Brunn-Minkowski inequalities for two functionals involving the $p$-Laplace operator

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Abstract

In the family of $n$-dimensional convex bodies, we prove a Brunn-Minkowski type inequality for the first eigenvalue of the $p$-Laplace operator, or Poincaré constant, and for a functional extending the notion of torsional rigidity. In the latter case we also characterize equality conditions.

Keywords and phrases: Brunn-Minkowski inequality, $p$-Laplace operator, torsional rigidity, first eigenvalue.

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1 Introduction

This paper represents a continuation of our previous work, contained in [9], [8], [19] and [7], on Brunn-Minkowski type inequalities for functionals of the Calculus of Variations.

We start by recalling the Brunn-Minkowski inequality in its classical formulation. Let $K_0$ and $K_1$ be compact convex sets in $\mathbb{R}^n$ with non-empty interior, i.e. convex bodies, and fix $t \in [0, 1]$; then consider the convex linear combination of these sets:

$$K_t = (1 - t)K_0 + tK_1 = \{(1 - t)x + ty \mid x \in K_0, y \in K_1\},$$

which is still a convex body.

The Brunn-Minkowski inequality claims that

$$V(K_t)^{1/n} \geq (1 - t)V(K_0)^{1/n} + tV(K_1)^{1/n}, \quad (1)$$

where $V$ denotes the $n$-dimensional volume (i.e. the Lebesgue measure). Moreover, equality holds in (1) if and only if $K_0$ and $K_1$ are homothetic, i.e. they are equal up to translation and dilatation.

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As bibliographical references about this inequality, we suggest the book by Schneider [20] (cfr Chapter 6) and the survey paper by Gardner [11]. The Brunn-Minkowski inequality has a fundamental role in the theory of convex bodies, moreover it is strongly connected to other inequalities like the isoperimetric inequality and the Sobolev inequality. Notice that the validity of (1) goes far beyond the family of convex bodies: namely it can be extended to the class of measurable sets (see [11] for more details).

Let $K^n$ denote the class of convex bodies in $\mathbb{R}^n$; $K^n$ is endowed with a scalar multiplication for positive numbers:

$$s K = \{s x \mid x \in K\}, \quad K \in K^n, \quad s > 0,$$

and with the Minkowski addition:

$$K_0 + K_1 = \{x + y \mid x \in K_0, \ y \in K_1\}, \quad K_0, K_1 \in K^n.$$

The Brunn-Minkowski inequality is equivalent to the fact that the $n$-dimensional volume raised to the power $1/n$ is concave in $K^n$; note that $V$ is positively homogeneous and its order of homogeneity is precisely $n$:

$$V(s K) = s^n V(K), \quad \forall s > 0.$$

These considerations lead us to the following:

**Definition 1.1** Let $F : K^n \to \mathbb{R}_+$ be a functional, invariant under rigid motions of $\mathbb{R}^n$. If $F$ is positively homogeneous of some order $\alpha \neq 0$ and $F^{1/\alpha}$ is concave in $K^n$, then we say that $F$ satisfies a Brunn-Minkowski type inequality.

In convex geometry there are many examples of functionals satisfying a Brunn-Minkowski inequality: the $(n-1)$-dimensional measure of the boundary, the quermassintegrals, etc. (see [20] and [11]). On the other hand, inequalities of Brunn-Minkowski type have been proved for functionals coming from a quite different area: the Calculus of Variations.

The first example in this sense is due to Brascamp and Lieb who proved that the first eigenvalue of the Laplace operator satisfies a Brunn-Minkowski inequality (cfr [5]). Subsequently, Borell proved the same result for the Newton capacity, the logarithmic capacity (in dimension $n = 2$) and the torsional rigidity (cfr [2], [3] and [4] respectively). In [6] Caffarelli, Jerison and Lieb established equality conditions for the Newton capacity. Let us notice that for the first eigenvalue of the Laplacian and the torsional rigidity, the Brunn-Minkowski inequality extends to the class of connected domains (with sufficiently smooth boundary) while so far this has not been proved for the capacity.

These results have been generalized, improved and developed in various directions: in [9] it is proved that the $p$-capacity, $p \in (1, n)$, satisfies a Brunn-Minkowski inequality (including equality conditions); the same result is proved in [8] for a $n$-dimensional version of the logarithmic capacity and in [19] for the eigenvalue of the Monge-Ampère operator. Moreover in [7] equality conditions are established for the first eigenvalue of the Laplacian (only in the case of convex bodies) and the torsional rigidity. Notice that in all known cases, equality conditions are the same as in the Brunn-Minkowski inequality for the volume, i.e. equality holds if and only if the involved sets are (convex and) homothetic.
In this paper we make a further progress, proving a Brunn-Minkowski inequality for the first eigenvalue of the $p$-Laplace operator, or Poincaré constant, $p > 1$, and a for functional which extends the notion of torsional rigidity, that we will call $p$-torsional rigidity.

Let $K$ be a convex body and assume that its boundary is of class $C^2$, let $\Omega$ be the interior of $K$ and fix $p > 1$. The Poincaré constant $\lambda(K)$ of $K$ is defined as follows

$$\lambda(K) = \inf \left\{ \frac{\int_{\Omega} |\nabla v|^p dx}{\int_{\Omega} |v|^p dx} : v \in W^{1,p}_0(\Omega), \int_{\Omega} |v|^p dx > 0 \right\}.$$  

(2)

Sakaguchi [18] proved that the infimum is attained for functions belonging to $W^{1,p}_0(\Omega)$, which are weak solutions of

$$\begin{cases} \Delta_p u = -\lambda(K)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where

$$\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$$

is the $p$-Laplace operator. For this reason $\lambda$ is also termed the first eigenvalue of the $p$-Laplace operator; in particular, for $p = 2$ we have the first eigenvalue of the ordinary Laplace operator.

An immediate consequence of its definition is that $\lambda$ is positively homogeneous of order $-p$.

We prove the following

**Theorem 1.1** Let $K_0$ and $K_1$ be convex bodies in $\mathbb{R}^n$ with boundary of class $C^2$ and let $p > 1$. For $t \in [0, 1]$, let $K_t = (1-t)K_0 + tK_1$. Then, the following inequality holds

$$\lambda(K_t)^{-\frac{1}{p}} \geq (1-t)\lambda(K_0)^{-\frac{1}{p}} + t\lambda(K_1)^{-\frac{1}{p}}.$$  

(4)

The proof of this result follows the idea used in [7] for the case $p = 2$. Our method allows to obtain a partial result regarding equality conditions: if there is equality in (4) and at least one of the following conditions holds:

1. $n = 2$,
2. $K_0$ and $K_1$ have positive Gauss curvature at each point of their boundary,

then $K_0$ and $K_1$ are homothetic.

Now we describe the result for the $p$-torsional rigidity. Let $K$ be a convex body in $\mathbb{R}^n$ and let $\Omega$ represents its interior. For $p > 1$, we define the $p$-torsional rigidity of $K$, $\tau(K)$, through the following formula:

$$\frac{1}{\tau(K)} = \inf \left\{ \frac{\int_{\Omega} |\nabla w(x)|^p dx}{\left(\int_{\Omega} |w(x)| dx\right)^p} : w \in W^{1,p}_0(\Omega), \int_{\Omega} |w(x)| dx > 0 \right\}.$$  

(5)
For $n = p = 2$ we obtain the usual notion of torsional rigidity, see for instance the book by Pólya and Szegö [16].

The $p$-torsional rigidity is homogeneous of degree $p + n(p - 1)$.

We have the following:

**Theorem 1.2** Let $K_0$ and $K_1$ be convex bodies in $\mathbb{R}^n$, and $p > 1$; for $t \in [0, 1]$, let $K_t = (1 - t)K_0 + tK_1$. Then the following inequality holds

$$\tau(K_t)^{\frac{1}{p + n(p - 1)}} \geq (1 - t)\tau(K_0)^{\frac{1}{p + n(p - 1)}} + t\tau(K_1)^{\frac{1}{p + n(p - 1)}}.$$  \hspace{1cm} (6)

Equality occurs if and only if $K_1$ is homothetic to $K_0$.

Note that in this case we do not need any additional assumption on the regularity of $\partial K$.

This result holds also under different assumptions on the geometry of the involved sets; see the remark at the end of §4.

As in the case $p = 2$ (see [7]), the proof of Theorems 1.2 is based on a technique employed to prove quasi-concavity (more precisely, power-concavity) of solutions to elliptic equations, due to Korevaar and Kennington (cfr [14] and [13]). In particular, we extend the Korevaar concavity maximum principle to the case of three functions, instead of one. One technical difficulty is the degeneracy of the operator $\Delta_p$, for which we use the approximation argument presented by Sakaguchi in [18].

