# Classes of centrally symmetric convex bodies


Paul Goodey

University of Oklahoma



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**Projections of zonotopes are again zonotopes. If all the 3-dimensional projections of a polytope $P$ are zonotopes, then so is $P$.**

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$$h(K,u) = \int_{S^{d-1}} |\langle u, v\rangle| \, \rho(dv)$$

**for some even measure $\rho$ on the unit sphere $S^{d-1}$. Here, an even measure is one that assigns equal measure to antipodal sets.**

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This uniqueness is equivalent to the statement that the space spanned by functions of the form $u \mapsto |\langle u, v\rangle|$, for some $v \in S^{d-1}$, are dense amongst the even functions in $C(S^{d-1})$. **This can be rephrased to say that it is dense in $C(\mathcal{L}_1^d)$, the continuous functions on the Grassmannian of lines in $\mathrm{E}^d$.**

# Projections of Zonoids

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are all strict. **This was shown by Weil [1982], it follows from the construction of a body $K$ which is not a zonoid and yet all its projections onto hyperplanes are zonoids.**

# Projection bodies

**If $K$ is a convex body in $\mathrm{E}^d$ and $u^\perp$ is the space orthogonal to $u \in S^{d-1}$, we denote by $V_{d-1}(K|u^\perp)$, the volume of the projection of $K$ onto $u^\perp$.**

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$$V_{d-1}(K|u^\perp) = \frac{1}{2}\int_{S^{d-1}} |\langle u, v\rangle|\, S_{d-1}(K, dv),$$

**where** $S_{d-1}(K,\cdot)$ **denotes the surface area measure of** $K$**.**

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$$V_1(Z|[u]) = V_{d-1}(K|u^\perp).$$

**Here $V_1(Z|[u])$ denotes the length of the projection of $Z$ onto the line spanned by $u$. We will call $Z$ the projection body of $K$.**

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Here $V_1(Z|[u])$ denotes the length of the projection of $Z$ onto the line spanned by $u$. We will call $Z$ the projection body of $K$. **As we have seen, each projection body is a zonoid. However, zonoids need not be full dimensional. Full dimensional zonoids are projection bodies. In fact, the uniqueness of their generating measures shows that they are projection bodies of a unique centrally symmetric body.**

# Extensions

**A convex body $K$ for which**

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*Assume* $K_1,\ K_2$ *are centrally symmetric bodies such that*

$$V_{d-1}(K_1|u^{\perp}) \geq V_{d-1}(K_2|u^{\perp}) \qquad \textit{for all } u \in S^{d-1}.$$

*If* $K_1$ *is a zoonoid, then* $V_d(K_1) \geq V_d(K_2)$.

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*If $K_1$ is a zonoid, then $V_d(K_1) \geq V_d(K_2)$. However, if $K$ is sufficiently smooth and centrally symmetric but not a zonoid, there is a centrally symmetric body $C$ such that*

$$V_{d-1}(C|u^\perp) > V_{d-1}(K|u^\perp) \qquad \textit{for all } u \in S^{d-1}$$

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**A strong form of the uniqueness property of the generating measure** $\rho$ **was given by Schneider [1970]. This was used to establish many characterizations of centrally symmetric sets. For example:** *If every shadow boundary bisects the surface area then the body is centrally symmetric.*

# Higher rank Grassmannians

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**Weil [1976] showed that if $K$ is a generalized zonoid with generating measure $\rho$, then there is a $j$-fold tensor product, $\rho_j$, of $\rho$ with itself, such that**

$$V_j(K|E) = \int_{\mathcal{L}_j^d} |\langle E, F\rangle| \, \rho_j(dF) \qquad \textbf{for all } E \in \mathcal{L}_j^d.$$

**$\rho_j$ is called the $j$-th projection generating measure, and $|\langle E, F\rangle|$ denotes the absolute value of the determinant of the projection of $E$ onto $F$.**

