



ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG

Daniel Hug, Rolf Schneider, Ralph Schuster
Mathematisches Institut
Universität Freiburg

Valuations, Integral Geometry and Linear Dependences

Daniel Hug, Rolf Schneider, Ralph Schuster

Firenze

May 2005

Contents

1. Minkowski Functionals
2. Valuations and Integral Geometry
3. Tensor Valuations
4. Special Results
5. A General Approach
6. Linear Dependences

1. Minkowski Functionals

Minkowski functionals (intrinsic volumes, quermassintegrals) are basic functionals of convex bodies in \mathbb{R}^n :

1. Minkowski Functionals

Minkowski functionals (intrinsic volumes, quermassintegrals) are basic functionals of convex bodies in \mathbb{R}^n :

- *Coefficients of a Steiner formula.* Let $K \in \mathcal{K}^n$ and $\varepsilon > 0$:

$$\mathcal{H}^n(K + \varepsilon B^n) = \sum_{i=0}^n \varepsilon^i \kappa_{n-i} V_i(K),$$

where B^n denotes the unit ball in \mathbb{R}^n , κ_n its volume, and \mathcal{H}^n is the Hausdorff measure (volume).

- **Axiomatic characterization.** The functionals

$$V_0, V_1, \dots, V_{n-1}, V_n$$

are real-valued additive (valuations), continuous and motion invariant; V_k is homogeneous of degree k .

- **Axiomatic characterization.** The functionals

$$V_0, V_1, \dots, V_{n-1}, V_n$$

are real-valued additive (valuations), continuous and motion invariant; V_k is homogeneous of degree k .

Theorem. [Hadwiger] Let $\psi : \mathcal{K}^n \rightarrow \mathbb{R}$ be additive, continuous and motion invariant. Then there exist constants c_0, \dots, c_n such that

$$\psi = \sum_{i=0}^n c_i V_i.$$

- **Objects of integral geometry.** Let \mathcal{E}_k^n be the Grassmannian of k -flats in \mathbb{R}^n , let μ_k^n be a Haar measure on \mathcal{E}_k^n . Then, for $K \in \mathcal{K}^n$ and $0 \leq j \leq k \leq n$, we have the *Crofton formula*

$$\int_{\mathcal{E}_k^n} V_j(K \cap E) \mu_k^n(dE) = \alpha_{njk} V_{n+j-k}(K).$$

- **Objects of integral geometry.** Let \mathcal{E}_k^n be the Grassmannian of k -flats in \mathbb{R}^n , let μ_k^n be a Haar measure on \mathcal{E}_k^n . Then, for $K \in \mathcal{K}^n$ and $0 \leq j \leq k \leq n$, we have the *Crofton formula*

$$\int_{\mathcal{E}_k^n} V_j(K \cap E) \mu_k^n(dE) = \alpha_{njk} V_{n+j-k}(K).$$

Let $\mathbb{G}(n)$ be the motion group, and μ a Haar measure on $\mathbb{G}(n)$. For $K, L \in \mathcal{K}^n$ and $j \in \{0, \dots, n\}$, the *kinematic formula* states

$$\int_{\mathbb{G}(n)} V_j(K \cap gL) \mu(dg) = \sum_{k=j}^n \alpha_{njk} V_k(K) V_{n+j-k}(L).$$

2. Valuations and integral geometry

Results about valuations have been used for the proofs of integral geometric results.

Hadwiger's abstract integral geometric formula

Let $\varphi : \mathcal{K}^n \rightarrow \mathbb{R}$ be an additive, continuous function. Put

$$\varphi_{n-q}(K) := \int_{\mathcal{E}_q^n} \varphi(K \cap E) \mu_q^n(dE), \quad K \in \mathcal{K}^n.$$

2. Valuations and integral geometry

Results about valuations have been used for the proofs of integral geometric results.

Hadwiger's abstract integral geometric formula

Let $\varphi : \mathcal{K}^n \rightarrow \mathbb{R}$ be an additive, continuous function. Put

$$\varphi_{n-q}(K) := \int_{\mathcal{E}_q^n} \varphi(K \cap E) \mu_q^n(dE), \quad K \in \mathcal{K}^n.$$

Theorem. [Hadwiger] For any $K, L \in \mathcal{K}^n$ and φ as above,

$$\int_{\mathbb{G}(n)} \varphi(K \cap gL) \mu(dg) = \sum_{q=0}^n \varphi_{n-q}(K) V_q(L).$$

Hints to the literature

3. Tensor Valuations

- Schneider, Schneider & Hadwiger ('71, '72)
- McMullen ('97)
- Alesker ('99)
- Schneider, Schneider & Schuster ('00, '02, '04)
- Beisbart, Mecke ... ('00 - ?)

