Convexity of level sets for solutions to nonlinear elliptic problems in convex rings

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Let u be a solution of a Dirichlet problem of elliptic type in a domain Ω:

$$\begin{cases} F(x, u, \nabla u, D^2 u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

- F is *degenerate elliptic*, i.e.

$$F(x, t, p, A) \leq F(x, t, p, B)\,,$$

for every A, $B \in \Gamma$ such that $A \leq B$, where $A \leq B$ means that $B - A$ is positive semidefinite.

- F is *proper*, i.e. $F(x, t, p, A) \leq F(x, s, p, A)$ if $s \leq t$.

A quite natural question is the following:
is there any relationship between the geometry of the domain Ω and the geometry of the solution u?

Is it possible to find assumptions on the elliptic operator F which ensure that some geometric property of the domain is inherited, in some sense, by the solution?

For instance, if the domain Ω is convex and the Dirichlet data is a constant, we may ask whether the function u must be convex too.

Precisely, the problem we are here involved with is the following:

– Ω is a convex ring, i.e. $\Omega = \Omega_0 \setminus \overline{\Omega_1}$ where Ω_0 and Ω_1 are convex, bounded and open subsets of $\mathbb{R}^n$ such that $\overline{\Omega_1} \subset \Omega_0$.

– u is a solution of the following Dirichlet problem

$$\begin{cases} F(x, u, \nabla u, D^2 u) = 0 & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega_0 \\ u = 1 & \text{on } \ \partial\Omega_1 \ . \end{cases} \tag{1}$$

Hence, two level sets of the solution, are convex surfaces: namely, $\{u = 0\} = \partial\Omega_0$ and $\{u = 1\} = \partial\Omega_1$.

MAIN QUESTION: *Is it possible to find suitable assumptions on F which ensure that all the level sets of u are convex?*

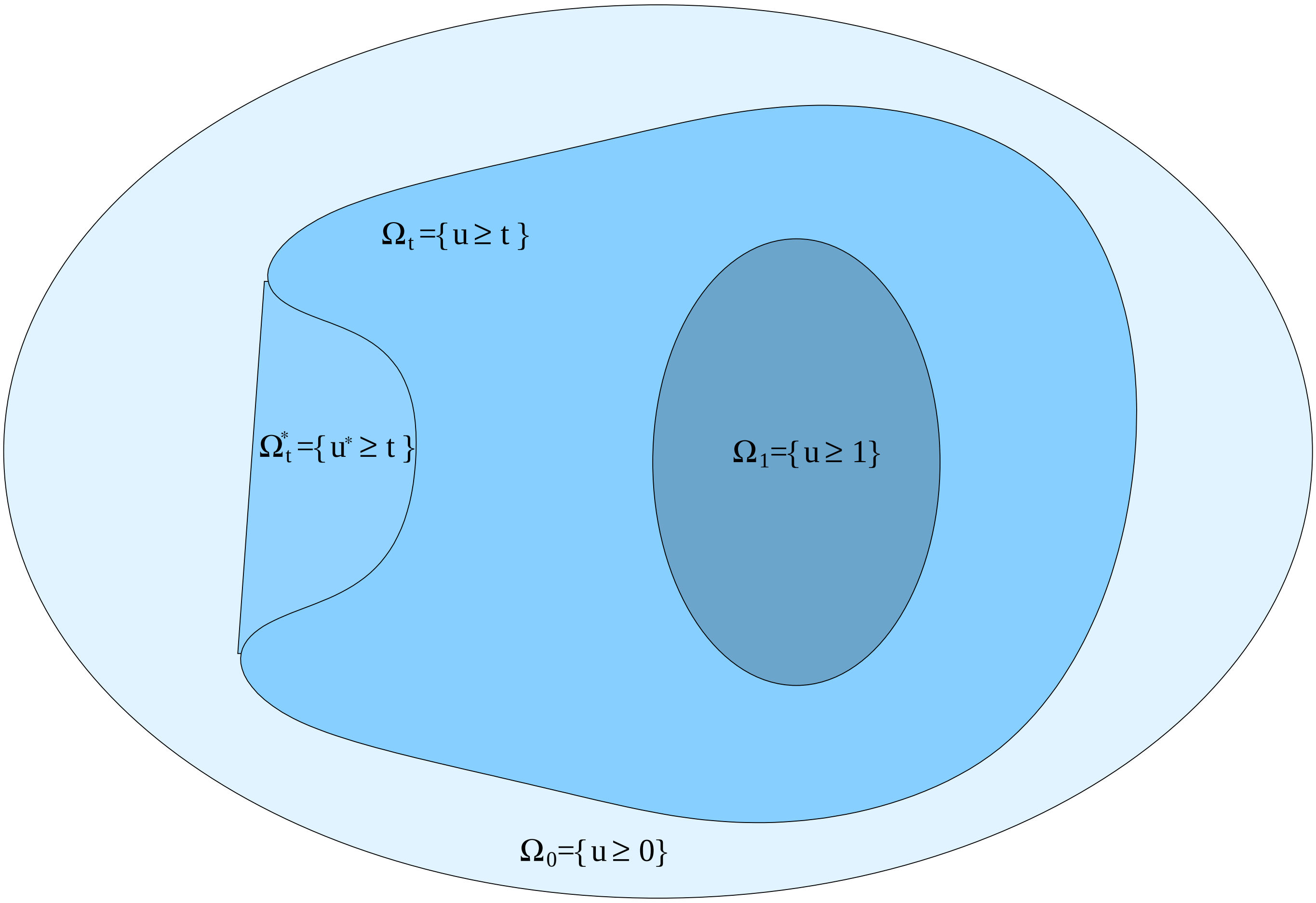
This problem and related questions have been faced by many authors; we recall, for instance, the classic work of Gabriel (1957) and more recent important contributions by Acker, Caffarelli, Korevaar, Lewis, Spruck, as well as many papers and a well-known monograph by Kawohl.

