On inequalities for convex bodies and the geometry of linear normed spaces

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Notations

- $\mathbb{E}^2$ - the Euclidean plane
- V - the area in $\mathbb{E}^2$
- K - convex body in $\mathbb{E}^2$
- B - convex body in $\mathbb{E}^2$ centered at the origin
- T - triangle in $\mathbb{E}^2$

Basic notions from Minkowski geometry

- $\mathcal{M}^2(B)$ is the Minkowski plane with the unit ball B
- Minkowskian ball – homothetical copy of a unit ball B
- Minkowskian radius – the corresponding homothety coefficient

Formulation of the problem

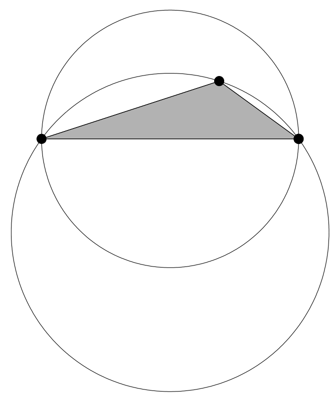
For a triangle T in a Minkowski plane $\mathcal{M}^2(B)$ we denote by $R_B(T)$ the least possible radius of a Minkowskian ball enclosing T. We wish to find the lower and the upper bounds of $R_B(T)$ for the case when B is an arbitrary planar convex body centered at the origin and T with given Minkowskian side lengths a_1, a_2, and a_3.

The minimum of $R_B(T)$

We remark that $R_B(T) \geq \frac{1}{2}\,\mathrm{diam}_B(T) = \frac{1}{2}\max\{a_1, a_2, a_3\}$ with equality if and only if two vertices of T lie in different parallel supporting lines of a minimal enclosing Minkowskian ball of T. Thus, the lower bound of $R_B(T)$ together with the characterization of the equality case are relatively simple to obtain.

Minkowskian circumradius

A Minkowskian ball containing all three vertices of a triangle T in $\mathcal{M}^2(B)$ is called a *Minkowskian circumball* of T. We remark that in most sources the measure $R_B(K)$ is called the Minkowskian circumradius of the convex body K. However, for a triangle T the measure $R_B(T)$ is not always the Minkowskian circumradius of T according to the definition introduced above.



Existence and uniqueness of circumradii

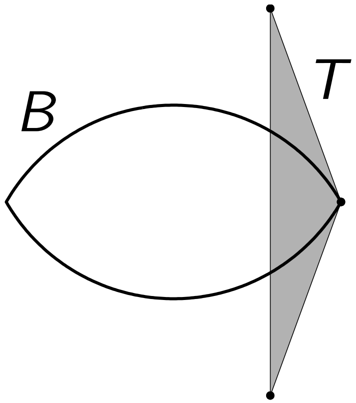


Figure 1

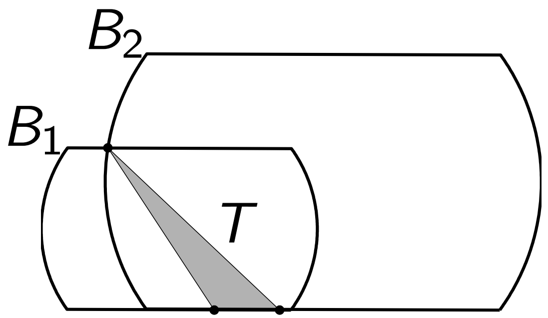


Figure 2

Theorem

Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then the following statements hold.

I. *Every triangle T in $\mathcal{M}^2(B)$ possesses at least one Minkowskian circumradius if and only if B is smooth.*

II. *Every triangle T in $\mathcal{M}^2(B)$ possesses at most one Minkowskian circumradius if and only if the unit Minkowskian ball B is strictly convex.*

Possible values of Minkowskian side lengths

Theorem

Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane, a_1, a_2, a_3 be positive scalars, and $u \in \mathcal{M}^2(B) \setminus \{o\}$ be an arbitrary direction. Then the following conditions are equivalent.

(i) *There exists a triangle $T \subseteq \mathcal{M}^2(B)$ with one side having direction u and Minkowskian length a_1, and two other sides having Minkowskian lengths a_2 and a_3.*

(ii) *The scalars a_1, a_2, a_3 satisfy the the triangle inequalities $2a_i \leq a_1 + a_2 + a_3$, where $i = 1, 2, 3$.*

□

The case $R_B(T) > \mathrm{diam}_B(T)/2$

Theorem

Let T be a triangle in a Minkowski plane $\mathcal{M}^2(B)$ such that $R_B(T) > \frac{1}{2}\,\mathrm{diam}_B(T)$. Then the Minkowskian circumball and the minimal enclosing Minkowskian ball of T are determined uniquely and coincide with each other. Moreover, the center of these (coinciding) Minkowskian balls lies in T. $\square$

Representation of convex hexagons

Lemma

Let b_1, b_2, b_3 be non-zero vectors in $\mathbb{E}^2$. Then the following conditions are equivalent:

(i) *The points b_1, b_2, b_3 are alternating vertices of a convex hexagon centered at the origin (i.e., $b_1, -b_3, b_2, -b_1, b_3, -b_2$ are vertices of a convex hexagon B traversing the boundary of B in that order).*

(ii) *There exist positive scalars a_1, a_2, a_3 satisfying the triangle inequalities $2a_i < a_1 + a_2 + a_3$ and the vector equality $a_1 b_1 + a_2 b_2 + a_3 b_3 = o$.*

The main result

Theorem

Let T be a triangle in a Minkowski plane $\mathcal{M}^2(B)$ and let a_1, a_2, a_3 be Minkowskian side lengths of T. Then

$$R_B(T) \leq \frac{2a_1a_2a_3}{a_1(a_2+a_3-a_1)+a_2(a_1+a_3-a_2)+a_3(a_1+a_2-a_3)}.$$

with equality if and only if B is a hexagon with vertices $\pm b_1, \pm b_2, \pm b_3$, where $b_1, b_2, b_3 \in \mathbb{E}^2$ are such that $a_1b_1 + a_2b_2 + a_3b_3 = 0$, and T is a triangle whose sides are parallel to vectors b_1, b_2, b_3.

A corollary

Corollary

Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane, and T be a triangle in $\mathcal{M}^2(B)$. Then

$$4R_B(T) \le \mathrm{perim}_B(T) - \frac{(\mathrm{perim}_B(T) - 2\,\mathrm{diam}_B(T))^3}{\mathrm{perim}_B(T)^2}.$$

□

Related papers

- Zamfirescu & Maehara (2005) - $4R_B(T) \leq \mathrm{perim}_B(T)$.
- Chakerian (1966) - Range of $R_B(T)$ for the case $a_1 = a_2 = a_3$.
- Papers by Lassak & Fabinska, Eggleston and some other authors - relation between $R_B(T)$ and $\mathrm{diam}_B(T)$.