Background

- Space of symmetric tensors of rank r : \mathbb{T}^r ; \mathbb{T}
symmetric tensor product of $a, b \in \mathbb{T}$: ab
- **Tensor valuation**: $\varphi : \mathcal{K}^n \rightarrow \bigcup_{s=0}^r \mathbb{T}^s$

Background

- Space of symmetric tensors of rank r : \mathbb{T}^r ; \mathbb{T}
symmetric tensor product of $a, b \in \mathbb{T}$: ab
- **Tensor valuation**: $\varphi : \mathcal{K}^n \rightarrow \bigcup_{s=0}^r \mathbb{T}^s$
- **Translation covariance**: there are $\varphi_0, \dots, \varphi_r : \mathcal{K}^n \rightarrow \mathbb{T}$ such that, for all $K \in \mathcal{K}^n$ and $t \in \mathbb{R}^n$,

$$\varphi(K + t) = \sum_{j=0}^r \varphi_{r-j}(K) \frac{t^j}{j!}.$$

Background

- Space of symmetric tensors of rank r : \mathbb{T}^r ; \mathbb{T} symmetric tensor product of $a, b \in \mathbb{T}$: ab
- **Tensor valuation**: $\varphi : \mathcal{K}^n \rightarrow \bigcup_{s=0}^r \mathbb{T}^s$
- **Translation covariance**: there are $\varphi_0, \dots, \varphi_r : \mathcal{K}^n \rightarrow \mathbb{T}$ such that, for all $K \in \mathcal{K}^n$ and $t \in \mathbb{R}^n$,

$$\varphi(K + t) = \sum_{j=0}^r \varphi_{r-j}(K) \frac{t^j}{j!}.$$

- **Rotation covariance**: for all $K \in \mathcal{K}^n$ and $U \in \mathbb{O}(n)$,

$$\varphi(UK) = U\varphi(K).$$

Background

- Space of symmetric tensors of rank r : \mathbb{T}^r ; \mathbb{T}
symmetric tensor product of $a, b \in \mathbb{T}$: ab
- **Tensor valuation**: $\varphi : \mathcal{K}^n \rightarrow \bigcup_{s=0}^r \mathbb{T}^s$
- **Translation covariance**: there are $\varphi_0, \dots, \varphi_r : \mathcal{K}^n \rightarrow \mathbb{T}$ such that, for all $K \in \mathcal{K}^n$ and $t \in \mathbb{R}^n$,

$$\varphi(K + t) = \sum_{j=0}^r \varphi_{r-j}(K) \frac{t^j}{j!}.$$

- **Rotation covariance**: for all $K \in \mathcal{K}^n$ and $U \in \mathbb{O}(n)$,

$$\varphi(UK) = U\varphi(K).$$

- **Isometry covariance**

A detour: Support measures

Fix $K \in \mathcal{K}^n$, $\varepsilon > 0$, $\eta \in \Sigma := \mathbb{R}^n \times \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n \setminus K$.

- Metric projection: $p(K, x)$
- Direction vector: $u(K, x) := (x - p(K, x)) / \|x - p(K, x)\|$
- Local parallel set:

$$M_\varepsilon(K, \eta) := \{x \in (K + \varepsilon B^n) \setminus K : (p(K, x), u(K, x)) \in \eta\}$$

A detour: Support measures

Fix $K \in \mathcal{K}^n$, $\varepsilon > 0$, $\eta \in \Sigma := \mathbb{R}^n \times \mathbb{S}^{n-1}$, $x \in \mathbb{R}^n \setminus K$.

- Metric projection: $p(K, x)$
- Direction vector: $u(K, x) := (x - p(K, x)) / \|x - p(K, x)\|$
- Local parallel set:

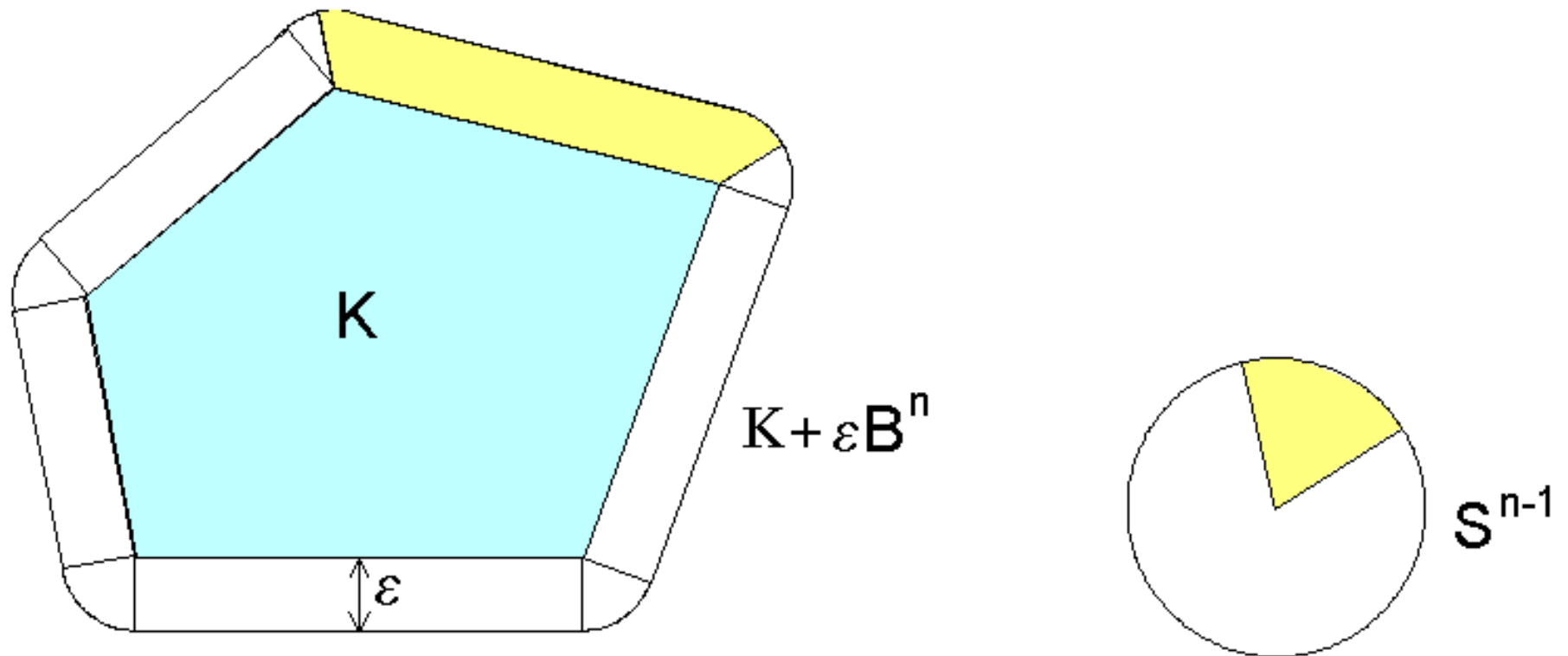
$$M_\varepsilon(K, \eta) := \{x \in (K + \varepsilon B^n) \setminus K : (p(K, x), u(K, x)) \in \eta\}$$

- Local Steiner formula:

$$\mathcal{H}^n(M_\varepsilon(K, \eta)) = \sum_{i=0}^{n-1} \varepsilon^{n-i} \kappa_{n-i} \Lambda_i(K, \eta);$$

$$\Lambda_n(K, \cdot) := \lambda_{n \perp} K \quad \text{on } \mathbb{R}^n$$

A local parallel set:



Basic Examples. $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$, $0 \leq k \leq n - 1$:

$$\Phi_{k,r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma} x^r u^s \Lambda_k(K, d(x, u))$$

and

$$\Phi_{n,r,0}(K) := \frac{1}{r!} \int_{\mathbb{R}^n} x^r \Lambda_n(K, dx);$$

in all other cases, $\Phi_{k,r,s} := 0$. Further, $Q \in \mathbb{T}^2$ is defined by $Q(x, y) := \langle x, y \rangle$.

Basic Examples. $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$, $0 \leq k \leq n - 1$:

$$\Phi_{k,r,s}(K) := \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma} x^r u^s \Lambda_k(K, d(x, u))$$

and

$$\Phi_{n,r,0}(K) := \frac{1}{r!} \int_{\mathbb{R}^n} x^r \Lambda_n(K, dx);$$

in all other cases, $\Phi_{k,r,s} := 0$. Further, $Q \in \mathbb{T}^2$ is defined by $Q(x, y) := \langle x, y \rangle$.

Theorem. [Alesker] *Let $p \in \mathbb{N}_0$, and let $\varphi : \mathcal{K}^n \rightarrow \mathbb{T}^p$ be a continuous, isometry covariant valuation. Then φ is a linear combination, with constant real coefficients, of the basic valuations $Q^m \Phi_{k,r,s}$, where $m, k, r, s \in \mathbb{N}_0$ are such that $2m + r + s = p$.*

Example. For $p = 2$ one can show that the tensor valuations

- $QV_j, j = 0, \dots, n,$
- $\Phi_{j,2,0}, j = 0, \dots, n,$ and
- $\Phi_{j,0,2}, j = 1, \dots, n - 1.$

form a basis ...

