The Steiner polynomial and its consequences on the Blaschke diagram

M. A. Hernández Cifre

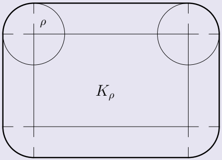
(joint work with E. Saorín)

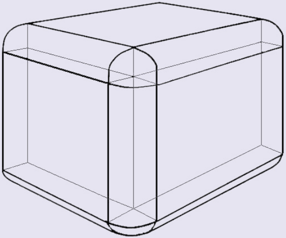
16 Mayo 2005

The outer parallel body

The outer parallel body

$$\left.\begin{array}{l} K \text{ convex body} \\ \rho \geq 0 \end{array}\right\} \rightsquigarrow K + \rho\mathbb{B}^n = \textit{outer parallel body of } K \text{ at distance } \rho$$





The Steiner formula

Theorem (Steiner's formula, 1840)

The volume of the outer parallel body of K at distance $\rho > 0$, $K_\rho = K + \rho\mathbb{B}^n$, can be expressed as a polynomial of degree the dimension n in the parameter ρ, where its coefficients are, up to a constant, the **quermassintegrals** of the body K, $W_i(K)$, for $0 \leq i \leq n$:

$$V(K_\rho) = V(K + \rho\mathbb{B}^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i.$$

This expression is known as the **Steiner polynomial**

The Steiner formula

Theorem (Steiner's formula, 1840)

The volume of the outer parallel body of K at distance $\rho > 0$, $K_\rho = K + \rho\mathbb{B}^n$, can be expressed as a polynomial of degree the dimension n in the parameter ρ, where its coefficients are, up to a constant, the **quermassintegrals** of the body K, $W_i(K)$, for $0 \leq i \leq n$:

$$V(K_\rho) = V(K + \rho\mathbb{B}^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i.$$

This expression is known as the **Steiner polynomial**

The 3-dimensional case

$$V(K + \rho\mathbb{B}^3) = V(K) + S(K)\rho + M(K)\rho^2 + \frac{4}{3}\pi\rho^3$$

Coefficients of $V(K_\rho) \longrightarrow$ Magnitudes in **Blaschke's problem**: V, S, M

The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

- GEOMETRIC INEQUALITIES OF MINKOWSKI:

$S(K)^2 \geq 3V(K)M(K)$	(Cap-body)
$M(K)^2 \geq 4\pi S(K)$	(Ball $\mathbb{B}^3$)
$M(K)^3 \geq 48\pi^2 V(K)$	(Ball $\mathbb{B}^3$)

The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

- GEOMETRIC INEQUALITIES OF MINKOWSKI:

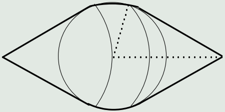
$$S(K)^2 \geq 3V(K)M(K) \qquad \text{(Cap-body)}$$

$$M(K)^2 \geq 4\pi S(K)$$

$$M(K)^3 \geq 48\pi^2 V(K)$$

Cap-Body

Convex hull of a ball and countably many points exterior to it such that the line segment joining any two of these points intersects the ball



The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

- GEOMETRIC INEQUALITIES OF MINKOWSKI:

$$S(K)^2 \geq 3V(K)M(K) \qquad \text{(Cap-body)}$$

$$M(K)^2 \geq 4\pi S(K) \qquad (\text{Ball } \mathbb{B}^3)$$

$$M(K)^3 \geq 48\pi^2 V(K) \qquad (\text{Ball } \mathbb{B}^3)$$

- ISOPERIMETRIC INEQUALITY IN $\mathbb{R}^3$:

$$S(K)^3 \geq 36\pi V(K)^2 \qquad (\text{Ball } \mathbb{B}^3)$$

The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

- GEOMETRIC INEQUALITIES OF MINKOWSKI:

$S(K)^2 \geq 3V(K)M(K)$	(Cap-body)
$M(K)^2 \geq 4\pi S(K)$	(Ball $\mathbb{B}^3$)
$M(K)^3 \geq 48\pi^2 V(K)$	(Ball $\mathbb{B}^3$)

- ISOPERIMETRIC INEQUALITY IN $\mathbb{R}^3$:

$$S(K)^3 \geq 36\pi V(K)^2 \qquad \text{(Ball } \mathbb{B}^3\text{)}$$

- ISOPERIMETRIC INEQUALITY FOR PLANAR SETS IN $\mathbb{R}^3$:

$$2M(K)^2 \geq \pi^3 S(K) \quad \text{where} \quad V(K) = 0 \qquad \text{(Circle } C\text{)}$$

The Blaschke problem

1916: Blaschke considered a compact convex set K in $\mathbb{R}^3$ with volume V, surface area S and integral mean curvature M

- GEOMETRIC INEQUALITIES OF MINKOWSKI:

$$S(K)^2 \geq 3V(K)M(K)$$

$$M(K)^2 \geq 4\pi S(K)$$

$$M(K)^3 \geq 48\pi^2 V(K)$$

- ISOPERIMETRIC INEQUALITY

$$S(K)^3 \geq 36\pi V(K)^2$$

Blaschke's problem, 1916

Given three positive real numbers V_0, S_0 and M_0 verifying the above five inequalities, can the existence of a convex set $K \subset \mathbb{R}^3$ with volume V_0, surface area S_0 and integral mean curvature M_0 be assured?