## 2 Preliminaries

Throughout the paper, $K$ (possibly with subscripts) denotes a convex body in $\mathbb{R}^n$, that is a convex, compact set with non-empty interior, and $\Omega$ (possibly with subscripts) represents the interior of $K$. We say that a convex body is of class $C^{2+}$ if its boundary $\partial K$ is of class $C^2$ and the Gauss curvature at every point of $\partial K$ is strictly positive.

Let $u : K \to \mathbb{R}$ be a twice differentiable function; we denote by $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$ the gradient of $u$ and by $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{ij}$ its Hessian matrix. We say that $u$ is of class $C^{2-}(\Omega)$, for some open set $\Omega$, if $u \in C^2(\Omega)$ and $D^2u(x) < 0$ for every $x \in \Omega$.

Let us consider two open convex sets $\Omega_0$ and $\Omega_1$ in $\mathbb{R}^n$ and $t \in [0, 1]$. For $i = 0, 1$, let $u_i : \Omega_i \to \mathbb{R}$ be a concave function.

The sup-convolution $\tilde{u}$ of $u_0$ and $u_1$ is defined in $\Omega_t = (1 - t)\Omega_0 + t\Omega_1$ as follows

$$\tilde{u}(z) = \sup \left\{ (1 - t)u_0(x) + tu_1(y) : x \in \Omega_0, y \in \Omega_1, z = (1 - t)x + ty \right\}.$$  

**Remark 2.1** Observe that $-\tilde{u}$ coincides with the infimal convolution of $-u_0$ and $-u_1$, as it is defined in Section 5 of [17]. In particular, as the infimal convolution of convex functions is convex, $\tilde{u}$ turns out to be concave.

We prove two auxiliary results that will be used in the proof of Theorem 1.1.
Lemma 2.1 For \( i = 0, 1 \), let \( \Omega_i \) be an open, bounded, convex set and \( u_i \in C^1(\Omega_i) \) be a strictly concave function such that \( \lim_{x \to \partial \Omega_i} u_i(x) = -\infty \). Then, for \( t \in [0,1] \) (with the notation introduced above), \( \tilde{u} \in C^1(\Omega_t) \) and it is strictly concave; moreover, for every \( z \in \Omega_t \), there exists a unique couple of points \( (x,y) \in \Omega_0 \times \Omega_1 \) such that

\[
\begin{align*}
z &= (1-t)x + ty, \\
\tilde{u}(z) &= (1-t)u_0(x) + tu_1(y), \\
\nabla\tilde{u}(z) &= \nabla u_0(x) = \nabla u_1(y).
\end{align*}
\]

If in addition \( z \in \Omega_t \) is such that \( u_0 \) and \( u_1 \) are twice differentiable at the corresponding point \( x \) and \( y \) respectively, and \( D^2 u_0(x), D^2 u_1(y) < 0 \), then \( \tilde{u} \) is twice differentiable at \( z \) and

\[
D^2 \tilde{u}(z) = \left[ (1-t) \left( D^2 u_0(x) \right)^{-1} + t \left( D^2 u_1(y) \right)^{-1} \right]^{-1}.
\]

Proof. We fix \( z \in \Omega_t \). As, for \( i = 0, 1 \), \( u_i \) goes to \(-\infty\) on the boundary of \( \Omega_i \), the supremum in the definition of \( \tilde{u} \) is in fact a maximum. Now, assume that there exist two distinct couples \((x_a, y_a), (x_b, y_b) \in \Omega_0 \times \Omega_1 \) such that (7) and (8) hold and define a third couple

\[
(x_c, y_c) = \frac{1}{2} [(x_a, y_a) + (x_b, y_b)] \in \Omega_0 \times \Omega_1.
\]

From the definition of \( \tilde{u} \) and by the strict concavity of \( u_0 \) and \( u_1 \) it follows

\[
\tilde{u}(z) \geq (1-t)u_0(x_c) + tu_1(y_c) > (1-t)\frac{1}{2}[u_0(x_a) + u_0(x_b)] + t\frac{1}{2}[u_1(y_a) + u_1(y_b)] = \tilde{u}(z)
\]
i.e. a contradiction. The function \( \tilde{u} \) goes to \(-\infty\) on the boundary of \( \Omega_t \); indeed, let \( z_k, k \in \mathbb{N} \), be a sequence of points in \( \Omega_t \) tending to \( \partial \Omega_t \) and let \( x_k \) and \( y_k \) be determined correspondingly in \( \Omega_0 \) and \( \Omega_1 \) as above. Up to subsequences, we may assume that \( z_k \to z \in K_t, x_k \to x \in K_0 \) and \( y_k \to y \in K_1 \). In particular, \( z \in \partial K_t \) so that, by (7), \( x \in \partial K_0 \) and \( y \in \partial K_1 \) and then (8) implies that \( \tilde{u}(z_k) \) tends to \(-\infty \) as \( k \to +\infty \). A further consequence is that

\[
(\partial(-\tilde{u}))(\Omega_t) = \mathbb{R}^n,
\]

(here \( \partial \) denotes the usual sub-differential of convex functions); similarly \( (\nabla u_i)(\Omega_i) = \mathbb{R}^n \), for \( i = 0, 1 \). From Theorem 16.4 in [17] (see also Remark 2.1 in this section) we have that

\[
(-\tilde{u})^* = (1-t)(-u_0)^* + t(-u_1)^*,
\]

where the symbol \( ^* \) means the usual conjugation of convex functions (cfr Section 12 in [17]). Following the notation of [17], we have that \(-u_0 \) and \(-u_1 \) are essentially smooth and strictly convex (by the assumptions of the lemma) so that by [17, Theorem 26.3] the same properties also hold for their conjugate functions. Now, by (12) and [17, Theorem 26.3], \( -\tilde{u} \) is strictly convex and essentially smooth and from [17, Corollary 25.5.1], \( \tilde{u} \in C^1(\Omega_t) \). Let \( z \in \Omega_t \) and \( x \) and \( y \) be such that (7) and (8) hold; a straightforward consequence of the Lagrange multipliers Theorem is that

\[
\nabla u_0(x) = \nabla u_1(y).
\]
Let $X = \nabla u_0(x) = \nabla u_1(y)$; by [17, Theorem 26.5] and (12), we have

$$(\nabla \tilde{u})^{-1}(X) = (1 - t)(\nabla u_0)^{-1}(X) + t(\nabla u_1)^{-1}(X) = (1 - t)x + ty = z.$$  \hfill (13)

Since we have proved that $\tilde{u}$ is strictly concave, its gradient map is invertible and the last formula implies (9).

Now let $z \in \Omega_t$ be such that $u_0$ and $u_1$ are twice differentiable at the corresponding points $x$ and $y$ respectively, and assume that $D^2 u_0(x), D^2 u_1(y) < 0$; let $X = \nabla u_0(x) = \nabla u_1(y) = \nabla \tilde{u}(z)$. Then the mappings $(\nabla u_0)^{-1}$ and $(\nabla u_1)^{-1}$ are differentiable at $X$ and

$$D(\nabla u_0)^{-1}(X) = [D^2 u_0(x)]^{-1} < 0, \quad D(\nabla u_1)^{-1}(X) = [D^2 u_1(y)]^{-1} < 0.$$  \hfill (14)

Now using (13) we have that $(\nabla \tilde{u})^{-1}$ is differentiable at $X$ and the determinant of its Jacobian matrix at $X$ does not vanish, so that $\tilde{u}$ is twice differentiable at $z$. Formula (10) is again a simple consequence of the first equality in (13). \hfill \square

**Lemma 2.2** Let $\Omega_0$ and $\Omega_1$ be two open bounded convex sets and let $u_i : \Omega_i \to \mathbb{R}$, $u_i \in C^1(\Omega_i)$, for $i = 0, 1$, be a strictly concave function. Fix $t \in [0, 1]$, for a point $z \in \Omega_t$ let $(x, y) \in \Omega_0 \times \Omega_1$ be the unique points of points determined by Lemma 2.1. If $\nabla \tilde{u}(z) \neq 0$, and $u_0$ and $u_1$ are twice differentiable at $x$ and $y$ respectively, with $D^2 u_0(x), D^2 u_1(y) < 0$, then

$$\Delta_p \tilde{u}(z) \geq (1 - t)\Delta_p u_0(x) + t\Delta_p u_1(y).$$  \hfill (15)