Example. For $p = 2$ one can show that the tensor valuations

- $QV_j, j = 0, \dots, n,$
- $\Phi_{j,2,0}, j = 0, \dots, n,$ and
- $\Phi_{j,0,2}, j = 1, \dots, n - 1.$

form a basis ...

The special linear relationships

$$2\pi \sum_s s \Phi_{k-r+s, r-s, s} = Q \sum_s \Phi_{k-r+s, r-s, s-2},$$

for $k, r \in \mathbb{N}_0$, have been found by McMullen.

Two main tasks to be discussed here:

1. Provide a complete system of integral geometric formulae for tensor valuations. Problem: linear dependences

Two main problems to be discussed here:

1. Provide a complete system of integral geometric formulae for tensor valuations.
2. Find all linear relationships between the basic tensor valuations, determine the dimension of the corresponding vector space.

4. Special Results

The case $s = 0$:

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,0}(K \cap E) \mu_k^n(dE) = \alpha_{njk} \Phi_{n+j-k,r,0}(K),$$

$$\int_{\mathbb{G}(n)} \Phi_{j,r,0}(K \cap gL) \mu(dg) = \sum_{k=j}^n \alpha_{njk} \Phi_{k,r,0}(K) V_{n+j-k}(L).$$

4. Special Results

The case $s = 0$:

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,0}(K \cap E) \mu_k^n(dE) = \alpha_{njk} \Phi_{n+j-k,r,0}(K),$$

$$\int_{\mathbb{G}(n)} \Phi_{j,r,0}(K \cap gL) \mu(dg) = \sum_{k=j}^n \alpha_{njk} \Phi_{k,r,0}(K) V_{n+j-k}(L).$$

The case $j = n - 1$:

$$\int_{\mathcal{E}_{n-1}^n} \Phi_{n-1,r,s}(K \cap E) \mu_{n-1}^n(dE) = \delta(n, s) Q^{\frac{s}{2}} \Phi_{n,r,0}(K),$$

$$\int_{\mathbb{G}(n)} \Phi_{n-1,r,s}(K \cap gL) \mu(dg) = \Phi_{n-1,r,s}(K) V_n(L) + \delta(n, s) Q^{\frac{s}{2}} \Phi_{n,r,0}(K) V_{n-1}(L).$$

The case $j = n - 2$:

$$\int_{\mathcal{E}_{n-1}^n} \Phi_{n-2,r,s}(K \cap E) \mu_{n-1}^n(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \alpha(n, s, m) Q^m \Phi_{n-1,r,s-2m}(K)$$

and

$$\int_{\mathcal{E}_{n-2}^n} \Phi_{n-2,r,s}(K \cap E) \mu_{n-2}^n(dE) = \beta(n, s) Q^{\frac{s}{2}} \Phi_{n,r,0}(K).$$

Further results follow from McMullen's relationships.

5. A General Approach. Let $r, s \in \mathbb{N}_0$, $0 \leq j \leq k \leq n - 1$.

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) \mu_k^n(dE)$$

Theorem. For $K \in \mathcal{K}^n$, $r, s \in \mathbb{N}_0$ and $0 \leq k \leq n - 1$,

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) = \begin{cases} 0, & \text{if } s \text{ is odd,} \\ \alpha Q^{\frac{s}{2}} \Phi_{n,r,0}(K), & \text{if } s \text{ is even.} \end{cases}$$

Theorem. For $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leq j < k \leq n - 1$,

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) d\mu_k^n(E)$$

$$= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) +$$

with explicitly known constants $\chi_{n,j,k,s,z}^{(1)}$ and $\chi_{n,j,k,s,z}^{(2)}$ independent of r .

Theorem. For $K \in \mathcal{K}^n$ and $k, j, r, s \in \mathbb{N}_0$ with $0 \leq j < k \leq n - 1$,

$$\begin{aligned} & \int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) d\mu_k^n(E) \\ &= \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \chi_{n,j,k,s,z}^{(1)} Q^z \Phi_{n+j-k,r,s-2z}(K) + \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \chi_{n,j,k,s,z}^{(2)} Q^z \\ & \quad \times \sum_{l=0}^{s-2z-1} (2\pi l \Phi_{n+j-k-s+2z+l,r+s-2z-l,l}(K) \\ & \quad - Q \Phi_{n+j-k-s+2z+l,r+s-2z-l,l-2}(K)) \end{aligned}$$

with explicitly known constants $\chi_{n,j,k,s,z}^{(1)}$ and $\chi_{n,j,k,s,z}^{(2)}$ independent of r .