- ISOPERIMETRIC INEQUALITY FOR PLANAR SETS IN $\mathbb{R}^3$:

$$2M(K)^2 \geq \pi^3 S(K) \quad \text{where} \quad V(K) = 0 \qquad (\text{Circle } C)$$

The Blaschke map

Blaschke proposed a method for representing K as a point of $\mathbb{R}^2$ from the triple of magnitudes $(V,\ S,\ M)$, mapping it to a point $(x, y) \in [0, 1]^2$:

Blaschke's map

$$\mathcal{K}^3 = \left\{ \text{Convex bodies} \right\} \longrightarrow [0,1] \times [0,1]$$

$$K\,(V,\ S,\ M) \rightsquigarrow (x, y) = \left(\frac{4\pi S}{M^2}, \frac{48\pi^2 V}{M^3} \right)$$

The Blaschke map

Blaschke proposed a method for representing K as a point of $\mathbb{R}^2$ from the triple of magnitudes $(V,\ S,\ M)$, mapping it to a point $(x,y) \in [0,1]^2$:

Blaschke's map

$$\mathcal{K}^3 = \left\{ \text{Convex bodies} \right\} \longrightarrow [0,1] \times [0,1]$$

$$K\,(V,\ S,\ M) \rightsquigarrow (x,y) = \left(\frac{4\pi S}{M^2}, \frac{48\pi^2 V}{M^3} \right)$$

- The functionals $\frac{4\pi S}{M^2}$ and $\frac{48\pi^2 V}{M^3}$ are invariant under dilatations

The Blaschke map

Blaschke proposed a method for representing K as a point of $\mathbb{R}^2$ from the triple of magnitudes $(V,\ S,\ M)$, mapping it to a point $(x, y) \in [0,1]^2$:

Blaschke's map

$$\mathcal{K}^3 = \left\{ \text{Convex bodies} \right\} \longrightarrow [0,1]\times[0,1]$$

$$K\,(V,\ S,\ M) \rightsquigarrow (x,y) = \left(\frac{4\pi S}{M^2}, \frac{48\pi^2 V}{M^3}\right)$$

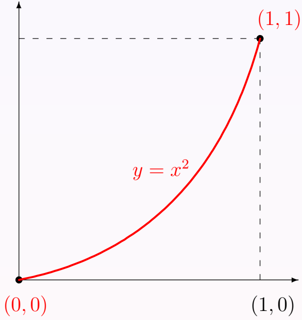
- The functionals $\dfrac{4\pi S}{M^2}$ and $\dfrac{48\pi^2 V}{M^3}$ are invariant under dilatations
- This choice of the coordinates allows us to assure that **all the sets with the same shape are mapped to the same point in the diagram**

Blaschke's diagram=Range of Blaschke's map

Each inequality relating V, M and S determines part of the boundary of Blaschke's Diagram. Inequality missing $\longrightarrow$ part of the boundary unknown.

Blaschke's diagram=Range of Blaschke's map

Each inequality relating V, M and S determines part of the boundary of Blaschke's Diagram. Inequality missing $\rightarrow$ part of the boundary unknown.

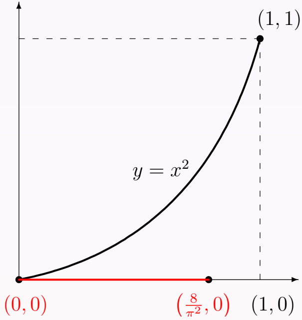


- $\boxed{S^2 \geq 3VM} \longrightarrow y \leq x^2$

  $y = x^2 \equiv$ Cap-Bodies

Blaschke's diagram=Range of Blaschke's map

Each inequality relating V, M and S determines part of the boundary of Blaschke's Diagram. Inequality missing $\rightarrow$ part of the boundary unknown.



- $\boxed{S^2 \geq 3VM} \longrightarrow y \leq x^2$

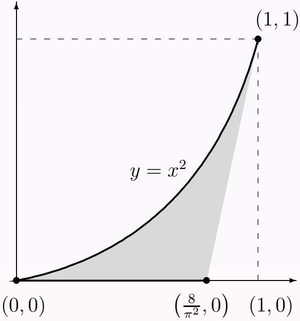
  $y = x^2 \equiv$ Cap-Bodies

- $\boxed{\begin{array}{c} V = 0 \\ 2M^2 \geq \pi^3 S \end{array}} \longrightarrow \left[\begin{array}{l} y = 0 \\ x \leq 8/\pi^2 \end{array} \right.$

  Planar convex bodies

Blaschke's diagram=Range of Blaschke's map

Each inequality relating V, M and S determines part of the boundary of Blaschke's Diagram. Inequality missing $\longrightarrow$ part of the boundary unknown.



- $\boxed{S^2 \geq 3VM} \longrightarrow y \leq x^2$

  $y = x^2 \equiv$ Cap-Bodies

- $\boxed{\begin{array}{l} V = 0 \\ 2M^2 \geq \pi^3 S \end{array}} \longrightarrow \left[\begin{array}{l} y = 0 \\ x \leq 8/\pi^2 \end{array} \right.$

  Planar convex bodies

If $8/\pi^2 < x \leq 1$ then y is strictly positive. *Solution over this range?*

The Missing Boundary of Blaschke's Diagram

Some comments on the interior of the diagram

- (x_0, y_0) = coordinates of a convex body K
- (x_ρ, y_ρ) = coordinates of the outer parallel body of K

Coordinates of the outer parallel body

$$x_\rho = \frac{x_0 + 2\lambda + \lambda^2}{(1+\lambda)^2}, \qquad y_\rho = \frac{y_0 + 3x_0\lambda + 3\lambda^2 + \lambda^3}{(1+\lambda)^3}$$

where $\lambda = 4\pi\rho/M$

Some comments on the interior of the diagram

- $(x_0, y_0) =$ coordinates of a convex body K
- $(x_\rho, y_\rho) =$ coordinates of the outer parallel body of K
- $h(\rho) = (x_\rho, y_\rho)$ is the equation of an algebraic curve which connects any given point (x_0, y_0) of Blaschke's diagram with the point $(1, 1)$

Coordinates of the outer parallel body

$$x_\rho = \frac{x_0 + 2\lambda + \lambda^2}{(1+\lambda)^2}, \qquad y_\rho = \frac{y_0 + 3x_0\lambda + 3\lambda^2 + \lambda^3}{(1+\lambda)^3}$$

where $\lambda = 4\pi\rho/M$

Some comments on the interior of the diagram

- $(x_0, y_0) =$ coordinates of a convex body K
- $(x_\rho, y_\rho) =$ coordinates of the outer parallel body of K
- $h(\rho) = (x_\rho, y_\rho)$ is the equation of an algebraic curve which connects any given point (x_0, y_0) of Blaschke's diagram with the point $(1, 1)$
- The curves $h(\rho)$ depend continuously on the given initial point