The method here adopted is a generalization of the method introduced in

[CS] - A. Colesanti & P. S.
Quasi-concave Envelope of a Function and Convexity of Level Sets of Solutions to Elliptic Equations
Math. Nach. 258 (2003), 3–15.

It makes use of the *quasi-concave envelope* u^* of a function u.

Roughly speaking, u^* is the function whose superlevel sets are the convex hulls of the corresponding superlevel sets of u (we sistematically extend $u \equiv 1$ in Ω_1), i.e. $\{u^* \geq t\} = \text{conv}(\{u \geq t\}$.



More explicitely, we can define u^* in the following way:

$$u^*(x) = \max\Big\{ \ \min\{u(x_1), ..., u(x_{n+1})\} : \\ x_1, ..., x_{n+1} \in \overline{\Omega},\ x = \textstyle\sum_{i=1}^{n+1} \lambda_i x_i, \\ \lambda_1 \geq 0, \ldots, \lambda_{n+1} \geq 0\,,\ \textstyle\sum_{i=1}^{n+1} \lambda_i = 1\Big\}.$$

We look for conditions that imply $u = u^*$.

Notice that $u^* \geq u$ by definition (to obtain u^* we enlarge the superlevel sets of u), then it suffices to prove the reverse inequality; the latter can be obtained by a suitable *Comparison Principle*, if u^* is a viscosity subsolution of problem (1).

In this way we reduce ourselves to the following question:

Can we find suitable assumptions on F that force u^* to be a viscosity subsolution of (1)?

A positive answer was given in [CS], but only for operators whose principal can be decomposed in a normal and a tangential part (normal and tangential with respect to level sets of the solution, we mean).

Basically, the operators which share such a structure are:

- the laplacian
  $\Delta u = f(x, u, |\nabla u|)$;
- the p-laplacian
  $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) = f(x, u, |\nabla u|)$;
- the mean curvature operator
  $\mathcal{M}u = n(1 + \|\nabla u\|^2)^{3/2}H = f(x, u, |\nabla u|)$.

There, we proved that under suitable concavity assumption on f, the solutions to Dirichlet problems in convex rings for these equations have convex level sets.

Now we are able to treat much more general operator!

In particular, we are able to treat the *Hessian operators*, defined, for $k = 1, \ldots, n$, as the k-th elementary symmetric function of the eigenvalues of the Hessian matrix of u, i.e. for $k = 1, \ldots, n$

$$S_k(D^2u) = S_k(\lambda_1, ..., \lambda_n) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D^2u.

MAIN THEOREM.

Let $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ be a convex ring and let $F(x, u, \theta, A)$ be a proper, continuous and degenerate elliptic operator in $\Omega \times (0, 1) \times \mathbb{R}^n \times \Gamma_F$.

Assume that there exists $\widetilde{p} < 0$ such that, for every $p \leq \widetilde{p}$ and for every $\theta \in \mathbb{R}^n$, the application

$$(x, t, A) \to F\left(x,\, t^{\frac{1}{p}},\, t^{\frac{1}{p}-1}\theta,\, t^{\frac{1}{p}-3}A\right) \tag{2}$$

is concave in $\Omega \times (1, +\infty) \times \Gamma_F$.

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is an admissible classical solution of (1) such that $|\nabla u| > 0$ in Ω, then u^* is a viscosity subsolution of (1).

Remarks

1. The above theorem holds also for viscosity solutions.

2. The assumption $|\nabla u| > 0$ is typical for this kind of investigation; in fact, we can relax it a little bit, but not in an essential way. To prove geometric properties of solutions to elliptic problems without this assumption it is a difficult task. In fact, this assumption is often automatically satisfied by classical solutions, thanks to some maximum principle applied to $|\nabla u|$.

3. *Admissible solution* means that $D^2u(x) \in \Gamma_F$ for every $x \in \Omega$, i.e. u is an *elliptic* solution.

To prove the main theorem, we need some preliminaries.

First of all we recall the notion of p-means of m non-negative real numbers.