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$\rho_j$ is called the $j$-th projection generating measure, and $|\langle E, F\rangle|$ denotes the absolute value of the determinant of the projection of $E$ onto $F$. **In the case that $K$ is a zonoid, $\rho_j$ is a positive measure. It, therefore, seems natural to introduce the class $\mathcal{K}(j)$ of those centrally symmetric convex bodies (of dimension $\geq j+1$) for which there is a positive measure $\rho$ on $\mathcal{L}_j^d$ such that**

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**In joint work with Zhang [1998], we considered an extension of the Shephard problem. In analogy with Schneider's work on the original problem we showed:**

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**The integral transform**

$$f \mapsto \int_{\mathcal{L}_j^d} |\langle \cdot, E \rangle| f(E)\, dE \qquad f \in L^2(\mathcal{L}_j^d)$$

**arises in the study of stationary Poisson processes of** $j$**-flats (see Matheron [1974,75]).**

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arises in the study of stationary Poisson processes of $j$-flats (see Matheron [1974,75]). Each such process induces a stationary point process on any $(d-j)$-flat $G$. **If $f$ is the directional part of the intensity measure of the original $j$-flat process, then the integral transform yields the intensity of the induced point process on $G^{\perp}$.**

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He conjectured that, as is the case for $j = 1$ or $d - 1$, the kernel of this transform is trivial for all $j$. This was shown to be false in joint work with Howard [1990]. An example of a non-trivial function in the kernel, in the case $d = 4,\ j = 2$, is given by

$$f(E) =$$

$$7\left[\langle E, E_{12}\rangle^4 + \langle E, E_{34}\rangle^4 - \langle E, E_{13}\rangle^4 - \langle E, E_{24}\rangle^4\right]$$

$$+ 42\left[\langle E, E_{12}\rangle^2\langle E, E_{34}\rangle^2 - \langle E, E_{13}\rangle^2\langle E, E_{24}\rangle^2\right]$$

$$+24\left[\langle E, E_{13}\rangle^2 + \langle E, E_{24}\rangle^2 - \langle E, E_{12}\rangle^2 - \langle E, E_{34}\rangle^2\right]$$

Here $E_{12}$, for example, denotes the plane spanned by the first two coordinate vectors in a basis of $\mathbf{E}^4$.

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Here $E_{12}$, for example, denotes the plane spanned by the first two coordinate vectors in a basis of $\mathbf{E}^4$.

**It follows that, the $j$-th projection generating measure of a body in $\mathcal{K}(j)$ is not unique. This is a phenomenon that causes considerable difficulties in the study of $\mathcal{K}(j)$ for $1 < j < d - 1$.**

We will think of the classes $\mathcal{K}(j)$ as a natural extension of the notion of zonoid. It follows from Weil's [1976] results that

$$\mathcal{K}(1) \subseteq \mathcal{K}(j) \subseteq \mathcal{K}(d-1) \qquad 1 < j < d-1.$$

We propose to study these inclusions further and to compare the classes $\mathcal{K}(j)$ with others.

# The classes $\mathcal{Z}_j^d$ – revisited

**Recall that** $K \in \mathcal{Z}_j^d$ **means that all** $j$**-dimensional projections of** $K$ **are zonoids. For a generalized zonoid** $K$ **with generating measure** $\rho$**, Weil [1982] showed that this is equivalent to**

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**In the case of a smooth generalized zonoid whose generating measure is a function $\rho$, the projection generating functions $\rho_j$ and $\rho_{j-1}$, obtained from $\rho$ via tensor products, satisfy**

$$\rho_j(E) = c_{d,j} \int_{S^{d-1}\cap E} \rho_{j-1}(E \cap v^{\perp}) \int_{S^{d-1}\cap E} \rho(u) |\langle u, v\rangle|^{d-j+1} \, du \, dv,$$

