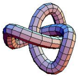
On characterizations of Euclidean spaces

Nico Düvelmeyer
Nico.Duevelmeyer@Mathematik.TU-Chemnitz.de

CHEMNITZ UNIVERSITY
OF TECHNOLOGY

Fakultät für Mathematik
TU Chemnitz
Germany



Educational Workshop on Geometric Inequalities
Firenze (Italy), May 2005, 16th-20th

Outline

CHEMNITZ UNIVERSITY OF TECHNOLOGY

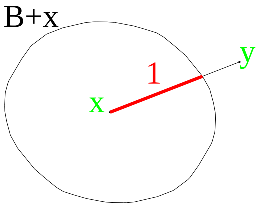
Outline

1. Introduction
   - Geometric Background (Banach)-Minkowski Geometry.
   - Well Known Characterization: Parallelogram Equality
   - Well Known Characterization: Symmetry of Orthogonality
2. Area and Arc Length Measure of Angles
   - Defining Measures and Bisectors of Angles
   - Relation Between Area and Arc Length Measure
   - Characterization Generalizing the Symmetry of Orthogonality
3. Further Angular Bisectors
   - Definitions
   - Characterizations

CHEMNITZ UNIVERSITY OF TECHNOLOGY

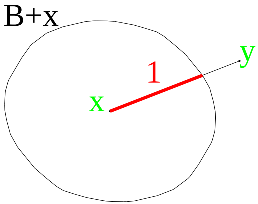
Geometric Background of (Banach)-Minkowski Geometry

Minkowski Space finite dimensional real linear normed space (finite dimensional Banach space) $\mathbb{M}^d$ with unit ball B



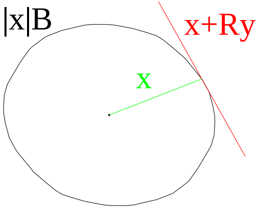
Geometric Background of (Banach)-Minkowski Geometry

Minkowski Space finite dimensional real linear normed space (finite dimensional Banach space) $\mathbb{M}^d$ with unit ball B



$\mathbb{M}^d$ is Euclidean iff B is an ellipsoid (ellipse).

Birkhoffs Orthogonality Relation.



$x \dashv y$

Parallelogram Equality.

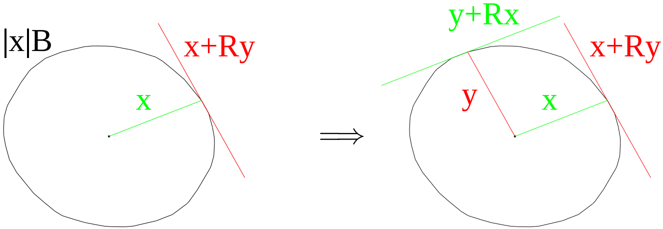
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in \mathbb{M}^d$.

Symmetry of Orthogonality.

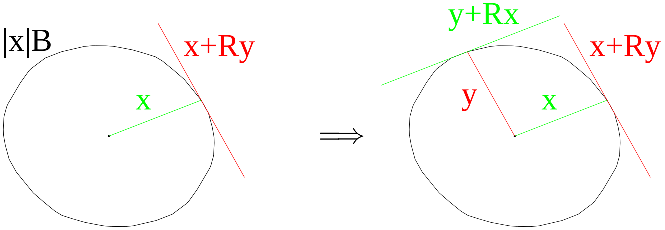
$\mathbb{M}^d$ is Euclidean if

- dimension d is at least 3, and
- $x \dashv y$ always implies $y \dashv x$ ($x, y \in \mathbb{M}^d$).



$\mathbb{M}^d$ is Euclidean if

- dimension d is at least 3, and
- $x \dashv y$ always implies $y \dashv x$ ($x, y \in \mathbb{M}^d$).

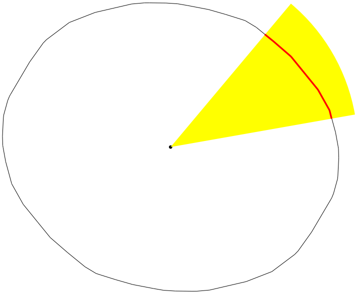


For $d = 2$ this property characterizes Radon planes, whose unit circles are Radon curves.