Given $a = (a_1, ..., a_m) > 0$, $\lambda \in \Lambda_m$ and $p \in [-\infty, +\infty]$, the quantity

$$M_p(a, \lambda) = \begin{cases} [\lambda_1 a_1^p + \cdots + \lambda_m a_m^p]^{1/p} & \text{for } p \neq -\infty, 0, +\infty \\ \max\{a_1, ..., a_m\} & p = +\infty \\ a_1^{\lambda_1} \cdots a_m^{\lambda_m} & p = 0 \\ \min\{a_1, a_2, ..., a_m\} & p = -\infty \end{cases}$$

is the p-(weighted) mean of a. For $a \geq 0$, we define $M_p(a, \lambda)$ as above if $p \geq 0$ and we set $M_p(a, \lambda) = 0$ if $p < 0$ and $a_i = 0$, for some $i = 1, ..., m$.

Let us fix $\lambda \in \Lambda_{n+1}$ and consider $p \in [-\infty, +\infty]$.

Definition 0.1 *The* (p, λ)–envelope *of u is the function $u_{p,\lambda} : \overline{\Omega} \to [0, 1]$ defined as follows*

$$u_{p,\lambda}(x) = \sup\Big\{ M_p\left(u(x_1), ..., u(x_{n+1}), \lambda\right) : x_i \in \overline{\Omega},\ x = \textstyle\sum_{i=1}^{n+1} \lambda_i x_i \Big\}. \quad (3)$$

Notice that, as $\overline{\Omega}$ is compact and M_p is continuous, the supremum of the definition is in fact a maximum.

Hence, for every $x \in \overline{\Omega}$, there exist $(x_{1,p}, ..., x_{n+1,p}) \in \overline{\Omega}^{n+1}$ such that

$$x = \sum_{i=1}^{n+1} \lambda_i x_{i,p}, \quad u_{p,\lambda}(x) = M_p\left(u(x_1), ..., u(x_{n+1}), \lambda\right) .$$

An immediate consequence of the definition is that

$$u_{p,\lambda}(x) \geq u(x), \quad \forall x \in \overline{\Omega}, \quad p \in [-\infty, , +\infty];$$

moreover, from the monotonicity of means, we have

$$u_{p,\lambda}(x) \leq u_{q,\lambda}(x), \quad \text{for } p \leq q, \quad x \in \Omega.$$

For convenience, we will refer to $u_{-\infty,\lambda}$ as u^*_λ.
Notice that

$$u^*(x) = \sup\left\{u^*_\lambda(x) \, : \, \lambda \in \Lambda_{n+1}\right\} \tag{4}$$

and the above supremum is in fact a maximum as Λ_{n+1} is compact.

Theorem. $u_{p,\lambda}$ converge uniformly to u^*_λ as $p \to \infty$.

Sketch of the proof of the main thm.

i. We fix $\lambda \in \Lambda_{n+1}$ and we prove that, for every compact subset K of Ω, $u_{p,\lambda}$ is a viscosity subsolution, for $-p$ large enough (depending on K).

ii. Then passing to the limit for $p \to -\infty$, thanks to uniform convergence, we obtain that u^*_λ is a subsolution.

iii. Thanks to (4) and since the supremum of subsolutions is a subsolution, the proof is concluded.

How do we prove the first item?

By definition, $u_{p,\lambda}$ is a viscosity subsolution of $F = 0$ if, for every C^2 function ϕ touching $u_{p,\lambda}$ from above at any point $\overline{x} \in \Omega$, it holds

$$F(\overline{x}, \phi(\overline{x}), \nabla\phi(\overline{x}), D^2\phi(\overline{x})) \geq 0\,.$$

If u is an admissible classical solution, this task is somewhat simplified, since u can be used as a test function.

Indeed, for every point $\overline{x} \in \Omega$, we can use u to construct a C^2 function $\varphi_{p,\lambda}$ that touches $u_{p,\lambda}$ from below at $\overline{x}$.

This function is defined as follows:

$$\varphi_{p,\lambda}(x) = \left[\sum_{i=1}^{n+1} \lambda_i u\left(x_{i,p} + a_{i,p}(x - \overline{x})\right)^p\right]^{1/p}, \qquad (5)$$

where

$$a_{i,p} = \frac{u(x_{i,p})^p}{u_{p,\lambda}(\overline{x})^p}, \quad \text{for } i = 1, ..., n+1\,, \qquad (6)$$

and the points $x_{i,p}$ are the ones which realize the maximum in the definition of $u_{p,\lambda}(\overline{x})$.

If all the points $x_{1,p}, \ldots, x_{n+1,p}$ belong to Ω, we can write the gradient and the Hessian matrix of $\phi_{p,\lambda}$ at the point $\overline{x}$ as a convex combination of the gradients and the Hessian matrices of u at these points and our concavity assumption on F makes the rest.

But we cannot be sure, in general, that any one of these points belongs to $\partial\Omega$; notice that, if for some p we could be able to prove this, we would obtain that u coincide with its p-concave envelope and finally it is p-concave, which is of course much more than quasi-concave.

On the other hand, it is easily seen that this actually happens for $p = -\infty$, hence restricting to any compact subset of Ω, we can prove that it must also happen for $-p$ enough large and this conclude the proof.