**where $c_{d,j}$ is a positive constant.**

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where $c_{d,j}$ is a positive constant. **It is tempting, therefore, to hope for a connection between $\mathcal{K}(j)$ and $\mathcal{Z}_{d-j+1}^d$ and for the inclusion**

$$\mathcal{K}(j) \subset \mathcal{K}(j+1).$$

# The classes $\mathcal{Z}_j^d$ – revisited

Recall that $K \in \mathcal{Z}_j^d$ means that all $j$-dimensional projections of $K$ are zonoids. For a generalized zonoid $K$ with generating measure $\rho$, Weil [1982] showed that this is equivalent to

$$\int_{S^{d-1}} |\langle u, v\rangle|^j \,\rho(dv) \geq 0 \qquad \text{for all } u \in S^{d-1}.$$

In the case of a smooth generalized zonoid whose generating measure is a function $\rho$, the projection generating functions $\rho_j$ and $\rho_{j-1}$, obtained from $\rho$ via tensor products, satisfy

$$\rho_j(E) = c_{d,j} \int_{S^{d-1}\cap E} \rho_{j-1}(E \cap v^{\perp}) \int_{S^{d-1}\cap E} \rho(u)|\langle u, v\rangle|^{d-j+1}\, du\, dv,$$

where $c_{d,j}$ is a positive constant. It is tempting, therefore, to hope for a connection between $\mathcal{K}(j)$ and $\mathcal{Z}_{d-j+1}^d$ and for the inclusion

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**Certainly we have $\mathcal{K}(1) = \mathcal{Z}_d^d$, $\mathcal{K}(d-1) = \mathcal{Z}_2^d$ and $\mathcal{Z}_{d-j+1}^d \subset \mathcal{Z}_{d-j}^d$.**

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**If $P$ is a $d$-polytope with centrally symmetric $(j+1)$-faces, there are positive numbers $\alpha_i$ and $j$-subspaces $E_i$ such that**

$$V_j(P|E) = \sum_{i=1}^{n} \alpha_i |\langle E, E_i \rangle|$$

**for all $E \in \mathcal{L}_j^d$. Thus, $P \in \mathcal{K}(j)$.**

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for all $E \in \mathcal{L}_j^d$. Thus, $P \in \mathcal{K}(j)$. **McMullen's example of a non-zonotope with centrally symmetric facets then provides us with a polytope in $\mathcal{K}(d-2)\backslash\mathcal{Z}_3^d$.**

# The classes $\mathcal{O}_j^d$

**For each** $1 \leq j \leq d-1$**, we say that** $K \in \mathcal{O}_j^d$ **if** $K$ **is centrally symmetric,** $\dim K \geq j+1$ **and there is a centrally symmetric body** $K^\perp$ **such that**

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McMullen [1984] showed that the unit cube, $C_d$ is in every $\mathcal{O}_j^d$ and that $C_d^\perp = C_d$ for each value of $j$. **Schneider [1996] showed that, for a polytope $P$, there is the equivalence**

$$P \in \mathcal{O}_1^d \text{ with } P = P^\perp \iff P \in \mathcal{O}_j^d \text{ with } P = P^\perp,$$

**for all $1 \le j \le d-1$, and gave a characterization of the polytopes which satisfy this property.**

# Harmonic analysis aspects

**We will use the (generalized) Radon and cosine transforms,**

$$R^d_{i,j} : L^2(\mathcal{L}^d_i) \to L^2(\mathcal{L}^d_j) \qquad i < j,$$
$$R_{i,j} f(E) = \int_{F \subset E} f(F)\, dF$$

**and**

$$T^d_j : L^2(\mathcal{L}^d_j) \to L^2(\mathcal{L}^d_j)$$
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**In particular, we note that**

$$R^d_{j,d-1} V_j(K|\cdot) = c_{d,j} T^d_{d-1} S_j(K,\cdot)^\perp,$$

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**Both these transforms intertwine the action of the rotation group $SO(d)$.**

**The irreducible invariant subspaces of $L^2(\mathcal{L}_1^d)$ are the spaces of spherical harmonics $\mathcal{H}_{2n}^{d,1}$, thus**

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**The transforms $R_{1,j}^d$ and $T_1^d$ are both injective on $L^2(\mathcal{L}_1^d)$ which we will identify, by orthogonality, with $L^2(\mathcal{L}_{d-1}^d)$.**