Outline

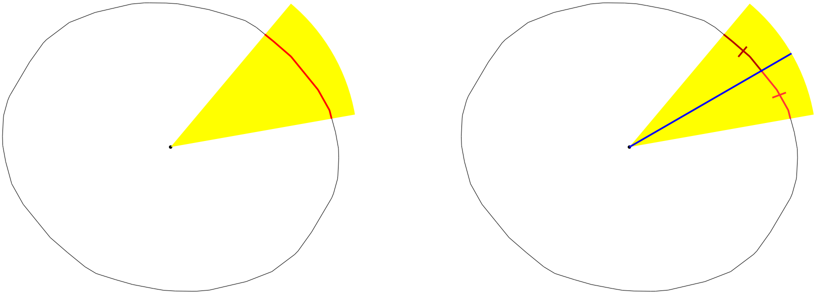
CHEMNITZ UNIVERSITY OF TECHNOLOGY

Define measure μ_I of an angle proportional to the length (measured in $\mathbb{M}^d$) of the corresponding arc of the unit circle (normalized to 2π).



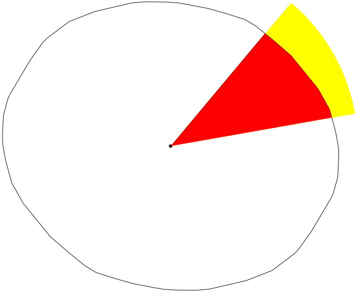
Measuring Angles by Arc Length

Define measure μ_l of an angle proportional to the length (measured in $\mathbb{M}^d$) of the corresponding arc of the unit circle (normalized to 2π).
This also defines an angular bisector.



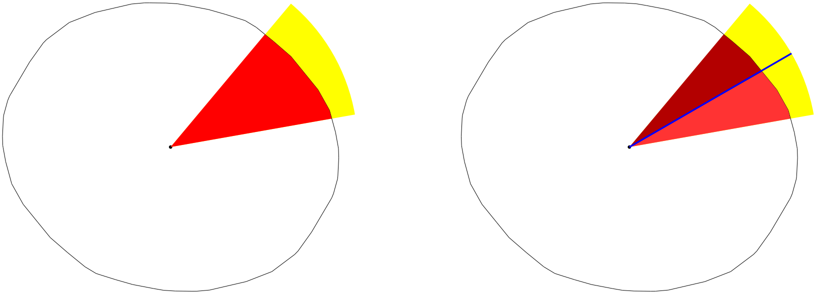
CHEMNITZ UNIVERSITY OF TECHNOLOGY

Define measure μ_a of an angle proportional to the area (with arbitrarily chosen unit) of the corresponding sector of the unit circle (normalized to 2π).



Measuring Angles by Area

Define measure μ_a of an angle proportional to the area (with arbitrarily chosen unit) of the corresponding sector of the unit circle (normalized to 2π).
This also defines an angular bisector.



When Are these Two Measures (Length and Area) Identical?

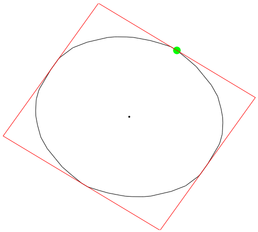
Theorem

The two measures μ_a and μ_l are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.

When Are these Two Measures (Length and Area) Identical?

Theorem

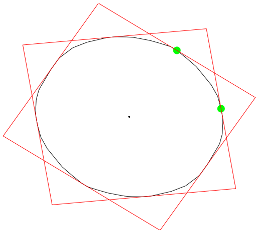
The two measures μ_a and μ_l are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.



When Are these Two Measures (Length and Area) Identical?

Theorem

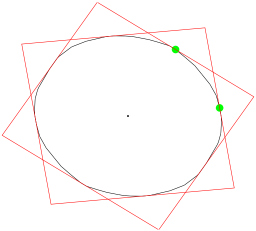
The two measures μ_a and μ_l are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.



When Are these Two Measures (Length and Area) Identical?

Theorem

The two measures μ_a and μ_l are identical for all angles of some Minkowski plane $\mathbb{M}^2$ iff its unit ball is equiframed, i.e., if each point of the unit circle belongs to the boundary of some circumscribed parallelogram of minimal area.



Especially, this holds for all planes with symmetric orthogonality (Radon planes).

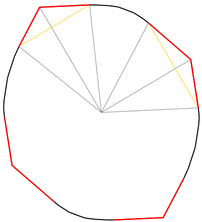
CHEMNITZ UNIVERSITY OF TECHNOLOGY

New Result.

Theorem

A Minkowski space $\mathbb{M}^d$ $(d \geq 3)$ is Euclidean iff for each two-dimensional subspace the unit disc is equiframed.