**Proof.** For a generic (smooth) function $u$ and a point $x$ such that $\nabla u(x) \neq 0$, we may write

$$\Delta_p u(x) = |\nabla u(x)|^{p-2} \left( \Delta u(x) + (p-2) \frac{\partial^2 u(x)}{\partial \nu^2} \right),$$  \hfill (16)

where $\nu = \nabla u(x)/|\nabla u(x)|$ and

$$\frac{\partial^2 u(x)}{\partial \nu^2} = \langle D^2 u(x) \nu, \nu \rangle.$$

We choose a coordinate frame such that

$$e_n = \frac{\nabla u_0(x)}{|\nabla u_0(x)|} = \frac{\nabla u_1(y)}{|\nabla u_1(y)|} = \frac{\nabla \tilde{u}(z)}{|\nabla \tilde{u}(z)|},$$

where $e_1, e_2, \ldots, e_n$ represent the standard basis of $\mathbb{R}^n$. Let $\alpha = |\nabla u_0(x)| = |\nabla u_1(y)| = |\nabla \tilde{u}(z)|$, from (15) we have

$$\Delta_p \tilde{u}(z) = \alpha^{p-2} \left( \text{trace } D^2 \tilde{u}(z) + (p-2) \langle D^2 \tilde{u}(z)e_n, e_n \rangle \right),$$
$$\Delta_p u_0(x) = \alpha^{p-2} \left( \text{trace } D^2 u_0(x) + (p-2) \langle D^2 u_0(x)e_n, e_n \rangle \right),$$
$$\Delta_p u_1(y) = \alpha^{p-2} \left( \text{trace } D^2 u_1(y) + (p-2) \langle D^2 u_1(y)e_n, e_n \rangle \right).$$

So, to establish (14) it suffices to prove:

\begin{enumerate}
  \item[i)] $\text{trace } D^2 \tilde{u}(z) \geq (1 - t)\text{trace } D^2 u_0(x) + t \text{trace } D^2 u_1(y),$
  \item[ii)] $(D^2 \tilde{u}(z)e_n, e_n) \geq (1 - t)(D^2 u_0(x)e_n, e_n) + t(D^2 u_1(y)e_n, e_n).$
\end{enumerate}

These formulas easily follows from (10) and from inequality (37) in [1] (notice that here the inequalities are reversed with respect to those in [1] since we are considering negative definite matrices). \hfill \square

6
Proof of Theorem 1.1

Let $K$ be a convex body in $\mathbb{R}^n$, with boundary of class $C^2$ and denote by $u$ a weak solution of problem (3), the existence of such solution is proved in [18]. We start by some considerations regarding $u$. Firstly, any multiple of $u$ is also a solution of (3); on the other hand, all the solutions are proportional to each other, i.e. the family of the solutions is a one-dimensional vector space. Concerning the regularity of $u$, we have $u \in C^{1,\alpha}(K)$ for some constant $\alpha \in (0, 1)$ (cfr Theorem A.1 in [18]). Moreover, Theorem 1.1 in [18] claims that if $u$ is any positive solution of (3), then the function $v = \log u$ is concave.

Remark 3.1 Consider the set

$$\hat{\Omega} = \{ x \in \Omega \mid \nabla u = 0 \},$$

where $u$ is a non-trivial solution of (3) in $\Omega$; as all the solutions are proportional to each other, this set depends only on $\Omega$. $\hat{\Omega}$ is convex because it is a level set of a log-concave function; more precisely it is the set where $u$ attains its maximum value in $K$.

Remark 3.2 If $u$ is any non-trivial solution of (3), then the $p$-Laplace operator (applied to $u$) is uniformly elliptic on compact subsets of $\Omega \setminus \hat{\Omega}$. By standard regularity results for solution of elliptic equations, we have that $u \in C^2(\Omega \setminus \hat{\Omega})$; hence the function $v = \log u$ solves

$$\begin{cases}
\Delta_p v = -[\lambda + (p-1)|\nabla v|^p] & \text{in } \Omega \setminus \hat{\Omega} \\
v \to -\infty & \text{on } \partial \Omega.
\end{cases} \quad (16)$$

Proof of Theorem 1.1. In order to prove (4), we first establish the following inequality

$$\lambda_t \leq (1 - t)\lambda_0 + t\lambda_1, \quad (17)$$

where $\lambda_i$ is the Poincaré constant of $K_i$, for $i = 0, 1, t$. Clearly, from (17), we have that

$$\lambda_t \leq \max\{\lambda_0, \lambda_1\}. \quad (18)$$

This fact, together with the homogeneity of $\lambda$, proves (4) by the following standard argument: for arbitrary $K_0$ and $K_1$ and $t \in [0, 1]$, let

$$K'_0 = [\lambda(K_0)]^{1/p}K_0, \quad K'_1 = [\lambda(K_1)]^{1/p}K_1, \quad t' = \frac{t[\lambda(K_1)]^{-1/p}}{(1 - t)[\lambda(K_0)]^{-1/p} + t[\lambda(K_1)]^{-1/p}}$$

and apply (18) to $K'_0$, $K'_1$ and $t'$; (4) follows.

In the sequel, we denote by $u_i$ a positive solution of (3) in $\Omega_i$, for $i = 0, 1, t$. We know that the function

$$v_i = \log u_i$$

is concave in $\Omega_i$, it belongs to $C^2(\Omega_i \setminus \hat{\Omega}_i)$ and it is a solution of (16) in $\Omega_i \setminus \hat{\Omega}_i$. Let us denote by $\tilde{v}$ the sup-convolution of $v_0$ and $v_1$, as defined in §2:

$$\tilde{v}(z) = \sup \left\{ (1 - t)v_0(x) + tv_1(y) : x \in \Omega_0, y \in \Omega_1, z = (1 - t)x + ty \right\}.$$
From [17, Corollaries 26.3.2 and 25.5.1], it follows that \( \tilde{v} \in C^1(\Omega_t) \). We construct a sequence of new functions approximating \( v_i, i = 0, 1 \), and which satisfies the assumptions of Lemma 2.2. For \( \varepsilon > 0 \) we define
\[
v_{i,\varepsilon}(x) = v_i(x) - \varepsilon \frac{|x|^2}{2}, \quad x \in \Omega_i.
\]
The function \( v_{i,\varepsilon} \) is strictly concave in \( \Omega_i \) and
\[
v_{i,\varepsilon} \in C^{2}\,_{-}(\Omega_i \setminus \hat{\Omega}_i), \quad \text{for } i = 0, 1.
\] (19)

We consider the sup-convolution \( \tilde{v}_\varepsilon \) of \( v_{0,\varepsilon}, v_{1,\varepsilon} \), that is
\[
\tilde{v}_\varepsilon(z) = \sup \left\{ (1 - t)v_{0,\varepsilon}(x) + tv_{1,\varepsilon}(y) : x \in \Omega_0, y \in \Omega_1, z = (1 - t)x + ty \right\}.
\] (20)

Clearly \( v_{i,\varepsilon} \) converges uniformly to \( v_i \) in \( \Omega_i \), for \( i = 0, 1 \), but we can also see that \( \tilde{v}_\varepsilon \) converges uniformly to \( \tilde{v} \) in \( \Omega_t \). Indeed, from the definition of \( \tilde{v}_\varepsilon \) and \( v_{i,\varepsilon} \), we have that
\[
\tilde{v}_\varepsilon \leq \tilde{v}; \quad \text{or}
\]
\[
\tilde{v}_\varepsilon \geq \tilde{v}(z) - \frac{\varepsilon}{2} \sup \left\{ (1 - t)|x|^2 + t|y|^2 : x \in \Omega_0, y \in \Omega_1, z = (1 - t)x + ty \right\} = \tilde{v}(z) - C\varepsilon,
\]
where \( C > 0 \) is a constant independent of \( \varepsilon \). The last equality together with (21) gives us
\[
|\tilde{v}_\varepsilon(z) - \tilde{v}(z)| \leq C\varepsilon, \quad \forall z \in \Omega_t,
\]
that is the uniform convergence. Actually, we can say more: from [17, Theorem 25.7] we conclude that
\[
\nabla v_{i,\varepsilon} \text{ converges uniformly to } \nabla v_i \text{ on every compact subset of } \Omega_i, \quad i = 0, 1, \quad (22)
\]
\[
\nabla \tilde{v}_\varepsilon \text{ converges uniformly to } \nabla \tilde{v} \text{ on every compact subset of } \Omega_t. \quad (23)
\]