Coordinates of the outer parallel body

$$x_\rho = \frac{x_0 + 2\lambda + \lambda^2}{(1+\lambda)^2}, \qquad y_\rho = \frac{y_0 + 3x_0\lambda + 3\lambda^2 + \lambda^3}{(1+\lambda)^3}$$

where $\lambda = 4\pi\rho/M$

Some comments on the interior of the diagram

- (x_0, y_0) = coordinates of a convex body K
- (x_ρ, y_ρ) = coordinates of the outer parallel body of K
- $h(\rho) = (x_\rho, y_\rho)$ is the equation of an algebraic curve which connects any given point (x_0, y_0) of Blaschke's diagram with the point $(1, 1)$
- The curves $h(\rho)$ depend continuously on the given initial point
- Blaschke's diagram is **simply connected**

Some comments on the interior of the diagram

- (x_0, y_0) = coordinates of a convex body K
- (x_ρ, y_ρ) = coordinates of the outer parallel body of K
- $h(\rho) = (x_\rho, y_\rho)$ is the equation of an algebraic curve which connects any given point (x_0, y_0) of Blaschke's diagram with the point $(1, 1)$
- The curves $h(\rho)$ depend continuously on the given initial point
- Blaschke's diagram is **simply connected**

More important:

The interior of Blaschke's diagram can be "filled" with the images of outer parallel bodies

Steiner's polynomial in $\mathbb{R}^3$

Steiner's polynomial in $\mathbb{R}^3$

$$f(\rho) = V(K) + S(K)\rho + M(K)\rho^2 + \frac{4}{3}\pi\rho^3$$

Steiner's polynomial in $\mathbb{R}^3$

Steiner's polynomial in $\mathbb{R}^3$

$$f(\rho) = V(K) + S(K)\rho + M(K)\rho^2 + \frac{4}{3}\pi\rho^3$$

Some properties:

- All the possibilities are allowed for the roots of $f(\rho)$: $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$
- If $\rho_i \in \mathbb{R} \Rightarrow \rho_i < 0$

Steiner's polynomial in $\mathbb{R}^3$

Steiner's polynomial in $\mathbb{R}^3$

$$f(\rho) = V(K) + S(K)\rho + M(K)\rho^2 + \frac{4}{3}\pi\rho^3$$

Some properties:

- All the possibilities are allowed for the roots of $f(\rho)$: $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$
- If $\rho_i \in \mathbb{R} \Rightarrow \rho_i < 0$

- A triple real root $\Longleftrightarrow$ $K = \mathbb{B}^3$ (*Characterization of the ball*)

Steiner's polynomial in $\mathbb{R}^3$

Steiner's polynomial in $\mathbb{R}^3$

$$f(\rho) = V(K) + S(K)\rho + M(K)\rho^2 + \frac{4}{3}\pi\rho^3$$

Some properties:

- All the possibilities are allowed for the roots of $f(\rho)$: $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$
- If $\rho_i \in \mathbb{R} \Rightarrow \rho_i < 0$

- A triple real root $\Longleftrightarrow K = \mathbb{B}^3$ (*Characterization of the ball*)

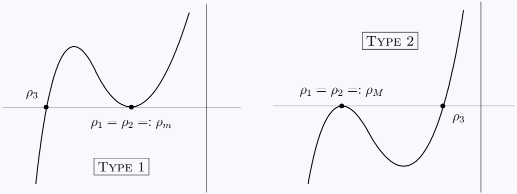
✓ The triple real root is

$$\rho = -\frac{M}{4\pi} = -R = -r$$

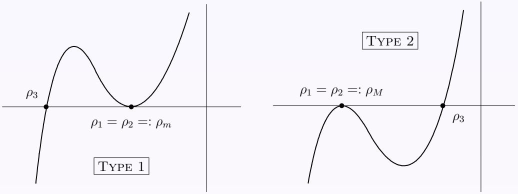
✓ The Steiner polynomial is

$$f(\rho) = \frac{4\pi}{3}(\rho + R)^3$$

A double real root + a simple real root



A double real root + a simple real root



Theorem (characterization of double roots)

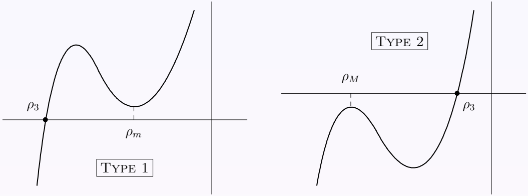
The Steiner polynomial of a convex body K has a double real root of TYPE 1 if, and only if, V, S and M satisfy the equation

$$M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$$

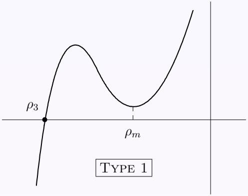
The Steiner polynomial of a convex body K has a double real root of TYPE 2 if, and only if, V, S and M satisfy the equation

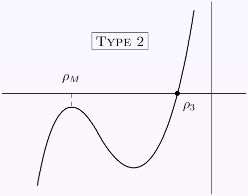
$$M^3 - 6\pi(MS - 4\pi V) + (M^2 - 4\pi S)^{3/2} = 0$$

Two complex roots + a simple real root



Two complex roots + a simple real root





Theorem (characterization of complex roots)

The Steiner polynomial of a convex body K has two conjugated complex roots of TYPE 1 if, and only if, V, S and M satisfy the inequality

$$M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} > 0$$

The Steiner polynomial of a convex body K has two conjugated complex roots of TYPE 2 if, and only if, V, S and M satisfy the inequality

$$M^3 - 6\pi(MS - 4\pi V) + (M^2 - 4\pi S)^{3/2} < 0$$

Three simple (*different*) real roots

Theorem (characterization of simple real roots)

