A Classification of SL(n) invariant valuations

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Hadwiger Characterization Theorem (1952):

A functional $\mu : \mathcal{K}^n \to \mathbb{R}$ is a continuous rigid motion invariant valuation

$$\Longleftrightarrow$$

$\exists \;\; c_0, c_1, \ldots, c_n \in \mathbb{R}:$

$$\mu(K) = c_0\, V_0(K) + \ldots + c_n\, V_n(K)$$

$\forall\, K \in \mathcal{K}^n$

$\mathcal{K}^n$ convex bodies in $\mathbb{R}^n$

$\mu : \mathcal{K}^n \to \mathbb{R}$ valuation $\quad \Leftrightarrow \quad \forall\, K, L, K \cup L \in \mathcal{K}^n$

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

$V_0(K), \ldots, V_n(K)$ intrinsic volumes of K

SL(n) **invariant valuations I**

- V_0 Euler Characteristic
- V_n Volume
- Ω affine surface area

$$\Omega(K) = \int\limits_{\partial K} \kappa(K,x)^{\frac{1}{n+1}} dx$$

∂K smooth:
Blaschke: Vorlesungen über Differentialgeometrie II (1923)

$K \in \mathcal{K}^n$ (general):
Leichtweiß 1986, Schütt-Werner 1990, Lutwak 1991, Dolzmann-Hug 1995

Properties of $\Omega : \mathcal{K}^n \to \mathbb{R}$:

- Ω is equi-affine invariant:

$$\Omega(\phi K) = \Omega(K) \qquad \forall\, \phi \in \mathrm{SL}(n)$$

$$\Omega(K + x) = \Omega(K) \qquad \forall\, x \in \mathbb{R}^n$$

- Ω is upper semicontinuous (Lutwak 1991):

$$\Omega(K) \geq \limsup_{j\to\infty} \Omega(K_j) \qquad K_j \to K$$

- Ω is a valuation

Theorem (L. 1999; L.-Reitzner 1999):

A functional $\mu : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous and equi-affine invariant valuation

$$\Longleftrightarrow$$

$\exists \;\; c_0, c_1 \in \mathbb{R}, c_2 \geq 0:$

$$\mu(K) = c_0 + c_1 V_n(K) + c_2 \Omega(K)$$

$\forall K \in \mathcal{K}^n$

SL(n) **invariant valuations II**

$\Omega_n : \mathcal{K}_0^n \to \mathbb{R}$ centro-affine surface area

$$\Omega_n(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}} \, d\sigma_K(x)$$

$\mathcal{K}_0^n$ convex bodies in $\mathbb{R}^n$, $0 \in \operatorname{int} K$

$$\kappa_0(K,x) = \frac{\kappa(K,x)}{\langle x, u(K,x)\rangle^{n+1}}$$

$u(K,x)$ outer unit normal vector of K at x

$d\sigma_K(x) = \langle x, u(K,x)\rangle \, dx$ cone measure

Ω_n is GL(n) invariant

Theorem (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semi-continuous and SL(n)-invariant valuation which vanishes on polytopes

$$\Longleftrightarrow$$

$\exists\, f : [0,\infty) \to [0,\infty]$, concave,
$\lim_{t\to 0} f(t) = 0$, $\lim_{t\to\infty} f(t)/t = 0$:

$$\mu(K) = \int_{\partial K} f(\kappa_0(K,x))\, d\sigma_K(x)$$

$\forall\, K \in \mathcal{K}_0^n$

Corollary (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semi-continuous and GL(n) invariant valuation

$$\Longleftrightarrow$$

$\exists \;\; c_0 \in \mathbb{R}, c_1 \geq 0:$

$$\mu(K) = c_0 + c_1 \, \Omega_n(K)$$

$\forall \, K \in \mathcal{K}^n$

Theorem (L. 2002):

A functional $\mu : \mathcal{P}_0^n \to \mathbb{R}$ is a measurable and $\mathrm{SL}(n)$ invariant valuation which is homogeneous of degree q

$$\Longleftrightarrow$$

$\exists\, c \in \mathbb{R}$:

$$\mu(P) = \begin{cases} c & q = 0 \\ c\,V(P) & q = n \\ c\,V(P^*) & q = -n \\ 0 & \text{otherwise} \end{cases}$$

$\forall\, P \in \mathcal{P}_0^n$

$\mathcal{P}_0^n$ convex polytopes in $\mathbb{R}^n$, $0 \in \operatorname{int} P$

P^* polar body of P

Theorem (L.-Reitzner 2005):

A functional $\mu : \mathcal{K}_0^n \to \mathbb{R}$ is an upper semicontinuous and SL(n) invariant valuation that is homogeneous of degree q

$$\Longleftrightarrow$$

$\exists\, c_0 \in \mathbb{R}, c_1 \geq 0$:

$$\mu(K) = \begin{cases} c_0 + c_1\ \Omega_n(K) & q = 0 \\ c_1\ \Omega_p(K) & -n < q < n,\ q \neq 0 \\ c_0\, V(K) & q = n \\ c_0\, V(K^*) & q = -n \\ 0 & \text{otherwise} \end{cases}$$

$\forall\, K \in \mathcal{K}_0^n$, $q = n(n-p)/(n+p)$

L_p-affine surface area

$$\Omega_p(K) = \int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}} \, d\sigma_K(x)$$

Definition:

Lutwak 1996 ($p \geq 1$), Hug 1996 ($p > 0$)

Applications:

Lutwak 1996, Lutwak-Oliker 1995, Meyer-Werner 2000, Schütt-Werner 2004