The next step is to express the \( p \)-Laplacian of \( v_{i,\varepsilon} \) in terms of the \( p \)-Laplacian of \( v_i \). For \( i = 0, 1 \) and \( x \in \Omega_i \setminus \hat{\Omega}_i \), we put
\[
\nu_{i,\varepsilon}(x) = \frac{\nabla v_{i,\varepsilon}(x)}{|
abla v_{i,\varepsilon}(x)|},
\]
so, by formula (15), we obtain
\[
\Delta_p v_{i,\varepsilon}(x) = |\nabla v_{i,\varepsilon}(x)|^{p-2} \left( \Delta v_{i,\varepsilon}(x) + (p - 2) \left( D^2 v_{i,\varepsilon}(x) \nu_{i,\varepsilon}, \nu_{i,\varepsilon} \right) \right)
\]
\[
= |\nabla v_i(x) - \varepsilon x|^{p-2} \left( \Delta v_i(x) - n\varepsilon + (p - 2) \left( D^2 v_i(x) \nu_{i,\varepsilon}, \nu_{i,\varepsilon} \right) - \varepsilon(p - 2) \left( I_n \nu_{i,\varepsilon}, \nu_{i,\varepsilon} \right) \right)
\]
\[
= |\nabla v_i(x) - \varepsilon x|^{p-2} \left( \Delta v_i(x) + (p - 2) \left( D^2 v_i(x) \nu_{i,\varepsilon}, \nu_{i,\varepsilon} \right) - \varepsilon(n + p - 2) \right); \quad \text{or}
\]
\[
\Delta_p v_{i, \varepsilon}(x) = \Delta_p v_i(x) + \left( |\nabla v_i(x) - \varepsilon x|^{p-2} - |\nabla v_i(x)|^{p-2} \right) \Delta v_i(x) +
\]
\[
+ (p-2) \left[ |\nabla v_i(x) - \varepsilon x|^{p-2} \langle D^2 v_i(x) \nu_{i,\varepsilon}, \nu_{i,\varepsilon} \rangle - |\nabla v_i(x)|^{p-2} \langle D^2 v_i(x) \nu_i, \nu_i \rangle \right] +
\]
\[
- \varepsilon(n+p-2) |\nabla v_i(x) - \varepsilon x|^{p-2}.
\]  

(24)

Let

\[ \hat{\Omega}_t = \{ x \in \Omega_t : \nabla \hat{v} = 0 \} ; \]

from [17, Theorems 26.5 and 23.8] and (12) we have that
\[
\hat{\Omega}_t = \partial(\hat{v}(0)) = \partial((1-t)(-v_0^*(0)) + t(-v_1^*(0)))
\]
\[
= (1-t)\partial(-v_0^*(0)) + t\partial(-v_1^*(0)) = (1-t)\hat{\Omega}_0 + t\hat{\Omega}_1,
\]
where, in the last equality, we used the fact that \( \hat{\Omega}_t = \{ \nabla u_i = 0 \} = \{ \nabla v_i = 0 \} \). From Lemma 2.1 and (19) it follows
\[
\hat{v}_\varepsilon \in C^2(\Omega_1 \setminus \hat{\Omega}_t).
\]
Moreover, from the same lemma, for a fixed \( z \in \Omega_t \setminus \hat{\Omega}_t \), there exists a unique \((x_\varepsilon, y_\varepsilon) \) in \( \Omega_0 \times \Omega_1 \), depending on \( z \), such that \( z = (1-t)x_\varepsilon + ty_\varepsilon \) and
\[
\nabla \hat{v}_\varepsilon(z) = \nabla v_{0,\varepsilon}(x_\varepsilon) = \nabla v_{1,\varepsilon}(y_\varepsilon).
\]  

(26)

As we know that \( \nabla \hat{v}(z) \neq 0 \), from (22), (23) and the last equality, we can deduce that there exists \( \varepsilon_1 > 0 \) such that, for \( 0 < \varepsilon < \varepsilon_1 \),
\[
(x_\varepsilon, y_\varepsilon) \in \left( (\Omega_0 \setminus \hat{\Omega}_0) \times (\Omega_1 \setminus \hat{\Omega}_1) \right).
\]

Now we apply Lemma 2.2, so
\[
\Delta_p \hat{v}_\varepsilon(z) \geq (1-t)\Delta_p v_{0,\varepsilon}(x_\varepsilon) + t\Delta_p v_{1,\varepsilon}(y_\varepsilon).
\]

With the help of (24) we get
\[
\Delta_p \hat{v}_\varepsilon(z) \geq (1-t)\Delta_p v_{0}(x_\varepsilon) + t\Delta_p v_{1}(y_\varepsilon) + (1-t)\Delta v_{0}(x_\varepsilon) \times
\]
\[
\times \left( |\nabla v_{0}(x_\varepsilon) - \varepsilon x_\varepsilon|^{p-2} - |\nabla v_{0}(x_\varepsilon)|^{p-2} \right) + (1-t)(p-2) \times
\]
\[
\times \left[ |\nabla v_{0}(x_\varepsilon) - \varepsilon x_\varepsilon|^{p-2} \langle D^2 v_{0}(x_\varepsilon) \nu_{0,\varepsilon}, \nu_{0,\varepsilon} \rangle - |\nabla v_{0}(x_\varepsilon)|^{p-2} \langle D^2 v_{0}(x_\varepsilon) \nu_0, \nu_0 \rangle \right] +
\]
\[
+ t\Delta v_{1}(y_\varepsilon) \left( |\nabla v_{1}(y_\varepsilon) - \varepsilon y_\varepsilon|^{p-2} - |\nabla v_{1}(y_\varepsilon)|^{p-2} \right) + t(p-2) \times
\]
\[
\times \left[ |\nabla v_{1}(y_\varepsilon) - \varepsilon y_\varepsilon|^{p-2} \langle D^2 v_{1}(y_\varepsilon) \nu_{1,\varepsilon}, \nu_{1,\varepsilon} \rangle - |\nabla v_{1}(y_\varepsilon)|^{p-2} \langle D^2 v_{1}(y_\varepsilon) \nu_1, \nu_1 \rangle \right] +
\]
\[
- \varepsilon(n+p-2) \left[ (1-t) \left( |\nabla v_{0}(x_\varepsilon) - \varepsilon x_\varepsilon|^{p-2} + t \left( |\nabla v_{1}(y_\varepsilon) - \varepsilon y_\varepsilon|^{p-2} \right) \right) \right].
\]
Set 
\[ F_\varepsilon(z) = (1 - t)\Delta v_0(x_\varepsilon) \left( |\nabla v_0(x_\varepsilon) - \varepsilon x_\varepsilon|^p - |\nabla v_0(x_\varepsilon)|^p \right) + (1 - t)(p - 2) \times \]
\[ \times \left[ |\nabla v_0(x_\varepsilon) - \varepsilon x_\varepsilon|^{p-2} \left( D^2 v_0(x_\varepsilon)\nu_0, \nu_0 \varepsilon \right) - |\nabla v_0(x_\varepsilon)|^{p-2} \left( D^2 v_0(x_\varepsilon)\nu_0, \nu_0 \varepsilon \right) \right] + \]
\[ + t\Delta v_1(y_\varepsilon) \left( |\nabla v_1(y_\varepsilon) - \varepsilon y_\varepsilon|^p - |\nabla v_1(y_\varepsilon)|^p \right) + t(p - 2) \times \]
\[ \times \left[ |\nabla v_1(y_\varepsilon) - \varepsilon y_\varepsilon|^{p-2} \left( D^2 v_1(y_\varepsilon)\nu_\varepsilon, \nu_\varepsilon \varepsilon \right) - |\nabla v_1(y_\varepsilon)|^{p-2} \left( D^2 v_1(y_\varepsilon)\nu_\varepsilon, \nu_\varepsilon \varepsilon \right) \right] + \]
\[ - \varepsilon(n + p - 2) \left[ (1 - t) \left( |\nabla v_0(x_\varepsilon) - \varepsilon x_\varepsilon|^{p-2} + t \left( |\nabla v_1(y_\varepsilon) - \varepsilon y_\varepsilon|^p \right) \right) \right], \]
so that
\[ \Delta_p \tilde{v}_\varepsilon(z) \geq (1 - t)\Delta v_0(x_\varepsilon) + t\Delta v_1(y_\varepsilon) + F_\varepsilon(z), \]
which, by (16), implies
\[ \Delta_p \tilde{v}_\varepsilon(z) \geq - [(1 - t)\lambda_0 + t\lambda_1 + (p - 1)((1 - t)|\nabla v_0(x_\varepsilon)|^p + t|\nabla v_1(y_\varepsilon)|^p)] + F_\varepsilon(z). \]
From the definition of \( v_{0,\varepsilon}, v_{1,\varepsilon} \) and (26) we get
\[ \Delta_p \tilde{v}_\varepsilon(z) \geq - [(1 - t)\lambda_0 + t\lambda_1 + (p - 1)((1 - t)|\nabla \tilde{v}_\varepsilon(z) + \varepsilon x_\varepsilon|^p + t|\nabla \tilde{v}_\varepsilon(z) + \varepsilon y_\varepsilon|^p)] + F_\varepsilon(z) \]
\[ = - [(1 - t)\lambda_0 + t\lambda_1 + (p - 1)|\nabla \tilde{v}_\varepsilon(z)|^p] + \]
\[ + (p - 1) \left[ |\nabla \tilde{v}_\varepsilon(z)|^p - (1 - t)|\nabla \tilde{v}_\varepsilon(z) + \varepsilon x_\varepsilon|^p - t|\nabla \tilde{v}_\varepsilon(z) + \varepsilon y_\varepsilon|^p \right] + F_\varepsilon(z). \]
We set 
\[ \tilde{F}_\varepsilon(z) = F_\varepsilon(z) + (p - 1) \left[ |\nabla \tilde{v}_\varepsilon(z)|^p - (1 - t)|\nabla \tilde{v}_\varepsilon(z) + \varepsilon x_\varepsilon|^p - t|\nabla \tilde{v}_\varepsilon(z) + \varepsilon y_\varepsilon|^p \right], \]
so that we can write
\[ \Delta_p \tilde{v}_\varepsilon(z) \geq - [(1 - t)\lambda_0 + t\lambda_1 + (p - 1)|\nabla \tilde{v}_\varepsilon(z)|^p] + \tilde{F}_\varepsilon(z). \] 
Clearly
\[ \tilde{F}_\varepsilon \to 0 \text{ point-wise in } \Omega_t \backslash \hat{\Omega}_t. \]
Moreover, if \( C \) is a compact subset of \( \Omega_t \backslash \hat{\Omega}_t \), then there exist \( \varepsilon = \varepsilon(C) \) and two compact subsets \( C_0 \) and \( C_1 \) of \( \Omega_0 \backslash \hat{\Omega}_0 \) and \( \Omega_1 \backslash \hat{\Omega}_1 \) respectively, such that, for every \( z \in C \) and for
\[ 0 < \varepsilon < \varepsilon, \]
\[ (x_\varepsilon, y_\varepsilon) \in C_0 \times C_1. \] 
This follows from (26) and the definition of \( \hat{\Omega}_t \). So, as \( v_i \in C^2(C_0) \), for \( i = 0, 1 \), we have that \( |\tilde{F}_\varepsilon| \) is uniformly bounded on \( C \) with respect to \( \varepsilon < \varepsilon \); consequently the sequence \( \tilde{F}_\varepsilon \) is uniformly bounded on compact subsets of \( \Omega_t \backslash \hat{\Omega}_t \).
We can now define the functions
\[ \tilde{u}_\varepsilon(z) = e^{\tilde{v}_\varepsilon(z)}, \quad z \in \Omega_t, \quad \varepsilon > 0. \]
Note that as $\tilde{v}_e \to -\infty$ on the boundary, $\tilde{u}_e \in C(K_t)$ and it vanishes on $\partial \Omega_t$. Furthermore, from Lemma 2.1, for every $z \in \Omega_t$,