The Steiner polynomial of a convex body K has three simple (*different*) real roots if, and only if, V, S and M satisfy *both* inequalities

$$M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} < 0,$$

$$M^3 - 6\pi(MS - 4\pi V) + (M^2 - 4\pi S)^{3/2} > 0$$

Three simple (*different*) real roots

Theorem (characterization of simple real roots)

The Steiner polynomial of a convex body K has three simple (*different*) real roots if, and only if, V, S and M satisfy *both* inequalities

$$M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} < 0,$$

$$M^3 - 6\pi(MS - 4\pi V) + (M^2 - 4\pi S)^{3/2} > 0$$

Notation

$$\phi_-(K) := M(K)^3 - 6\pi\big(M(K)S(K) - 4\pi V(K)\big) - \big(M(K)^2 - 4\pi S(K)\big)^{3/2},$$

$$\phi_+(K) := M(K)^3 - 6\pi\big(M(K)S(K) - 4\pi V(K)\big) + \big(M(K)^2 - 4\pi S(K)\big)^{3/2}$$

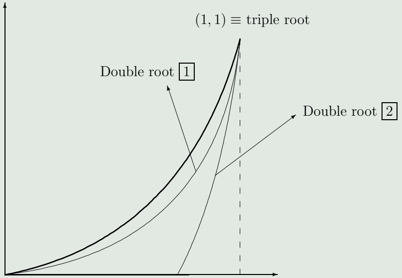
Translation in Blaschke's diagram

- Double TYPE 1: $\phi_-(K) = M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$
- Double TYPE 2: $\phi_+(K) = 0$

Translation in Blaschke's diagram

- Double TYPE 1: $\phi_-(K) = M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$
- Double TYPE 2: $\phi_+(K) = 0$

The different types of roots

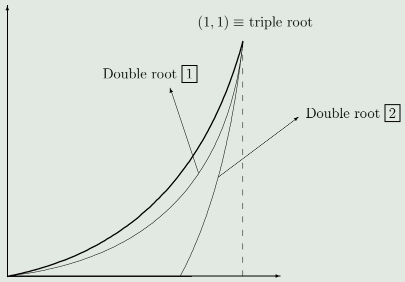


$$y = 3x + 2(1-x)^{3/2} - 2$$
$$y = 3x - 2(1-x)^{3/2} - 2$$

Translation in Blaschke's diagram

- Double TYPE 1: $\phi_-(K) = M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$
- Double TYPE 2: $\phi_+(K) = 0$

The different types of roots

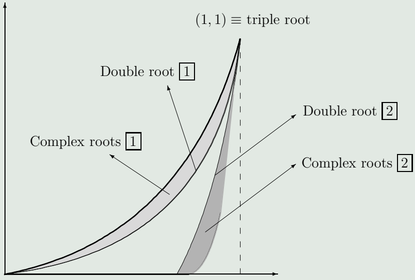


- Complex TYPE 1: $\phi_-(K) > 0$
- Complex TYPE 2: $\phi_+(K) < 0$

Translation in Blaschke's diagram

- Double TYPE 1: $\phi_-(K) = M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$
- Double TYPE 2: $\phi_+(K) = 0$

The different types of roots

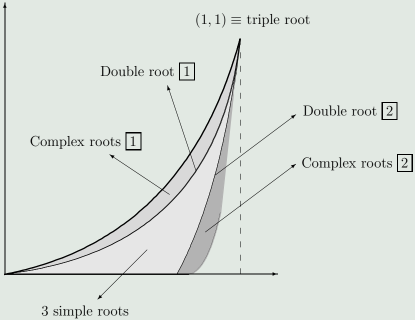


- Complex TYPE 1: $\phi_-(K) > 0$
- Complex TYPE 2: $\phi_+(K) < 0$

Translation in Blaschke's diagram

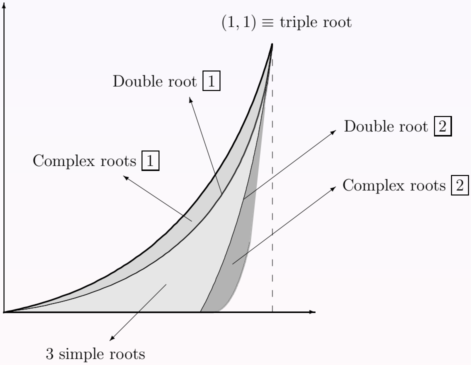
- Double TYPE 1: $\phi_-(K) = M^3 - 6\pi(MS - 4\pi V) - (M^2 - 4\pi S)^{3/2} = 0$
- Double TYPE 2: $\phi_+(K) = 0$

The different types of roots



- Complex TYPE 1: $\phi_-(K) > 0$
- Complex TYPE 2: $\phi_+(K) < 0$
- Simple real: $\begin{cases} \phi_-(K) < 0 \\ \phi_+(K) > 0 \end{cases}$

Steiner's polynomial and Blaschke's diagram: consequences



Steiner's polynomial and Blaschke's diagram: consequences

Corollary

The sets closing the boundary of Blaschke's diagram verify that their Steiner polynomial has conjugated complex roots of TYPE 2

Steiner's polynomial and Blaschke's diagram: consequences

Corollary

The sets closing the boundary of Blaschke's diagram verify that their Steiner polynomial has conjugated complex roots of TYPE 2

Classification of convex bodies: 3 mutually disjoint classes

1. Convex bodies whose Steiner polynomial has only **real roots** (simple or double) (class $\overline{\mathcal{R}} = \mathcal{R} \cup \mathcal{D}_1 \cup \mathcal{D}_2$)

Steiner's polynomial and Blaschke's diagram: consequences

Corollary

The sets closing the boundary of Blaschke's diagram verify that their Steiner polynomial has conjugated complex roots of $\text{Type } 2$

Classification of convex bodies: 3 mutually disjoint classes

1. Convex bodies whose Steiner polynomial has only **real roots** (simple or double) (class $\overline{\mathcal{R}} = \mathcal{R} \cup \mathcal{D}_1 \cup \mathcal{D}_2$)
2. Convex bodies whose Steiner polynomial has two conjugated **complex roots of Type 1** (class $\mathcal{C}_1$)

