

A Classification of $SL(n)$ invariant valuations

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Hadwiger Characterization Theorem (1952):

A functional $\mu : \mathcal{K}^n \rightarrow \mathbb{R}$ is a continuous rigid motion invariant valuation

\iff

$\exists c_0, c_1, \dots, c_n \in \mathbb{R} :$

$$\mu(K) = c_0 V_0(K) + \dots + c_n V_n(K)$$

$\forall K \in \mathcal{K}^n$

\mathcal{K}^n convex bodies in \mathbb{R}^n

$\mu : \mathcal{K}^n \rightarrow \mathbb{R}$ valuation $\iff \forall K, L, K \cup L \in \mathcal{K}^n$

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

$V_0(K), \dots, V_n(K)$ intrinsic volumes of K

$SL(n)$ invariant valuations I

- V_0 Euler Characteristic
- V_n Volume
- Ω affine surface area

$$\Omega(K) = \int_{\partial K} \kappa(K, x)^{\frac{1}{n+1}} dx$$

∂K smooth:

Blaschke: Vorlesungen über Differentialgeometrie II (1923)

$K \in \mathcal{K}^n$ (general):

Leichtweiß 1986, Schütt-Werner 1990,
Lutwak 1991, Dolzmann-Hug 1995

Properties of $\Omega : \mathcal{K}^n \rightarrow \mathbb{R}$:

- Ω is equi-affine invariant:

$$\Omega(\phi K) = \Omega(K) \quad \forall \phi \in \text{SL}(n)$$

$$\Omega(K + x) = \Omega(K) \quad \forall x \in \mathbb{R}^n$$

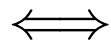
- Ω is upper semicontinuous (Lutwak 1991):

$$\Omega(K) \geq \limsup_{j \rightarrow \infty} \Omega(K_j) \quad K_j \rightarrow K$$

- Ω is a valuation

Theorem (L. 1999; L.-Reitzner 1999):

A functional $\mu : \mathcal{K}^n \rightarrow \mathbb{R}$ is an upper semicontinuous and equi-affine invariant valuation



$\exists c_0, c_1 \in \mathbb{R}, c_2 \geq 0:$

$$\mu(K) = c_0 + c_1 V_n(K) + c_2 \Omega(K)$$

$\forall K \in \mathcal{K}^n$

SL(n) invariant valuations II

$\Omega_n : \mathcal{K}_0^n \rightarrow \mathbb{R}$ centro-affine surface area

$$\Omega_n(K) = \int_{\partial K} \kappa_0(K, x)^{\frac{1}{2}} d\sigma_K(x)$$

\mathcal{K}_0^n convex bodies in \mathbb{R}^n , $0 \in \text{int } K$

$$\kappa_0(K, x) = \frac{\kappa(K, x)}{\langle x, u(K, x) \rangle^{n+1}}$$

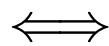
$u(K, x)$ outer unit normal vector of K at x

$$d\sigma_K(x) = \langle x, u(K, x) \rangle dx \quad \text{cone measure}$$

Ω_n is GL(n) invariant

Theorem (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \rightarrow \mathbb{R}$ is an upper semi-continuous and $SL(n)$ -invariant valuation which vanishes on polytopes



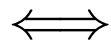
$\exists f : [0, \infty) \rightarrow [0, \infty]$, concave,
 $\lim_{t \rightarrow 0} f(t) = 0$, $\lim_{t \rightarrow \infty} f(t)/t = 0$:

$$\mu(K) = \int_{\partial K} f(\kappa_0(K, x)) d\sigma_K(x)$$

$\forall K \in \mathcal{K}_0^n$

Corollary (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \rightarrow \mathbb{R}$ is an upper semi-continuous and $GL(n)$ invariant valuation



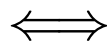
$\exists c_0 \in \mathbb{R}, c_1 \geq 0 :$

$$\mu(K) = c_0 + c_1 \Omega_n(K)$$

$\forall K \in \mathcal{K}^n$

Theorem (L. 2002):

A functional $\mu : \mathcal{P}_0^n \rightarrow \mathbb{R}$ is a measurable and $SL(n)$ invariant valuation which is homogeneous of degree q



$\exists c \in \mathbb{R} :$

$$\mu(P) = \begin{cases} c & q = 0 \\ cV(P) & q = n \\ cV(P^*) & q = -n \\ 0 & \text{otherwise} \end{cases}$$

$\forall P \in \mathcal{P}_0^n$

\mathcal{P}_0^n convex polytopes in \mathbb{R}^n , $0 \in \text{int } P$

P^* polar body of P

Theorem (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \rightarrow \mathbb{R}$ is an upper semicontinuous and $SL(n)$ invariant valuation that is homogeneous of degree q

\iff

$\exists c_0 \in \mathbb{R}, c_1 \geq 0 :$

$$\mu(K) = \begin{cases} c_0 + c_1 \Omega_n(K) & q = 0 \\ c_1 \Omega_p(K) & -n < q < n, q \neq 0 \\ c_0 V(K) & q = n \\ c_0 V(K^*) & q = -n \\ 0 & \text{otherwise} \end{cases}$$

$\forall K \in \mathcal{K}_0^n, q = n(n-p)/(n+p)$

L_p -affine surface area

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K, x)^{\frac{p}{n+p}} d\sigma_K(x)$$

Definition:

Lutwak 1996 ($p \geq 1$), Hug 1996 ($p > 0$)

Applications:

Lutwak 1996, Lutwak-Oliker 1995, Meyer-Werner 2000, Schütt-Werner 2004