$$|\nabla \tilde{u}_e(z)| = |\nabla \tilde{v}_0(z)|e^{\tilde{v}_e(z)} = |\nabla \tilde{v}_0(x_e)|^{1-\frac{t}{t_1}}|\nabla \tilde{v}_1(y_e)|^{1-\frac{t}{t_2}}e^{(1-t)\nu_0(x_e) + t\nu_1(y_e)}$$

$$= \left[|\nabla \tilde{v}_0(x_e)|e^{\nu_0(x_e)}\right]^{1-\frac{t}{t_1}} \left[|\nabla \tilde{v}_1(y_e)|e^{\nu_1(y_e)}\right]^{t}$$

$$= |\nabla (e^{\nu_0(x_e)})|^{1-\frac{t}{t_1}}|\nabla (e^{\nu_1(y_e)})|^{t}.$$

From the last equality, the definition of $v_0, v_1$ and the regularity of $v_0$ and $v_1$ (cfr [18]), we obtain that $|\nabla \tilde{u}_e|$ is bounded in $K_t$, and, in particular, $\tilde{u}_e \in W^{1,p}(\Omega_t)$.

From (28) we get

$$\Delta_p \tilde{u}_e(z) \geq -[(1-t)\lambda_0 + t\lambda_1] \tilde{u}_e(z)^{p-1} + \tilde{F}_e(z)\tilde{u}_e(z)^{p-1}, \quad z \in \Omega_t \setminus \hat{\Omega}_t. \quad (30)$$

Now, let us consider a sequence of compact sets $T_j = \overline{A}_j \setminus B_j$, where $A_j$ and $B_j$ are open convex sets so that $A_1 \subset \subset \Omega_t$, $B_1 \supset \hat{\Omega}_t$ and $B_j \subset A_j$. Moreover, assume that $A_j \to \Omega_t$ and $B_j \to \hat{\Omega}_t$ in the Hausdorff metric, as $j \to +\infty$. For every $j \in \mathbb{N}$, we multiply each side of inequality (30) by $\tilde{u}_e(z)$ and we integrate it over $T_j$:

$$\int_{T_j} \tilde{u}_e(z)\Delta_p \tilde{u}_e(z)dz \geq -[(1-t)\lambda_0 + t\lambda_1] \int_{T_j} \tilde{u}_e(z)^p dz + \int_{T_j} \tilde{F}_e(z)\tilde{u}_e(z)^p dz.$$

Integrating by parts we get

$$\int_{T_j} |\nabla \tilde{u}_e(z)|^p dz - \int_{\partial T_j} \tilde{u}_e(z)\nabla \tilde{u}_e(z)\frac{\partial \tilde{u}_e(z)}{\partial \nu} dz \leq$$

$$\leq [-(1-t)\lambda_0 + t\lambda_1] \int_{T_j} \tilde{u}_e(z)^p dz - \int_{T_j} \tilde{F}_e(z)\tilde{u}_e(z)^p dz. \quad (31)$$

Note that, as $\tilde{v}_e$ converges to $\tilde{v}$ uniformly, then $\tilde{u}_e$ converges to $\tilde{u} = e^{\tilde{v}}$ uniformly in $K_t$. Moreover, from (22) and (23), $\nabla \tilde{u}_e$ converges to $\nabla \tilde{u}$ uniformly on $T_j$. Passing to the limit for $\varepsilon \to 0$ in (31), using the uniform convergence of $\tilde{u}_e$ and $\nabla \tilde{u}_e$, the properties of $\tilde{F}_e$ and the Dominated Convergence Theorem, we find out that

$$\int_{T_j} |\nabla \tilde{u}(z)|^p dz - \int_{\partial T_j} \tilde{u}(z)\nabla \tilde{u}(z)\frac{\partial \tilde{u}(z)}{\partial \nu} dz \leq [-(1-t)\lambda_0 + t\lambda_1] \int_{T_j} \tilde{u}(z)^p dz$$

$$\leq [-(1-t)\lambda_0 + t\lambda_1] \int_{\Omega_t} \tilde{u}(z)^p dz.$$

We may rewrite the last inequality in the following way

$$\int_{A_j \setminus B_j} |\nabla \tilde{u}(z)|^p dz - \int_{\partial A_j \cup \partial B_j} \tilde{u}(z)\nabla \tilde{u}(z)\frac{\partial \tilde{u}(z)}{\partial \nu} dz \leq [-(1-t)\lambda_0 + t\lambda_1] \int_{\Omega_t} \tilde{u}(z)^p dz. \quad (32)$$
As \( \tilde{u} \to 0 \) for \( z \to \partial \Omega_i \) and \( \nabla \tilde{u} \to 0 \) for \( z \to \partial \Omega_i \) (recall that \( \Omega_i = \{ \nabla \tilde{v} = 0 \} = \{ \nabla \tilde{u} = 0 \} \)), passing to the limit for \( j \to +\infty \) we have

\[
\int_{\Omega_i \setminus \hat{\Omega}_i} |\nabla \tilde{u}(z)|^p dz \leq [(1 - t)\lambda_0 + t\lambda_1] \int_{\Omega_i} \tilde{u}(z)^p dz. \tag{33}
\]

But

\[
\int_{\Omega_i \setminus \hat{\Omega}_i} |\nabla \tilde{u}(z)|^p dz = \int_{\Omega_i} |\nabla \tilde{u}(z)|^p dz,
\]

so (33) becomes

\[
\int_{\Omega_i} |\nabla \tilde{u}(z)|^p dz \leq [(1 - t)\lambda_0 + t\lambda_1] \int_{\Omega_i} \tilde{u}(z)^p dz. \tag{34}
\]

Finally, we get

\[
\lambda_i \leq \frac{\int_{\Omega_i} |\nabla \tilde{u}(z)|^p dz}{\int_{\Omega_i} \tilde{u}(z)^p dz} \leq [(1 - t)\lambda_0 + t\lambda_1], \tag{35}
\]

i.e. (17). \( \square \)

**Remark 3.3** Concerning equality conditions in the Brunn-Minkowski inequality for \( \lambda \), we conjecture that they are the same as in the other known cases: if \( K_0 \) and \( K_1 \) are such that equality occurs in (4), then they are homothetic. This claim is true for \( p = 2 \) as showed in [7]; moreover we are able to prove it in some other special cases. We start with two remarks.