The first step. Let $r, s \in \mathbb{N}_0$, $0 \leq j \leq k \leq n - 1$.

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) \mu_k^n(dE)$$

The first step. Let $r, s \in \mathbb{N}_0$, $0 \leq j \leq k \leq n - 1$.

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) \mu_k^n(dE)$$

For $L \in \mathcal{L}_k^n$, $t \in L^\perp$ and $L_t := L + t$ (McMullen):

$$\Phi_{j,r,s}(K \cap L_t) = \sum_{m \geq 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi_{j,r,s-2m}^{(L_t)}(K \cap L_t).$$

The first step. Let $r, s \in \mathbb{N}_0$, $0 \leq j \leq k \leq n - 1$.

$$\int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) \mu_k^n(dE)$$

For $L \in \mathcal{L}_k^n$, $t \in L^\perp$ and $L_t := L + t$:

$$\Phi_{j,r,s}(K \cap L_t) = \sum_{m \geq 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi_{j,r,s-2m}^{(L_t)}(K \cap L_t).$$

Thus

$$\begin{aligned} \int_{\mathcal{E}_k^n} \Phi_{j,r,s}(K \cap E) \mu_k^n(dE) &= \int_{\mathcal{L}_k^n} \int_{L^\perp} \Phi_{j,r,s}(K \cap L_t) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \\ &= \sum_{m \geq 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \int_{\mathcal{L}_k^n} \int_{L^\perp} \Phi_{j,r,s-2m}^{(L_t)}(K \cap L_t) \mathcal{H}^{n-k}(dt) \nu_k^n(dL). \end{aligned}$$

Case 1: $j = k$

$$\Phi_{k,r,s-2m}^{(L_t)}(K \cap L_t) = 0 \quad \text{if } s - 2m \neq 0.$$

Case 1: $j = k$

$$\Phi_{k,r,s-2m}^{(L_t)}(K \cap L_t) = 0 \quad \text{if } s - 2m \neq 0.$$

If s is odd,

$$\Phi_{k,r,s}(K \cap L_t) = 0,$$

and thus

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) = 0.$$

Case 1: $j = k$

$$\Phi_{k,r,s-2m}^{(L_t)}(K \cap L_t) = 0 \quad \text{if } s - 2m \neq 0.$$

If s is odd,

$$\Phi_{k,r,s}(K \cap L_t) = 0,$$

and thus

$$\int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) = 0.$$

If s is even,

$$\Phi_{k,r,s}(K \cap L_t) = \frac{Q(L^\perp)^{\frac{s}{2}}}{(4\pi)^{\frac{s}{2}} (\frac{s}{2})! r!} \int_{K \cap L_t} x^r \mathcal{H}^k(dx).$$

Hence, if s is even,

$$\begin{aligned} & \int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) \\ &= \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)! r!} \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \int_{L^\perp} \int_{K \cap L_t} x^r \mathcal{H}^k(dx) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \end{aligned}$$

Hence, if s is even,

$$\begin{aligned}
& \int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)! r!} \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \int_{L^\perp} \int_{K \cap L_t} x^r \mathcal{H}^k(dx) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)!} \Phi_{n,r,0}(K) \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \nu_k^n(dL)
\end{aligned}$$

Hence, if s is even,

$$\begin{aligned}
& \int_{\mathcal{E}_k^n} \Phi_{k,r,s}(K \cap E) \mu_k^n(dE) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)! r!} \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \int_{L^\perp} \int_{K \cap L_t} x^r \mathcal{H}^k(dx) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \\
&= \frac{1}{(4\pi)^{\frac{s}{2}} \left(\frac{s}{2}\right)!} \Phi_{n,r,0}(K) \int_{\mathcal{L}_k^n} Q(L^\perp)^{\frac{s}{2}} \nu_k^n(dL) \\
&= \alpha Q^{\frac{s}{2}} \Phi_{n,r,0}(K),
\end{aligned}$$

where α is explicitly known.