Steiner's polynomial and Blaschke's diagram: consequences

Corollary

The sets closing the boundary of Blaschke's diagram verify that their Steiner polynomial has conjugated complex roots of TYPE 2

Classification of convex bodies: 3 mutually disjoint classes

1. Convex bodies whose Steiner polynomial has only **real roots** (simple or double) (class $\overline{\mathcal{R}} = \mathcal{R} \cup \mathcal{D}_1 \cup \mathcal{D}_2$)
2. Convex bodies whose Steiner polynomial has two conjugated **complex roots of Type 1** (class $\mathcal{C}_1$)
3. Convex bodies whose Steiner polynomial has two conjugated **complex roots of Type 2** (class $\mathcal{C}_2$)

Steiner's polynomial and Blaschke's diagram: consequences

Corollary

The sets closing the boundary of Blaschke's diagram verify that their Steiner polynomial has conjugated complex roots of TYPE 2

Classification of convex bodies: 3 mutually disjoint classes

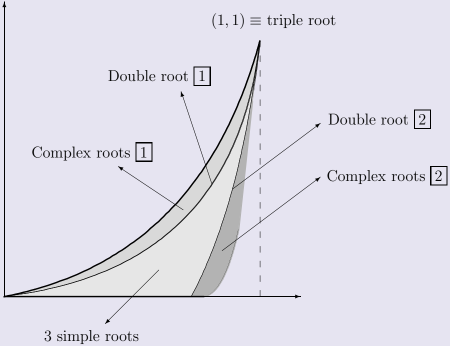
1. Convex bodies whose Steiner polynomial has only **real roots** (simple or double) (class $\overline{\mathcal{R}} = \mathcal{R} \cup \mathcal{D}_1 \cup \mathcal{D}_2$)
2. Convex bodies whose Steiner polynomial has two conjugated **complex roots of Type 1** (class $\mathcal{C}_1$)
3. Convex bodies whose Steiner polynomial has two conjugated **complex roots of Type 2** (class $\mathcal{C}_2$)

A map

We obtain a "map" of Blaschke's diagram

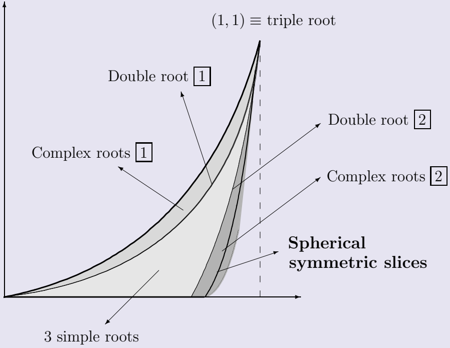
Special families of sets

- Cap-bodies $\longrightarrow$ Complex roots TYPE 1 (class $\mathcal{C}_1$)



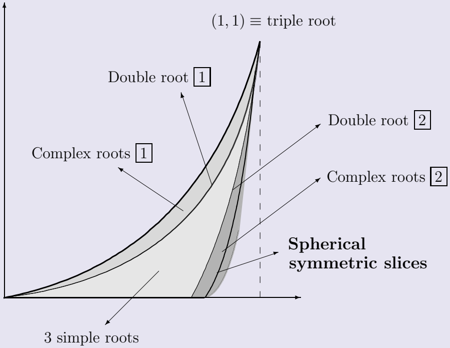
Special families of sets

- Cap-bodies $\longrightarrow$ Complex roots $\textsc{Type}$ 1 (class $\mathcal{C}_1$)
- Spherical symmetric slices



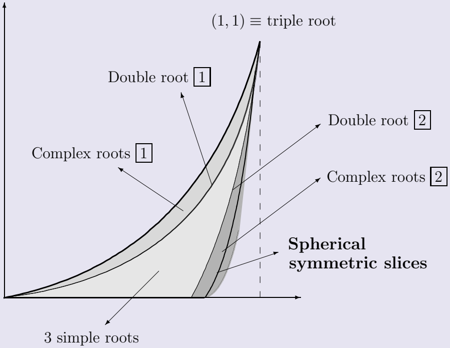
Special families of sets

- Cap-bodies $\longrightarrow$ Complex roots Type 1 (class $\mathcal{C}_1$)
- Spherical symmetric slices $\longleftrightarrow$ Complex roots Type 2 (class $\mathcal{C}_2$)



Special families of sets

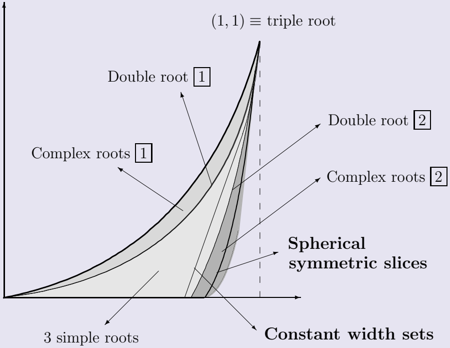
- Cap-bodies $\longrightarrow$ Complex roots TYPE 1 (class $\mathcal{C}_1$)
- Spherical symmetric slices $\longleftrightarrow$ Complex roots TYPE 2 (class $\mathcal{C}_2$)



- Constant width sets $\longleftrightarrow 2V = bS - \frac{2\pi}{3} b^3$

Special families of sets

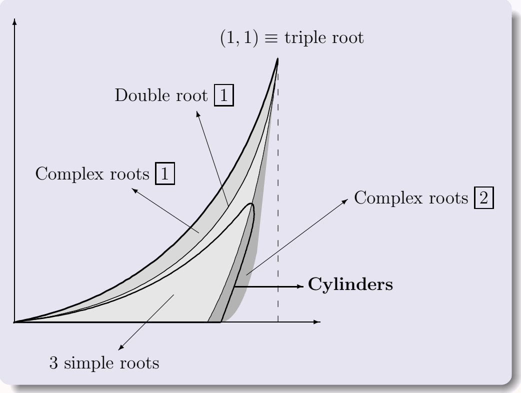
- Cap-bodies $\longrightarrow$ Complex roots TYPE 1 (class $\mathcal{C}_1$)
- Spherical symmetric slices $\longleftrightarrow$ Complex roots TYPE 2 (class $\mathcal{C}_2$)