(i) If \( K_0 \) and \( K_1 \) are convex bodies in \( \mathbb{R}^n \), with boundary of class \( C^2 \), such that there is equality in (4), we may assume, after a normalization, that \( \lambda_0 = \lambda_1 = \lambda_t = 1 \); indeed, for \( i = 0, 1 \), let

\[
T_i = \lambda_i^{1/p} K_i, \quad \eta = \frac{t\lambda_i^{-1/p}}{(1 - t)\lambda_0^{-1/p} + t\lambda_1^{-1/p}};
\]

it is easy to check that \( T_0, T_1 \) and \( T_\eta = (1 - \eta)T_0 + \eta T_1 \) still render (4) an equality and trivially \( \lambda(T_0) = \lambda(T_1) = \lambda(T_\eta) = 1 \). (ii) If equality holds in (4), then, by (35), the function \( \tilde{u} \) is a minimizer for the quotient in the definition of \( \lambda(K_i) \) and therefore it is a solution of problem (3) in \( \Omega_i \); in turn, \( \tilde{\nu} \) is a solution of (16) in \( \Omega_i \setminus \hat{\Omega}_i \). In view of this fact in the sequel we will write \( u_t \) and \( v_t \) instead of \( \tilde{u} \) and \( \tilde{\nu} \), respectively.

The argument that we use to characterize equality conditions is based on the fact that \( v_i \) is of class \( C^{2,\alpha} \) in a neighbourhoud of \( \partial \Omega_i, i = 0, 1 \). Under the assumption that \( \Omega_i \) is just convex, this is proved for \( n = 2 \) in [15] (where it is also conjectured for \( n > 2 \)). In order to have the same property in higher dimension, we make the stronger assumption that \( K_0 \) and \( K_1 \) are of class \( C^{2,\alpha} \).

**Case I:** \( n = 2 \). We consider \( v_0 \) and \( v_1 \) as in the proof of Theorem 1.1; from the previous considerations and from Remarks 1 and 2 in [15], we obtain that \( v_i \in C^{2,\alpha} (\Omega_i \setminus \hat{\Omega}_i), i = 0, 1, t \). For \( X \in \mathbb{R}^3 \setminus \{0\} \), let \( z \in \Omega_i \setminus \hat{\Omega}_i \) such that

\[
\nabla v_t^{-1}(X) = z.
\]

Here we can repeat the argument contained in the proof of Lemma 2.1 to conclude that (9) and (10) hold. Moreover, we know that the following formulas hold

\[
\Delta_p v_i(z) = -[1 + (p - 1)|\nabla v_i(z)|^p], \quad \Delta_p v_0(x) = -[1 + (p - 1)|\nabla v_0(x)|^p],
\]

\[
\Delta_p v_1(y) = -[1 + (p - 1)|\nabla v_1(y)|^p].
\]
In these conditions, we can apply Lemma 4.2 in [9] and deduce

\[ D^2v_t(z) = D^2v_0(x) = D^2v_1(y), \]

which implies

\[ D^2v_t((\nabla v_t)^{-1}(X)) = D^2v_0((\nabla v_0)^{-1}(X)) = D^2v_1((\nabla v_1)^{-1}(X)) \]

and, passing to the conjugate functions,

\[ D^2(-v_0)^*(X) = D^2(-v_1)^*(X), \text{ for every } X \in \mathbb{R}^2 \backslash \{0\}. \]

Hence, there exists \( \bar{X} \in \mathbb{R}^2 \) such that, for every \( X \in \mathbb{R}^2 \backslash \{0\}, \)

\[ \nabla(-v_0)^*(X) = \nabla(-v_1)^*(X) + \bar{X}. \]

Then,

\[ \Omega_0 \backslash \Omega_0 = \nabla(-v_0)^*(\mathbb{R}^2 \backslash \{0\}) = \nabla(-v_1)^*(\mathbb{R}^2 \backslash \{0\}) + \bar{X} = \Omega_1 \backslash \Omega_1 + \bar{X}, \quad \text{(36)} \]

from which one easily obtains that

\[ \Omega_0 = \Omega_1 + \bar{X}. \]

**Case II:** \( K_0 \) and \( K_1 \) are of class \( C^{2,+} \). From [18], for \( i = 0, 1 \), we know that there exists \( \delta_i > 0 \) such that \( v_i \in C^{2,-}(N_i) \), where \( N_i = \{ x \in \Omega_i : \text{dist} (x, \partial \Omega_i) < \delta_i \} \). Since \( |\nabla v_i| \) tends to \( +\infty \) as \( x \to \partial \Omega_i \), for \( i = 0, 1, t \), there exists \( R > 0 \) such that

\[ \nabla v_i(N_i) \subseteq \{ X \in \mathbb{R}^n : |X| > R \}. \]

Repeating the argument of the previous case, we find

\[ D^2(-v_0)^*(X) = D^2(-v_1)^*(X) \quad \forall X \text{ such that } |X| > R. \]

Therefore, for some \( \bar{X} \in \mathbb{R}^n, \)

\[ \nabla(-v_0)^*(X) = \nabla(-v_1)^*(X) + \bar{X}, \]

for every \( X \) such that \( |X| > R \). Now, if we put \( A_i = \{ x \in \Omega_i : |\nabla v_i(x)| \leq R \} \), we obtain

\[ \Omega_0 \backslash A_0 = \Omega_1 \backslash A_1 + \bar{X} \]

and then

\[ \Omega_0 = \Omega_1 + \bar{X}. \]
4 Proof of Theorem 1.2

Before proving Theorem 1.2 we need some preparatory facts. As we said in the introduction, the $p$-torsional rigidity of $K$, $\tau(K)$, is defined, for $p > 1$, through the following formula:

$$\frac{1}{\tau(K)} = \inf \left\{ \frac{\int_{\Omega} |\nabla w(x)|^p dx}{\left( \int_{\Omega} |w(x)| dx \right)^p} : w \in W^{1,p}_0(\Omega), \int_{\Omega} |w(x)| dx > 0 \right\}. \quad (37)$$

The above problem admits a minimizer. Indeed, consider the functional

$$F(w) = \frac{1}{p} \int_{\Omega} |\nabla w(x)|^p dx - \int_{\Omega} w(x) dx, \quad w \in W^{1,p}_0(\Omega); \quad (38)$$

a standard variational argument ensures that $F$ has a minimizer $u \in W^{1,p}_0(\Omega)$ and it can be immediately seen that $u$ provides a solution to problem (37) also; hence

$$\tau(K) = \left[ \frac{\int_{\Omega} u(x) dx}{\int_{\Omega} |\nabla u(x)|^p dx} \right]^p. \quad (39)$$

In addition, $u$ is a weak solution of

$$\begin{cases}
\Delta_p u = -1 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega. \quad (40)
\end{cases}$$

Applying the Gauss-Green formula we obtain

$$\int_{\Omega} |\nabla u(x)|^p dx = \int_{\Omega} u(x) dx, \quad (41)$$

and using (39) we find the following relation

$$\tau(K) = \left[ \int_{\Omega} u(x) dx \right]^{p-1}. \quad (42)$$

**Remark 4.1** Concerning specific properties of the solution $u$ of (40), let us recall that $u \in C^{1,\alpha}(K)$ for some $\alpha \in (0, 1)$ (cfr [21] and [22]), and it is unique for the strict convexity of the functional $F$. Moreover, Sakaguchi proved in [18] that $u$ is $\frac{p-1}{p}$-concave, i.e.

$$v(x) := u(x)^{\frac{p-1}{p}}$$

is a concave function.

Let $K_0, K_1 \in \mathcal{K}^n$, $t \in [0, 1]$ and consider $K_i = (1 - t)K_0 + tK_1$. For $i = 0, 1, t$, let $u_i : K_i \to \mathbb{R}$ be a bounded function. The concavity function $c$ of $u_0, u_1$ and $u_t$ is defined in $K_0 \times K_1$ by

$$c(x, y) = u_t((1 - t)x + ty) - [(1 - t)u_0(x) + tu_1(y)]. \quad (43)$$

The notion of concavity function, depending only on one function, was originally introduced by Koorevar in [14], where the so-called Korevaar concavity maximum principle is proved,
a tool which permits to establish the quasi-concavity of solutions of elliptic boundary-value problems in many important cases. Subsequently, the same technique was employed by various other authors; we mention in particular Kennington who improved in [13] the results of Korevaar. We now establish a version of concavity maximum principle for three functions.