Lemma. For $m \in \mathbb{N}_0$ and $k \in \{1, \dots, n\}$,

$$\int_{\mathcal{L}_k^n} Q(L)^m \nu_k^n(dL) = \frac{\Gamma(\frac{k}{2} + m)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2} + m)\Gamma(\frac{k}{2})} Q^m.$$

Case 2: $0 \leq j \leq k - 1, P \in \mathcal{P}^n$.

$$\begin{aligned} \Phi_{j,r,s}(P \cap L_t) &= \sum_{m \geq 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi_{j,r,s-2m}^{(L_t)}(P \cap L_t) \\ &= \sum_{m \geq 0} Q(L^\perp)^m \frac{\alpha_{j,k,s,m} \omega_{k-j}}{r!} \int_{L_t \times (\mathbb{S}^{n-1} \cap L)} x^r u^{s-2m} \Lambda_j^{(L_t)}(P \cap L_t, d(x, u)), \end{aligned}$$

Case 2: $0 \leq j \leq k - 1$, $P \in \mathcal{P}^n$.

$$\begin{aligned} \Phi_{j,r,s}(P \cap L_t) &= \sum_{m \geq 0} \frac{Q(L^\perp)^m}{(4\pi)^m m!} \Phi_{j,r,s-2m}^{(L_t)}(P \cap L_t) \\ &= \sum_{m \geq 0} Q(L^\perp)^m \frac{\alpha_{j,k,s,m} \omega_{k-j}}{r!} \int_{L_t \times (\mathbb{S}^{n-1} \cap L)} x^r u^{s-2m} \Lambda_j^{(L_t)}(P \cap L_t, d(x, u)), \end{aligned}$$

We need the **translative integral (Crofton) formula** (Rataj '99):

$$\begin{aligned} &\int_{L^\perp} \int_{L_t \times (\mathbb{S}^{n-1} \cap L)} g(x, v) \Lambda_j^{(L_t)}(P \cap L_t, d(x, v)) \mathcal{H}^{n-k}(dt) \\ &= \frac{1}{\omega_{k-j}} \sum_{F \in \mathcal{F}^{n+j-k}(P)} \int_{F \times (N(P,F) \cap \mathbb{S}^{n-1})} g(x, \pi_L(u)) \|p_L(u)\|^{j-k} \\ &\quad \times [F, L]^2 (\mathcal{H}^{n+j-k} \otimes \mathcal{H}^{k-j-1})(d(x, u)). \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}
& \int_{\mathcal{L}_k^n} \int_{L^\perp} \Phi_{j,r,s}(P \cap L_t) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \\
&= \sum_{m \geq 0} \sum_{F \in \mathcal{F}^{n+j-k}(P)} \alpha_{j,k,s,m} \frac{1}{r!} \int_F x^r \mathcal{H}^{n+j-k}(dx) \int_{N(P,F) \cap \mathbb{S}^{n-1}} \\
&\quad \times \int_{\mathcal{L}_k^n} Q(L^\perp)^m \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F, L]^2 \nu_k^n(dL).
\end{aligned}$$

Proposition.

$$\begin{aligned}
& \int_{\mathcal{L}_k^n} Q(L^\perp)^m \pi_L(u)^{s-2m} \|p_L(u)\|^{j-k} [F, L]^2 \nu_k^n(dL) \\
&= \beta_{n,j,k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \zeta_{n,j,k,s,z,m}^{(1)} Q^z u^{s-2z} \\
&+ \beta_{n,j,k} \frac{k-n}{n+j-k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \zeta_{n,j,k,s,z,m}^{(2)} Q^z u^{s-2z-2} Q(F).
\end{aligned}$$

Some constants:

$$\beta_{n,j,k} := \frac{(k-1)!(n+j-k)! \Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2})}{\sqrt{\pi}j!(n-1)! \Gamma(\frac{k}{2})\Gamma(\frac{k+1}{2})},$$

$$\gamma_{n,k,l,p,q}^{(1)} := \sum_{y=0}^q (-1)^{l+y} \binom{q}{y} \frac{\Gamma(\frac{k-1}{2} + l - p + y)}{\Gamma(\frac{n+1}{2} + l - p + y)} \left(\frac{k-1}{2} + l - p + y\right),$$

$$\gamma_{n,k,l,p,q}^{(2)} := \sum_{y=0}^q (-1)^{l+y} \binom{q}{y} \frac{\Gamma(\frac{k-1}{2} + l - p + y)}{\Gamma(\frac{n+1}{2} + l - p + y)} (l - p + y)$$

with $\gamma_{n,k,l,p,q}^{(2)} = 0$ if $l - p + q = 0$.

