On characterizations of Euclidean spaces

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Outline

1. Introduction
   - Geometric Background (Banach)-Minkowski Geometry.
   - Well Known Characterization: Parallelogram Equality
   - Well Known Characterization: Symmetry of Orthogonality

2. Area and Arc Length Measure of Angles
   - Defining Measures and Bisectors of Angles
   - Relation Between Area and Arc Length Measure
   - Characterization Generalizing the Symmetry of Orthogonality

3. Further Angular Bisectors
   - Definitions
   - Characterizations
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Minkowski Space finite dimensional real linear normed space (finite dimensional Banach space) $\mathbb{M}^d$ with unit ball $B$
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$\mathbb{M}^d$ is Euclidean iff $B$ is an ellipsoid (ellipse).
Birkhoff's Orthogonality Relation.

\[ |x|B \]

\[ x + Ry \]

\[ x \perp y \]
Parallelogram Equality.

\[ \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \]

for all \( x, y \in \mathbb{M}^d \).
\( \mathbb{M}^d \) is Euclidean if
- dimension \( d \) is at least 3, and
- \( x \nmid y \) always implies \( y \nmid x \) (\( x, y \in \mathbb{M}^d \)).
$\mathbb{M}^d$ is Euclidean if
- dimension $d$ is at least 3, and
- $x \perp y$ always implies $y \perp x$ ($x, y \in \mathbb{M}^d$).

For $d = 2$ this property characterizes Radon planes, whose unit circles are Radon curves.
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Define measure $\mu_l$ of an angle proportional to the length (measured in $\mathbb{M}^d$) of the corresponding arc of the unit circle (normalized to $2\pi$).
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This also defines an angular bisector.
Define measure $\mu_a$ of an angle proportional to the area (with arbitrarily chosen unit) of the corresponding sector of the unit circle (normalized to $2\pi$).
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This also defines an angular bisector.
The two measures $\mu_a$ and $\mu_l$ are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.
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The two measures $\mu_a$ and $\mu_l$ are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.
When Are these Two Measures (Length and Area) Identical?

Theorem

The two measures $\mu_a$ and $\mu_l$ are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.

Especially, this holds for all planes with symmetric orthogonality (Radon planes).
Theorem

A Minkowski space $\mathbb{M}^d$ ($d \geq 3$) is Euclidean iff for each two-dimensional subspace the unit disc is equiframed.
reduce to the case of symmetric orthogonality

uses *reductio ad absurdum*

local difference with adjacent straight segments in the unit circle

extends to planar part in three-dimensional unit ball

this subconfiguration has no end (unbounded cylinder)
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Further properties of angular bisectors in the Euclidean plane can be used to define Angular Bisectors:

set of points equidistant to the sides \((Glogovskij)\)
Further properties of angular bisectors in the Euclidean plane can be used to define Angular Bisectors:

- The set of points equidistant to the sides (Glogovskij).
- The ray dividing each secant in the ratio of the lengths of corresponding segments on the sides. (Busemann)
Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

<table>
<thead>
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and $\mu_1 = \mu_a = \mu_l$ | iff $\mu_1 = \mu_2$
for $\mu_l = \mu_a$: iff $\mathbb{M}^2$ has an equiframed unit circle |
Characterizations Using Equivalent Systems of Angular Bisectors in $\mathbb{M}^d$, $d \geq 3$.

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There are a lot of properties which characterize Euclidean spaces within the family of Minkowski spaces. They can be regarded as proofs that our world is “the best of all possible worlds...” (3-dimensional space)
We have seen four new such characterizations.
For Further Reading

Nico Düvelmeyer.
The new characterization of Radon curves via angular bisectors. 

Nico Düvelmeyer.
Angle measures and bisectors in Minkowski planes. 

Nico Düvelmeyer.
On convex bodies all whose two-dimensional sections are equiframed. 
to be submitted