We set
\[ c = \inf \{ c(x, y) : (x, y) \in K_0 \times K_1 \} . \]

**Theorem 4.1** Let \( K_0, K_1 \in \mathcal{K}^n \), \( t \in [0, 1] \), \( K_t = (1-t)K_0 + tK_1 \). For \( i = 0, 1, t \), let \( u_i \in C^2(\Omega_i) \), where \( \Omega_i \) denotes the interior of \( K_i \) and assume that \( u_i \) is solution of
\[
\sum_{r,s=1}^{n} a_{rs}(\nabla u_i(x))(u_i)_{rs}(x) + b(u_i(x), \nabla u_i(x)) = 0 ,
\]
for \( x \in \Omega_i \), where \( a_{rs}(p) \) is a real symmetric positive semidefinite matrix, for every \( p \in \mathbb{R}^n \) and \( b > 0 \). Assume that, for every \( p \in \mathbb{R}^n \), \( b(\cdot, p) \) is strictly decreasing and harmonic concave (i.e. \( (b(\cdot, p))^{-1} \) is concave). Under these assumptions, if \( c < 0 \), then \( c \) is not attained in \( \Omega_0 \times \Omega_1 \).

The proof of this result is a mere repetition (with the obvious modifications) of the one given by Kennington in the case of one function: see Theorem 3.1 in [13].

The last preliminary result is the Prékopa-Leindler inequality.

**Theorem 4.2 (Prékopa-Leindler Inequality)** Let \( f, g, h \in L^1(\mathbb{R}^n) \) be nonnegative functions and \( t \in (0, 1) \). Assume that
\[
h((1-t)x + ty) \geq f(x)^{1-t}g(y)^t ,
\]
for all \( x, y \in \mathbb{R}^n \). Then
\[
\int_{\mathbb{R}^n} h(x)dx \geq \left( \int_{\mathbb{R}^n} f(x)dx \right)^{1-t} \left( \int_{\mathbb{R}^n} g(x)dx \right)^{t} .
\]
In addition, if equality holds then \( f \) coincides a.e. with a log-concave function and there exist \( C \in \mathbb{R} \), \( a > 0 \) and \( y_0 \in \mathbb{R}^n \) such that
\[
g(y) = C f(ay + y_0) \quad \text{for almost every} \quad y \in \mathbb{R}^n .
\]

For the proof of the Prékopa-Leindler inequality we refer to [11]. The equality condition is contained in Theorem 12 in [10].

**Proof of Theorem 1.2.** We are going to prove the inequality
\[
\tau(K_t) \geq \tau(K_0)^{1-t}\tau(K_1)^t ; \quad (44)
\]
notice that this inequality implies in particular
\[
\tau(K_t) \geq \min\{\tau(K_0), \tau(K_1)\} \quad (45)
\]
15
and the latter implies (6) via the same argument that we have used for the Poincaré constant, at the beginning of the proof of Theorem 1.1. We denote by \( u_t \) the solution of (40) in \( \Omega_i \), for \( i = 0, 1, t \). The crucial part of the proof is to establish the following inequality

\[
 u_t((1-t)x + ty)^{\frac{p-1}{p}} \geq (1-t)u_0(x)^{\frac{p-1}{p}} + tu_1(y)^{\frac{p-1}{p}}, \quad \forall x \in \Omega_0, y \in \Omega_1. \tag{46}
\]

Before we prove it, let us see how this leads to the conclusion. First of all, by the arithmetic-geometric mean inequality it follows

\[
 u_t((1-t)x + ty) \geq u_0(x)^{1-t} u_1(y)^t, \quad \forall x \in \Omega_0, y \in \Omega_1. \tag{47}
\]

Now, extend \( u_t \) as zero in \( \mathbb{R}^n \setminus \Omega_i \), for \( i = 0, 1, t \). Inequality (47) continues to hold; indeed, if either \( x \notin \Omega_0 \) or \( y \notin \Omega_1 \), then the right hand-side vanishes and the left hand-side is nonnegative. Hence we may apply the Prékopa-Leindler inequality and (44) follows.

Firstly, we prove (46) assuming that \( K_0 \) and \( K_1 \) have boundary of class \( C^2 \). Let \( v_i = u_i^{(p-1)/p} \), \( i = 0, 1, t \), and consider the concavity function \( c : K_0 \times K_1 \rightarrow \mathbb{R}, \)

\[
 c(x, y) = v_t((1-t)x + ty) - [(1-t)v_0(x) + tv_1(y)]. \tag{48}
\]

As \( u_i \in C(K_i), \) for \( i = 0, 1, t \), the infimum of \( c \) is attained at some point \((\bar{x}, \bar{y}) \in K_0 \times K_1\). Once we have proved that

\[
 \bar{c} = c(\bar{x}, \bar{y}) = \min_{K_0 \times K_1} c(x, y) \geq 0,
\]

then we have (46). If \( t = 0 \) or \( t = 1 \) then

\[
 c(x, y) \equiv 0, \quad \text{for every } (x, y) \in K_0 \times K_1.
\]

So, from now on, we assume that \( t \in (0, 1) \). There are three cases we have to deal with:

i) \( (\bar{x}, \bar{y}) \in \partial K_0 \times \partial K_1; \)

ii) \( (\bar{x}, \bar{y}) \in (\partial K_0 \times \Omega_1) \cup (\Omega_0 \times \partial K_1); \)

iii) \( (\bar{x}, \bar{y}) \in \Omega_0 \times \Omega_1. \)

Case i) If \( (\bar{x}, \bar{y}) \in \partial K_0 \times \partial K_1, \) from the boundary conditions satisfied by \( v_0 \) and \( v_1 \), it follows

\[
 c(\bar{x}, \bar{y}) = v_t((1-t)\bar{x} + t\bar{y}) \geq 0.
\]

Case ii) This case leads to a contradiction; indeed suppose that \( (\bar{x}, \bar{y}) \in \partial K_0 \times \Omega_1 \). As \( \bar{x} \in \partial K_0 \), \( u_0(\bar{x}) = v_0(\bar{x}) = 0 \), and as \( K_0 \) has boundary of class \( C^2 \), we may apply the Hopf Lemma (see [22], Proposition 3.2.1) and obtain that

\[
 \frac{\partial v_0(\bar{x})}{\partial \nu} = -\infty, \tag{49}
\]

where \( \nu \) is the outer normal unit vector to \( \partial K_0 \) at \( \bar{x} \). Now, we consider

\[
 \varphi(s) = c(\bar{x} + s\nu, \bar{y} + s\nu), \quad \text{for } s \in (-\delta, 0),
\]

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where $\delta > 0$ is sufficiently small. According to our assumption, $\varphi$ attains its global minimum at $s = 0$, on the other hand (49) implies \[
\lim_{s \to 0^-} \varphi'(s) = +\infty,
\] these two facts are in contradiction. The case $(\bar{x}, \bar{y}) \in \Omega_0 \times \partial \Omega_1$ is completely analogous.

**Case iii)** Also in this case we argue by contradiction. Suppose that $\bar{c} = c(\bar{x}, \bar{y}) < 0$.

For $\eta > 0$ and for $i = 0, 1, t$ we set \[
\Omega_{i, \eta} = \{x \in \Omega_i \mid \text{dist}(x, \partial \Omega_i) > \eta\}.
\] As $(\bar{x}, \bar{y}) \in \Omega_0 \times \Omega_1$, there exist $\delta > 0$ such that $(\bar{x}, \bar{y}) \in \Omega_{0, \delta} \times \Omega_{1, \delta}$. Let $\delta' > 0$ be such that \[
(1 - t)\Omega_{0, \delta} + t\Omega_{1, \delta} \subset \Omega_{t, \delta'};
\] in particular $\bar{z} = (1 - t)\bar{x} + t\bar{y} \in \Omega_{t, \delta'}$. In the proof of Theorem 2 in [18] it is showed that, for $i = 0, 1$, we can construct a family of functions $u_{i, \varepsilon}$, for $i = 0, 1, t$,

- $u_{i, \varepsilon} \in C^\infty(\Omega_{i, \delta})$;
- $u_{i, \varepsilon}$ converges uniformly to $u_i$ in the closure $K_{i, \delta}$ of $\Omega_{i, \delta}$;
- $v_{i, \varepsilon} := u_{i, \varepsilon}^{(p-1)/p} \in C^\infty(\Omega_{i, \delta})$ and converges uniformly to $v_i$ in $K_{i, \delta}$;
- $v_{i, \varepsilon}$ are solution of \[
\sum_{r,s=1}^n a_{rs}(\nabla v_{i, \varepsilon})(v_{i, \varepsilon})_{rs} + b(v_{i, \varepsilon}, \nabla v_{i, \varepsilon}) = 0,
\] where $b(v_{i, \varepsilon}, \nabla v_{i, \varepsilon}) > 0$, $b(\cdot, p)$ is strictly decreasing and harmonic concave for every $p \in \mathbb{R}^n$.