Sketch of the Proof



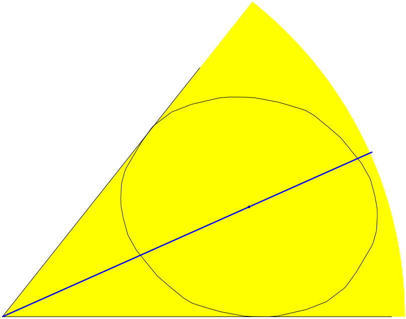
- reduce to the case of symmetric orthogonality
- uses *reductio ad absurdum*
- local difference with adjacent straight segments in the unit circle
- extends to planar part in three-dimensional unit ball
- this subconfiguration has no end (unbounded cylinder)

Outline

1. Introduction
   - Geometric Background (Banach)-Minkowski Geometry.
   - Well Known Characterization: Parallelogram Equality
   - Well Known Characterization: Symmetry of Orthogonality
2. Area and Arc Length Measure of Angles
   - Defining Measures and Bisectors of Angles
   - Relation Between Area and Arc Length Measure
   - Characterization Generalizing the Symmetry of Orthogonality
3. Further Angular Bisectors
   - Definitions
   - Characterizations

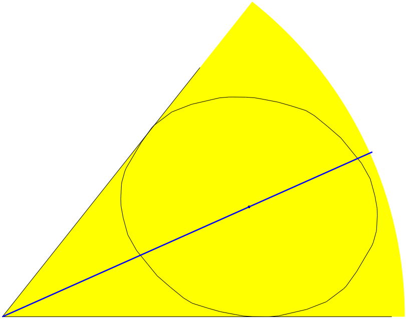
CHEMNITZ UNIVERSITY OF TECHNOLOGY

Further properties of angular bisectors in the Euclidean plane can be used to define Angular Bisectors:

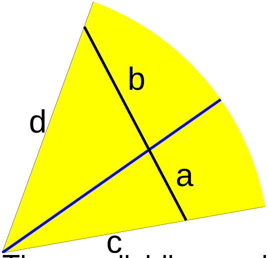


set of points equidistant to the sides (Glogovskij)

Further properties of angular bisectors in the Euclidean plane can be used to define Angular Bisectors:



set of points equidistant to the sides (Glogovskij)



$$\frac{a}{b} = \frac{c}{d}$$

The ray dividing each secant in the ratio of the lengths of corresponding segments on the sides. (Busemann)

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann		
measure μ_1		

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^2$ is Radon	
measure μ_1		

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^2$ is Radon	iff $\mathbb{M}^2$ is Euclidean and $\mu_2 = \mu_a = \mu_l$
measure μ_1		

CHEMNITZ UNIVERSITY OF TECHNOLOGY

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^2$ is Radon	iff $\mathbb{M}^2$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^2$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^2$ is Radon	iff $\mathbb{M}^2$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^2$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	iff $\mu_1 = \mu_2$

CHEMNITZ UNIVERSITY OF TECHNOLOGY

Characterizations Using Equivalent Systems of Angular Bisectors in the Plane

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^2$ is Radon	iff $\mathbb{M}^2$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^2$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	iff $\mu_1 = \mu_2$ for $\mu_I = \mu_a$: iff $\mathbb{M}^2$ has an equiframed unit circle

Characterizations Using Equivalent Systems of Angular Bisectors in $\mathbb{M}^d$, $d \geq 3$.

$\equiv$	Glogovskij	measure μ_2
Busemann		iff $\mathbb{M}^d$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^d$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	

CHEMNITZ UNIVERSITY OF TECHNOLOGY

Characterizations Using Equivalent Systems of Angular Bisectors in $\mathbb{M}^d$, $d \geq 3$.

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^d$ is Euclidean	iff $\mathbb{M}^d$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^d$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	

Characterizations Using Equivalent Systems of Angular Bisectors in $\mathbb{M}^d$, $d \geq 3$.

$\equiv$	Glogovskij	measure μ_2
Busemann	iff $\mathbb{M}^d$ is Euclidean	iff $\mathbb{M}^d$ is Euclidean and $\mu_2 = \mu_a = \mu_I$
measure μ_1	iff $\mathbb{M}^d$ is Euclidean and $\mu_1 = \mu_a = \mu_I$	iff $\mu_1 = \mu_2$ for $\mu_I = \mu_a$: iff $\mathbb{M}^d$ is Euclidean

Summary

There are a lot of properties which characterize Euclidean spaces within the family of Minkowski spaces. They can be regarded as proofs that our world is "the best of all possible worlds..." (3-dimensional space)
We have seen four new such characterizations.

CHEMNITZ UNIVERSITY
OF TECHNOLOGY

For Further Reading

- Nico Düvelmeyer.
  A new characterization of Radon curves via angular bisectors.
  Journal of Geometry, 80(1–2):75 – 81, 2004.
- Nico Düvelmeyer.
  Angle measures and bisectors in Minkowski planes.
  Canadian Mathematical Bulletin, to appear.
- Nico Düvelmeyer.
  On convex bodies all whose two-dimensional sections are equiframed.
  to be submitted

CHEMNITZ UNIVERSITY OF TECHNOLOGY