- Constant width sets $\longleftrightarrow$ $2V = bS - \frac{2\pi}{3}b^3$ $\longrightarrow$ Real roots (class $\mathcal{R}$)

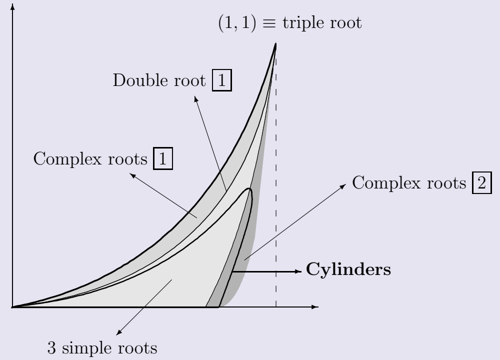
Special families of sets

- Cylinders $\longrightarrow x = \dfrac{8(h+1)}{(\pi+h)^2}$ and $y = \dfrac{48h}{(\pi+h)^3}$



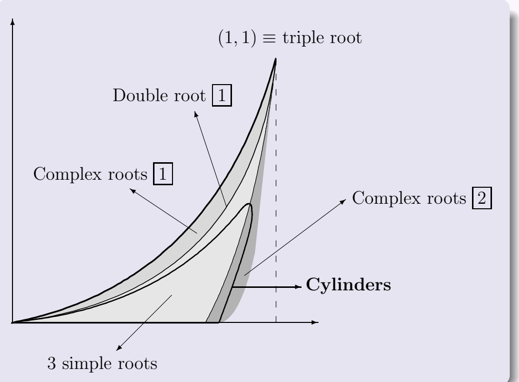
Special families of sets

- Cylinders $\longrightarrow$ $x = \dfrac{8(h+1)}{(\pi+h)^2}$ and $y = \dfrac{48h}{(\pi+h)^3}$
- Cylinders $\longrightarrow$ Real roots and complex roots TYPE 2 (classes $\overline{\mathcal{R}}$, $\mathcal{C}_2$)



Special families of sets

- Cylinders $\longrightarrow x = \dfrac{8(h+1)}{(\pi+h)^2}$ and $y = \dfrac{48h}{(\pi+h)^3}$
- Cylinders $\longrightarrow$ Real roots and complex roots TYPE 2 (classes $\overline{\mathcal{R}}$, $\mathcal{C}_2$)



- $h \approx 1.710653378 \rightsquigarrow \mathcal{D}_2$
- Limit case:
  $h \longrightarrow \infty \rightsquigarrow$ segment $\rightsquigarrow \mathcal{D}_1$
- No complex roots of TYPE 1

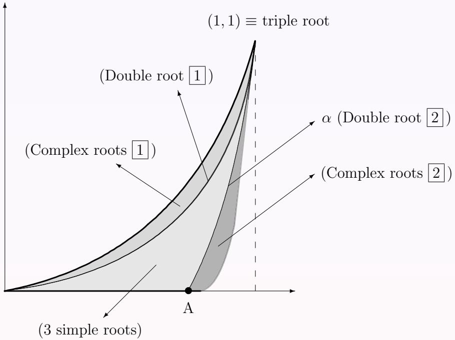
Particular families of sets for each type of roots

Particular families of sets for each type of roots

- If $K \in \mathcal{K}^3$, $\phi_-(K + \rho\mathbb{B}^3) = \phi_-(K)$ and $\phi_+(K + \rho\mathbb{B}^3) = \phi_+(K)$

Particular families of sets for each type of roots

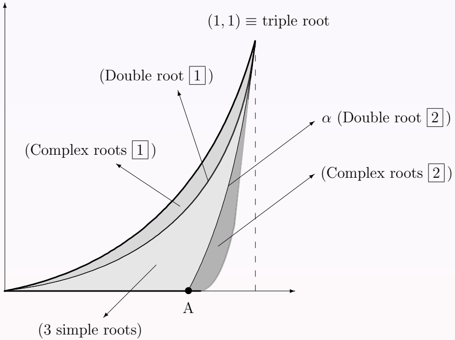
- If $K \in \mathcal{K}^3$, $\phi_-(K+\rho\mathbb{B}^3) = \phi_-(K)$ and $\phi_+(K+\rho\mathbb{B}^3) = \phi_+(K)$



- $K \rightsquigarrow A \equiv$ double roots of TYPE 2 $\iff \phi_+(K) = 0$

Particular families of sets for each type of roots

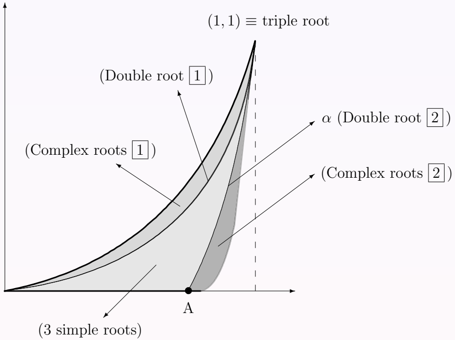
- If $K \in \mathcal{K}^3$, $\phi_-(K+\rho\mathbb{B}^3) = \phi_-(K)$ and $\phi_+(K+\rho\mathbb{B}^3) = \phi_+(K)$



- $K \rightsquigarrow A \equiv$ double roots of TYPE 2 $\iff \phi_+(K) = 0$
- $K_\lambda = \lambda K + (1-\lambda)\mathbb{B}^3, \quad 0 \le \lambda \le 1$