Clearly, the same can be done for $\Omega_t$ and we obtain a sequence $u_{t, \varepsilon}$ having properties a) - d) for $\varepsilon < \varepsilon_i(\delta')$ and with $\Omega_{i, \delta}$ replaced by $\Omega_{t, \delta'}$. Let $\varepsilon_{\delta} := \min\{\varepsilon_0(\delta), \varepsilon_1(\delta), \varepsilon_t(\delta')\}$. Now, we introduce the function $c_{\varepsilon} : K_{0, \delta} \times K_{1, \delta} \rightarrow \mathbb{R}$ \[
c_{\varepsilon}(x, y) = u_{t, \varepsilon}((1 - t)x + ty) - [(1 - t)v_{0, \varepsilon}(x) + tv_{1, \varepsilon}(y)].
\] For $0 < \varepsilon < \varepsilon_{\delta}$, $c_{\varepsilon} \in C^\infty(\Omega_{0, \delta} \times \Omega_{1, \delta})$ and \[
c_{\varepsilon}(x, y) \to c(x, y) \text{ uniformly in } K_{0, \delta} \times K_{1, \delta}.
\] From (50) and (51) it follows that $c_{\varepsilon}$ admits a negative global minimum, for $\varepsilon$ sufficiently small; let $(\bar{x}_{\varepsilon}, \bar{y}_{\varepsilon})$ be the point where such minimum is attained and let $\bar{c}_{\varepsilon} = c_{\varepsilon}(\bar{x}_{\varepsilon}, \bar{y}_{\varepsilon})$. We apply Theorem 4.1 and we deduce that $\bar{c}_{\varepsilon}$ can not be attained in $\Omega_{0, \delta} \times \Omega_{1, \delta}$. Consequently \[
(\bar{x}_{\varepsilon}, \bar{y}_{\varepsilon}) \in \partial(K_{0, \delta} \times K_{1, \delta})
\]
for \( \varepsilon \) sufficiently small. So
\[
\min_{K_0,\delta \times K_1,\delta} c_\varepsilon(x, y) = \min_{\partial(K_0,\delta \times K_1,\delta)} c_\varepsilon(x, y) = c_\varepsilon(\bar{x}_\varepsilon, \bar{y}_\varepsilon).
\]
As \( \varepsilon \) tends to \( 0^+ \) we obtain that
\[
\min_{K_0,\delta \times K_1,\delta} c(x, y) = \min_{\partial(K_0,\delta \times K_1,\delta)} c(x, y) = \bar{c} < 0,
\]
and this holds for every \( \delta \) sufficiently small. If we let \( \delta \) tend to \( 0^+ \) we have the following result
\[
\min_{K_0 \times K_1} c(x, y) = \min_{\partial(K_0 \times K_1)} c(x, y) < 0,
\]
which is in contradiction with the previous discussion of cases \( i \) and \( ii \). Inequality (46) is then proved.

Next, we remove the assumption on the regularity of \( \partial K_0 \) and \( \partial K_1 \). For an arbitrary \( K \in \mathcal{K}^n \), let \( u \) be the solution of (40) in \( \Omega \). There exists a sequence \( \{\Omega_j\}_{j \in \mathbb{N}} \), with boundary of class \( C^2 \), such that
\[
\bar{\Omega}_j \subset \Omega_{j+1}, \quad +\infty \bigcup_{j=1}^{+\infty} \Omega_j = \Omega.
\]
For every \( j \in \mathbb{N} \), there exists a unique function \( u_j \in W_0^{1,p}(\Omega_j) \) such that it solves (40) in \( \Omega_j \), or equivalently, it minimizes (38) in \( \Omega_j \). If we extend it in the following way
\[
u_j(x) = 0 \text{ in } \Omega \setminus \Omega_j,
\]
then \( u_j \in W_0^{1,p}(\Omega) \). So, for the minimizing properties of \( u \) in \( \Omega \),
\[
F(\Omega_j) \geq F(\Omega), \forall j \in \mathbb{N}
\]
(see (38) for the definition of \( F \)). From (41), (38) and the above inequality it follows
\[
\int_{\Omega_j} |\nabla u_j|^p dx \leq \int_{\Omega} |\nabla u|^p dx;
\]
this and the Poincaré inequality imply that the sequence \( u_j \) is bounded in \( W_0^{1,p}(\Omega) \). Therefore we can find a subsequence \( u_{j'} \) and a function \( \tilde{u} \in W_0^{1,p}(\Omega) \) satisfying \( u_{j'} \rightharpoonup \tilde{u} \) in \( W_0^{1,p}(\Omega) \) as \( j' \to +\infty \). As the weak limit of weak solutions is still a weak solution, \( \tilde{u} \) is a solution of (40) in \( \Omega \) and as such solution is unique \( \tilde{u} = u \) (in particular, the sequence \( u_j \) converges to \( u \) and not just a subsequence of it). From (55) and the lower semi-continuity of
\[
w \to \int_{\Omega} |\nabla w|^p dx, \quad w \in W_0^{1,p}(\Omega),
\]
it follows that
\[
\lim_{j \to +\infty} \int_{\Omega} |\nabla u_j|^p dx = \int_{\Omega} |\nabla u|^p dx.
\]
Using this fact and the weak convergence in $W_0^{1,p}$ we obtain
\[ \nabla u_j \to \nabla u \text{ in } L_p(\Omega) \]
and consequently
\[ u_j \to u \text{ in } L_p(\Omega). \]

In particular $u_j$ converges almost everywhere in $\Omega$, and, if we set $v_j(x) = u_j(x)^{\frac{p-1}{p}}$, we have that $v_j$ converges almost everywhere in $\Omega$. As $v$ and $v_j$ are concave, we have point-wise convergence in $\Omega$ and uniform convergence on compact subsets of $\Omega$.

Given $K_0$ and $K_1$ in $\mathcal{K}^n$, let $\Omega_{0,j}$ and $\Omega_{1,j}$ be two sequences of open sets approximating the interior of $K_0$ and $K_1$ respectively, constructed as above, and let
\[ \Omega_{t,j} = (1-t)\Omega_{0,j} + t\Omega_{1,j}. \]

With obvious extension of notation, for $i = 0, 1, t$, let $u_{i,j}$ be the solution of problem (40) in $\Omega_{i,j}$, and $v_{i,j} = u_{i,j}^{(p-1)/p}$. For $x \in \Omega_{0,j}$ and $y \in \Omega_{1,j}$, $z = (1-t)x + ty \in \Omega_{t,j}$ and, for the previous part of the proof,
\[ v_{t,j}(z) \geq (1-t)v_{0,j}(x) + tv_{1,j}(y). \]

As $j$ tends to $+\infty$, for $i = 0, 1, t$,
\[ u_{i,j} \to u_i, \]
where $u_i$ is the solution of (40) in $\Omega_i$. Letting $j \to +\infty$ we obtain (46). The proof of (6) is concluded.

Finally, we consider the equality case. Let $K_0, K_1$ and $t$ be such that equality holds; recalling the argument at the beginning of the proof, we have that $u_t, u_0$ and $u_1$, extended as zero in $\mathbb{R}^n \setminus \Omega_t, \mathbb{R}^n \setminus \Omega_0$ and $\mathbb{R}^n \setminus \Omega_1$ respectively, give equality in the Prékopa-Leindler inequality. Hence, by Proposition 4.2 we deduce that
\[ u_1(y) = C u_0(ay + y_0), \]
where $C, a > 0$ and $y_0 \in \mathbb{R}^n$. Since $u_i(x) > 0$ if and only if $x \in \Omega_i, i = 0, 1$, we deduce that $K_0$ and $K_1$ have to be homothetic. □

**Remark 4.2** Theorem 1.2 can be proved also in the class of compact sets with boundary of class $C^2$. This fact is proved in [7] (Theorem 2.9) in the special case $p = 2$ but the argument can be adapted to the general situation $p > 1$.

**Remark 4.3** We present an extension of Theorem 1.2. Let $K \in \mathcal{K}^n$ and let $\Omega$ denote its interior. For $0 \leq \alpha < p - 1$, and $p > 1$, we may consider the following problem
\[
\begin{cases}
\Delta_p u = -u^\alpha, & u > 0 \text{ in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By direct methods in the Calculus of Variations it is possible to prove the existence of a solution $u$ of (58) as minimizer of
\[ \tilde{F}(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \int_{\Omega} v^{1+\alpha} dx. \]
in $W^{1,p}_0(\Omega)$. Such solution is unique; this kind of result has been the subject of several papers, see for instance [12]. Then, one can define the functional

$$\tau_\alpha(K) = \left( \int_\Omega |\nabla u|^p dx \right)^{p-1},$$

(59)

where $u$ is the solution of (58). Following the lines of the proof of Theorem 1.2, it is possible to prove that $\tau_\alpha$ satisfies a Brunn-Minkowski type inequality. A similar result is proved in [7] for $p = 2$.

References