More constants:

$$\zeta_{n,j,k,s,z,m}^{(1)} := \sum_{l=\max\{0,m-z\}}^m \sum_{p=0}^l \sum_{q=\max\{0,z-m+p\}}^{\lfloor \frac{s}{2} \rfloor - m + p} (-1)^{m-p+q-z} \gamma_{n,k,l,p,q}^{(1)} \binom{m}{l} \binom{l}{p} \\ \times \binom{s-2m+2p}{2q} \binom{l-p+q}{z-m+l} \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1) \Gamma(q + \frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)}$$

and

$$\zeta_{n,j,k,s,z,m}^{(2)} := \sum_{l=\max\{0,m-z\}}^m \sum_{p=0}^l \sum_{q=\max\{0,z-m+p+1\}}^{\lfloor \frac{s}{2} \rfloor - m + p} (-1)^{m-p+q-z-1} \gamma_{n,k,l,p,q}^{(2)} \binom{m}{l} \binom{l}{p} \\ \times \binom{s-2m+2p}{2q} \binom{l-p+q-1}{z-m+l} \frac{\Gamma(\frac{s+j}{2} - m + p - q + 1) \Gamma(q + \frac{1}{2})}{\Gamma(\frac{s+n-k+j}{2} - m + p + 1)}.$$

Some combinatorial identities of the form:

Lemma. For $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $k \in \{1, \dots, n-1\}$ and $a \geq 0$,

$$\sum_{i=0}^m \binom{m}{i} \Gamma\left(\frac{k+a}{2} + m - i\right) \frac{\Gamma\left(\frac{n-k}{2} + i\right) \Gamma\left(\frac{a}{2} + 1\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right) \Gamma\left(\frac{a}{2} + 1 - i\right) \Gamma\left(\frac{n}{2} + i\right)} (-1)^i$$

$$= \frac{\Gamma\left(\frac{k}{2} + m\right) \Gamma\left(\frac{n+a}{2} + m\right) \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k+a}{2}\right)}{\Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{n+a}{2}\right) \Gamma\left(\frac{n}{2} + m\right)}.$$

Putting things together and by some additional calculations, we get

$$\begin{aligned}
& \int_{\mathcal{L}_k^n} \int_{L^\perp} \Phi_{j,r,s}(P \cap L_t) \mathcal{H}^{n-k}(dt) \nu_k^n(dL) \\
&= \beta_{n,j,k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \xi_{n,j,k,s,z}^{(1)} (s-2z)! \omega_{s-2z-j+k} Q^z \Phi_{n+j-k,r,s-2z}(P) \\
&+ \beta_{n,j,k} \frac{k-n}{n+j-k} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor - 1} \xi_{n,j,k,s,z}^{(2)} (s-2z-2)! \omega_{s-2z-2-j+k} \\
&\times Q^z \sum_{F \in \mathcal{F}^{n+j-k}(P)} Q(F) \Upsilon_r(F) \Theta_{s-2z-2}(P, F),
\end{aligned}$$

where, for a k -face F of P ,

$$\Upsilon_r(F) := \frac{1}{r!} \int_F x^r \mathcal{H}^k(dx)$$

and

$$\Theta_s(P, F) := \frac{1}{s!} \int_{N(P, F)} x^s e^{-\pi \|x\|^2} \mathcal{H}^{n-k}(dx).$$

Hence

$$\Phi_{k,r,s}(P) = \sum_{F \in \mathcal{F}^k(P)} \Upsilon_r(F) \Theta_s(P, F).$$

The final step: a lemma due to McMullen ...

$$\begin{aligned}
 & \sum_{F \in \mathcal{F}^{n+j-k}(P)} Q(F) \Upsilon_r(F) \Theta_{s-2z-2}(P, F) \\
 &= Q \Phi_{n+j-k, r, s-2z-2}(P) - 2\pi(s-2z) \Phi_{n+j-k, r, s-2z}(P) \\
 & \quad + \sum_{G \in \mathcal{F}^{n+j-k+1}(P)} Q(G) \Upsilon_{r-1}(G) \Theta_{s-2z-1}(P, G).
 \end{aligned}$$

... can be iterated to give

$$\begin{aligned}
& \sum_{F \in \mathcal{F}^{n+j-k}(P)} Q(F) \Upsilon_r(F) \Theta_{s-2z-2}(P, F) \\
&= \sum_{l \geq s-2z} Q \Phi_{n+j-k-s+2z+l, r+s-2z-l, l-2}(P) \\
&\quad - 2\pi \sum_{l \geq s-2z} l \Phi_{n+j-k-s+2z+l, r+s-2z-l, l}(P).
\end{aligned}$$

6. Linear Dependences

Theorem. For $k, r \in \mathbb{N}_0$,

$$2\pi \sum_s s \Phi_{k-r+s, r-s, s} - Q \sum_s \Phi_{k-r+s, r-s, s-2} = 0. \quad (*)$$