Particular families of sets for each type of roots

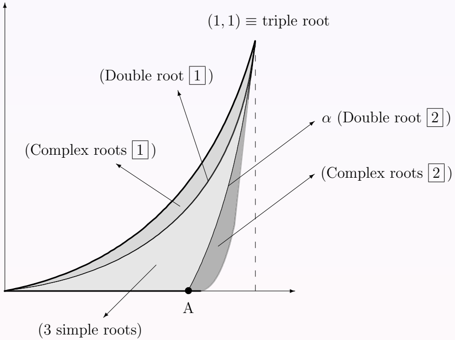
- If $K \in \mathcal{K}^3$, $\phi_-(K+\rho\mathbb{B}^3)=\phi_-(K)$ and $\phi_+(K+\rho\mathbb{B}^3)=\phi_+(K)$



- $K \rightsquigarrow A \equiv$ double roots of TYPE 2 $\Longleftrightarrow \phi_+(K)=0$
- $K_\lambda = \lambda K + (1-\lambda)\mathbb{B}^3, \quad 0 \le \lambda \le 1$
- $\phi_+(K_\lambda) = \lambda^3\,\phi_+(K) = 0$

Particular families of sets for each type of roots

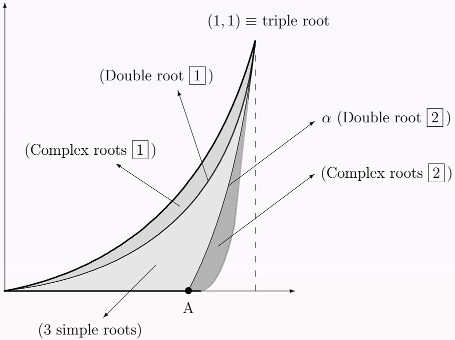
- If $K \in \mathcal{K}^3$, $\phi_-(K+\rho\mathbb{B}^3)=\phi_-(K)$ and $\phi_+(K+\rho\mathbb{B}^3)=\phi_+(K)$



- $K \rightsquigarrow A \equiv$ double roots of TYPE 2 $\iff \phi_+(K)=0$
- $K_\lambda = \lambda K + (1-\lambda)\mathbb{B}^3, \quad 0 \le \lambda \le 1$
- $\phi_+(K_\lambda) = \lambda^3\,\phi_+(K) = 0$
- $\{\lambda K + (1-\lambda)\mathbb{B}^3,\ 0 \le \lambda \le 1\} \rightsquigarrow \alpha$

Particular families of sets for each type of roots

- If $K \in \mathcal{K}^3$, $\phi_-(K+\rho\mathbb{B}^3)=\phi_-(K)$ and $\phi_+(K+\rho\mathbb{B}^3)=\phi_+(K)$



- $K \rightsquigarrow A \equiv$ double roots of TYPE 2 $\iff \phi_+(K)=0$
- $K_\lambda = \lambda K + (1-\lambda)\mathbb{B}^3, \quad 0 \le \lambda \le 1$
- $\phi_+(K_\lambda) = \lambda^3\, \phi_+(K) = 0$
- $\left\{\lambda K + (1-\lambda)\mathbb{B}^3,\ 0 \le \lambda \le 1\right\} \rightsquigarrow \alpha$

- Analogously for the other possibilities
- **Outer parallel bodies of the segment $\rightsquigarrow$ double roots TYPE 1**

Particular families of sets for each type of roots

Theorem

In general, if K is a convex body whose Steiner polynomial has its roots of a certain type (simple real, double real or complex), then *any* set in the family of all convex combinations of K and the Euclidean ball $\mathbb{B}^3$ verifies that its Steiner polynomial has its roots of the same type.

$$+$$

Particular families of sets for each type of roots

Theorem

In general, if K is a convex body whose Steiner polynomial has its roots of a certain type (simple real, double real or complex), then *any* set in the family of all convex combinations of K and the Euclidean ball $\mathbb{B}^3$ verifies that its Steiner polynomial has its roots of the same type.

$$+$$

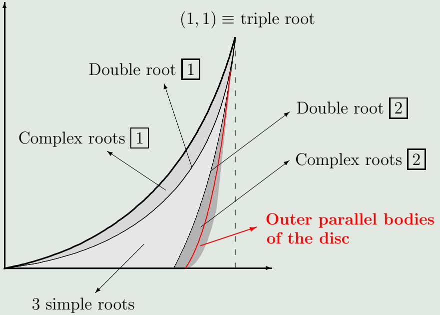
Property of the interior of Blaschke's diagram

The interior of Blaschke's diagram can be filled with the images of outer parallel bodies

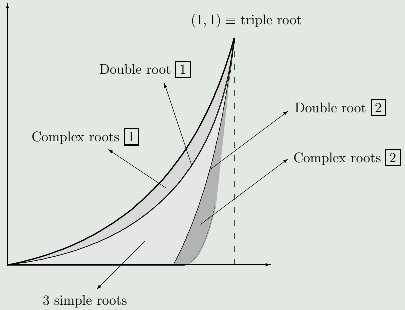
How to fill the Blaschke diagram

Consequence 1:

The outer parallel bodies of the planar convex bodies (classes $\overline{\mathcal{R}}$ and $\mathcal{C}_2$) fill the part of the diagram which corresponds to the real roots and a part of the complex roots of TYPE 2 (till the outer parallel bodies of the disc)



How to fill the Blaschke diagram



Consequence 2:

The outer parallel bodies of the cap-bodies (class $\mathcal{C}_1$) fill the part of the diagram corresponding to complex roots of TYPE 1 (for the limit case of a segment, the image curve of its outer parallel bodies is, precisely, the one corresponding to the double roots of TYPE 1)

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$
- **Steiner's polynomial:** $A + P\rho + \pi\rho^2$

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$
- **Steiner's polynomial:** $A + P\rho + \pi\rho^2$

Alternating Steiner's polynomial: $A - P\lambda + \pi\lambda^2$

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$
- **Steiner's polynomial:** $A + P\rho + \pi\rho^2$

  Alternating Steiner's polynomial: $A - P\lambda + \pi\lambda^2$
- Bonnesen's inequality $\Longrightarrow \lambda_1 \leq r \leq R \leq \lambda_2 \Longrightarrow$

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$
- **Steiner's polynomial:** $A + P\rho + \pi\rho^2$