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**Moving to the case $j = 2$, the irreducible invariant subspaces of $L^2(\mathcal{L}_2^d)$ form a two parameter family $\mathcal{H}_{(2n,2m)}^{d,2}$ $(n \geq |m|)$. Here we have,**

$$\ker R_{2,d-1}^d = \bigoplus_{n \geq |m| > 0} \mathcal{H}_{(2n,2m)}^{d,2}$$

$$\ker T_2^d = \bigoplus_{n \geq |m| > 1} \mathcal{H}_{(2n,2m)}^{d,2}$$

**also**

$$R_{1,2}^d \mathcal{H}_{2n}^{d,1} = \mathcal{H}_{(2n,0)}^{d,2}.$$

**This representation of the kernel of the cosine transform is a result of Alesker and Berenstein [2004], who obtained the kernel of $T_j^d$, in general. A key to this was the connection between the range of $T_j^d$ and Alesker's earlier work on translation invariant valuations.**

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**We will make significant use of the fact that, if $f \in L^2(\mathcal{L}_j^d)$ has rotational symmetry, then the harmonic components of $f$ lie in $\bigoplus_{n=0}^{\infty} R_{1,j}\mathcal{H}_{2n}^{d,1}$.**

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However, a rotationally symmetric function restricts to a constant function and so $m = 0$.

**A different formulation of this result states that, in some sense, a rotationally symmetric function only has spherical harmonic components.**

**This observation allows us, in the case of rotationally symmetric bodies, to use the equation**

$$R^d_{j,d-1} V_j(K|\cdot) = c_{d,j} T^d_{d-1} S_j(K,\cdot)^\perp,$$

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**It is an interesting question to decide whether $V_j(K|\cdot)$ is determined by $R^d_{j,d-1} V_j(K|\cdot)$. This is known to be true for centrally symmetric $K$, for polytopes and for rotationally symmetric bodies.**

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It is an interesting question to decide whether $V_j(K|\cdot)$ is determined by $R^d_{j,d-1} V_j(K|\cdot)$. This is known to be true for centrally symmetric $K$, for polytopes and for rotationally symmetric bodies.

For example, if $K$ is a convex body in $\mathbf{E}^4$ and all three dimensional projections of $K$ have the same surface area, it is not known if all two dimensional projections of $K$ necessarily have the same area.

# Bodies of revolution

**If $K$ is a generalized zonoid and a body of revolution, there is, for each $1 \leq j \leq d-1$, a unique $j$-th projection generating measure with rotation invariance, we will call this the standard $j$-th projection generating measure.**

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**We will study the families of convex bodies $K_\alpha,\ L_{\beta,j},\ W_{\gamma,j}$ defined by**

$$h(K_\alpha, u) = 1 + \alpha P_2^d(\langle e_d, u\rangle);$$

$$S_j(L_{\beta,j}, u) = 1 + \beta P_2^d(\langle e_d, u\rangle);$$

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**For the latter two, we will use results of Firey [1970] to determine the parameter values $\beta,\ \gamma$ which yield convex bodies.**

# Comparing the classes

**The bodies $K_\alpha$ exist precisely when**

$$-\frac{d-1}{2d-1} \le \alpha \le \frac{d-1}{d+1},$$

**and are in $\mathcal{Z}_j^d$ precisely when**

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**The bodies $L_{\beta,j}$ exist precisely when**

$$-1 \leq \beta \leq \frac{j(d+1)}{2d-j},$$

**and are in $\mathcal{K}(j)$ precisely when**

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**We can use the bodies $W_{\gamma,j}$ to find bodies in $\mathcal{O}_j^d\backslash\mathcal{K}(j)$. When the orthogonal body exists, it is a homothet of $W_{\gamma',d-j}$ where**

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**In conclusion, we note that the classes $\mathcal{K}(j)$, $\mathcal{O}_j^d$ and $\mathcal{Z}_{d-j+1}^d$ are all different if $1 < j < d-1$.**

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