Any linear dependence among those tensor valuations $Q^l \Phi_{k,r,s}$, which do not vanish trivially, can be obtained by multiplying by Q relations of the form $()$ and by taking linear combinations of relations obtained in this way.*

6. Linear Dependences

Theorem. For $k, r \in \mathbb{N}_0$,

$$2\pi \sum_s s \Phi_{k-r+s, r-s, s} - Q \sum_s \Phi_{k-r+s, r-s, s-2} = 0. \quad (*)$$

Any linear dependence among those tensor valuations $Q^l \Phi_{k,r,s}$, which do not vanish trivially, can be obtained by multiplying by Q relations of the form $()$ and by taking linear combinations of relations obtained in this way.*

Proof. Consider, for $r \in \mathbb{N}_0$ and $k = 0, \dots, n+r$,

$$G_{k,r} := \text{lin}\{Q^{l'} \Phi_{k',r',s'} : k' + r' = k, r' + s' + 2l' = r\}$$

“Tensors of rank r , homogeneous of degree k ”

Proof. Induction wrt r . Main case: $r \geq 2, k \geq 1$.

Assume

$$\sum_{l,s} \alpha_{l,s} Q^l \Phi_{k-r+s+2l, r-s-2l, s} = 0. \quad (+)$$

Proof. Induction wrt r . Main case: $r \geq 2$, $k \geq 1$.

Assume

$$\sum_{l,s} \alpha_{l,s} Q^l \Phi_{k-r+s+2l,r-s-2l,s} = 0. \quad (+)$$

Substitute $K + t$, for $t \in \mathbb{R}^n$, and use translation covariance:

$$\sum_{l,s} \alpha_{l,s} Q^l \Phi_{k-r+s+2l,r-s-2l-1,s} = 0. \quad (\text{in } G_{k-1,r-1})$$

Proof. Induction wrt r . Main case: $r \geq 2$, $k \geq 1$.

Assume

$$\sum_{l,s} \alpha_{l,s} Q^l \Phi_{k-r+s+2l,r-s-2l,s} = 0. \quad (+)$$

Substitute $K + t$, for $t \in \mathbb{R}^n$, and use translation covariance:

$$\sum_{l,s} \alpha_{l,s} Q^l \Phi_{k-r+s+2l,r-s-2l-1,s} = 0. \quad (\text{in } G_{k-1,r-1})$$

Via the induction hypothesis, (+) is equivalent to

$$\alpha 2\pi \sum_{s=1}^{r-1} s \Phi_{k-r+s,r-s,s} + \alpha 0_r \Phi_{k,0,r} + \sum_{l \geq 1, s \geq 0} \alpha_{l,s} Q^l \Phi_{k-r+s+2l,r-s-2l,s} = 0$$

or to

$$(\alpha_{0,r} - 2\pi r\alpha)\Phi_{k,0,r} + \sum_{l \geq 1, s \geq 0} \tilde{\alpha}_{l,s} Q^l \Phi_{k-r+s+2l, r-s-2l, s} = 0.$$

This is of the form

$$(\alpha_{0,r} - 2\pi r\alpha)\Phi_{k,0,r} = Qv, \quad v \in \mathbb{T}^{r-2}.$$

or to

$$(\alpha_{0,r} - 2\pi r\alpha)\Phi_{k,0,r} + \sum_{l \geq 1, s \geq 0} \tilde{\alpha}_{l,s} Q^l \Phi_{k-r+s+2l, r-s-2l, s} = 0.$$

This is of the form

$$(\alpha_{0,r} - 2\pi r\alpha)\Phi_{k,0,r} = Qv, \quad v \in \mathbb{T}^{r-2}.$$

Lemma. *For all $s, k \in \mathbb{N}$ with $1 \leq k \leq n - 1$ and $s \geq 2$, there exists a convex body $K \in \mathcal{K}^n$ such that*

$$\Phi_{k,0,s}(K) \neq Qv$$

for all $v \in \mathbb{T}^{s-2}$.

Theorem. Let $r \in \mathbb{N}_0$, $n \geq 1$ and $0 \leq k \leq n + r$. Put

$$j_0 := \min \left\{ \left\lfloor \frac{n + r - k}{2} \right\rfloor, \left\lfloor \frac{r}{2} \right\rfloor \right\}, \quad j_1 := \max \left\{ -1, \left\lfloor \frac{r - k}{2} \right\rfloor \right\}.$$

Then

$$\dim(G_{k,r}) = j_0(\min\{1, n - k\} + r - j_0) + 1 - (j_1 + 1)(r - k - j_1).$$

Happy

Birthday

Dear

Rolf

Schneider