  Alternating Steiner's polynomial: $A - P\lambda + \pi\lambda^2$
- Bonnesen's inequality $\Longrightarrow \lambda_1 \leq r \leq R \leq \lambda_2 \Longrightarrow$

  $\Longrightarrow \rho_1 = -\lambda_2 \leq -R \leq -r \leq -\lambda_1 = \rho_2$

Bonnesen's inequality

Let us recall:

- Bonnesen's inequality: $A - P\lambda + \pi\lambda^2 \leq 0$ if $r \leq \lambda \leq R$
- **Steiner's polynomial:** $A + P\rho + \pi\rho^2$

  Alternating Steiner's polynomial: $A - P\lambda + \pi\lambda^2$
- Bonnesen's inequality $\Longrightarrow \lambda_1 \leq r \leq R \leq \lambda_2 \Longrightarrow$

$$\Longrightarrow \rho_1 = -\lambda_2 \leq -R \leq -r \leq -\lambda_1 = \rho_2$$

Conjecture (Teissier, Oda)

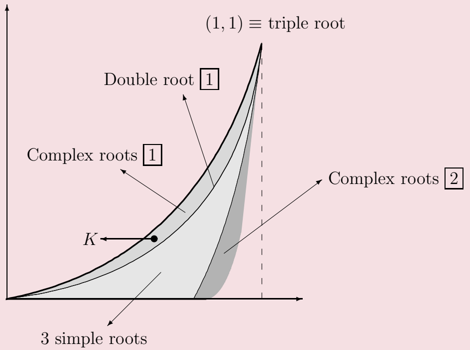
Let $K \in \mathcal{K}^n$. If $a_1 \leq \cdots \leq a_n$ are the real parts of the roots of the alternating Steiner polynomial $\sum_{i=0}^{n} \binom{n}{i} W_i(K)(-\rho)^i$, then

$$0 \leq a_1 \leq r \leq R \leq a_n$$

Teissier's conjecture in $\mathbb{R}^3$

- $K \equiv$ convex body in $\mathbb{R}^3$ whose Steiner polynomial has complex (or double real) roots of TYPE 1

$\rho :=$ complex or double real root $\qquad \rho_3 :=$ real root



Teissier's conjecture in $\mathbb{R}^3$

- $K \equiv$ convex body in $\mathbb{R}^3$ whose Steiner polynomial has complex (or double real) roots of $\mathrm{T}\textsc{ype}\ 1$

$\rho :=$ complex or double real root $\qquad \rho_3 :=$ real root

$$\rho_3 \leq -R \quad \text{and} \quad -r \leq \mathrm{Re}(\rho)?$$

Teissier's conjecture in $\mathbb{R}^3$

- $K \equiv$ convex body in $\mathbb{R}^3$ whose Steiner polynomial has complex (or double real) roots of $\text{TYPE } 1$

$$\rho := \text{complex or double real root} \qquad \rho_3 := \text{real root}$$

$$\rho_3 \leq -R \quad \text{and} \quad -r \leq \mathrm{Re}(\rho)?$$

- $\mathrm{Re}(\rho) \geq \rho_m = \dfrac{-M + \sqrt{M^2 - 4\pi S}}{4\pi}; \qquad \rho_m \geq -r?$

Teissier's conjecture in $\mathbb{R}^3$

- $K \equiv$ convex body in $\mathbb{R}^3$ whose Steiner polynomial has complex (or double real) roots of TYPE 1

$\rho :=$ complex or double real root $\qquad \rho_3 :=$ real root

$$\rho_3 \leq -R \quad \text{and} \quad -r \leq \mathrm{Re}(\rho)?$$

- $\mathrm{Re}(\rho) \geq \rho_m = \dfrac{-M + \sqrt{M^2 - 4\pi S}}{4\pi}; \qquad \rho_m \geq -r?$
- $\rho_m \geq -r \iff S - 2Mr + 4\pi r^2 \leq 0$

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies

How to fill the Blaschke diagram

The outer parallel bodies of the cap-bodies *fill* the part of Blaschke's diagram corresponding to complex and double real roots of TYPE 1

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies
- $K \in \mathcal{C}_1 \cup \mathcal{D}_1 \Longrightarrow$ there exist a cap-body K^c and $\rho > 0$, such that

$$K^c + \rho\mathbb{B}^3 \equiv K$$

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies
- $K \in \mathcal{C}_1 \cup \mathcal{D}_1 \Longrightarrow$ there exist a cap-body K^c and $\rho > 0$, such that

$$K^c + \rho\mathbb{B}^3 \equiv K$$

- If we choose $K^c + \rho\mathbb{B}^3$ with equal V, S, M, and *less* r than K

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies
- $K \in \mathcal{C}_1 \cup \mathcal{D}_1 \implies$ there exist a cap-body K^c and $\rho > 0$, such that

$$K^c + \rho\mathbb{B}^3 \equiv K$$

- If we choose $K^c + \rho\mathbb{B}^3$ with equal V, S, M, and *less* r than K
- $\varphi(K) \leq \varphi(K^c + \rho\mathbb{B}^3) = \varphi(K^c)$

Teissier's conjecture in $\mathbb{R}^3$: studying the functional $S - 2Mr + 4\pi r^2$

- $\varphi(K) = S - 2Mr + 4\pi r^2$ is invariant by outer parallel bodies
- $K \in \mathcal{C}_1 \cup \mathcal{D}_1 \implies$ there exist a cap-body K^c and $\rho > 0$, such that

$$K^c + \rho\mathbb{B}^3 \equiv K$$

- If we choose $K^c + \rho\mathbb{B}^3$ with equal V, S, M, and *less* r than K
- $\varphi(K) \leq \varphi(K^c + \rho\mathbb{B}^3) = \varphi(K^c)$
- $S - 2Mr + 4\pi r^2 \leq 0$ for the cap-bodies

The Steiner polynomial and its consequences on the Blaschke diagram

M. A. Hernández Cifre

(joint work with E. Saorín)

16 Mayo 2005