the number of lattice points on the edges and the difference of parallel edges of $K$ we introduce
  \[ m'(K) := \min \{ \#F(K, u) : u \in U(K) \} , \]
  \[ m''(K) := \min \{ \#F(K, u) - \#F(K, -u) + 1 : \]
  \[ u \in \mathbb{Z}^2 \setminus \{0\} \land \#F(K, u) > \#F(K, -u) > 1 \} \]
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Further notation

For a finite set $U$ of vectors in $\mathbb{R}^2$ linearly spanning $\mathbb{R}^2$ let

$$D(U) := \{| \det(u_1, u_2) | : u_1, u_2 \in U \} \setminus \{0\}$$

We call

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Theorem 4
Let $K, L \subseteq \mathbb{Z}^2$ be bounded and lattice-convex. Then

I. $m'(K), m''(K), m(K), U(K) \cup U(-K)$ and $\delta(K)$ are determined by $g_K$.

II. If

$$m(K) \geq \delta(K)^2 + \delta(K) + 1$$

and

$$g_K = g_L,$$

then $K$ and $L$ coincide up to translations and reflections.
Outlook

- How to detect the central symmetry of sets?
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