Minkowski Sums and Perfectly Centered Polytopes

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Plan

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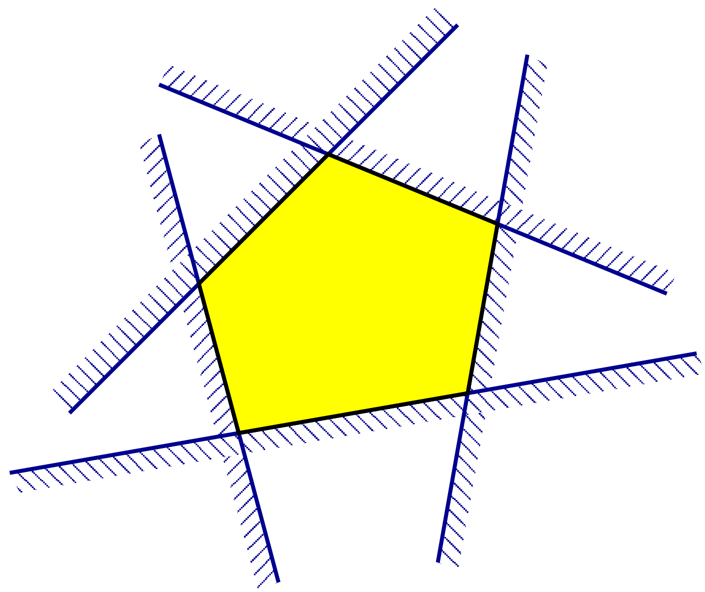
- Nesterov Rounding
- Perfectly Centered Polytopes

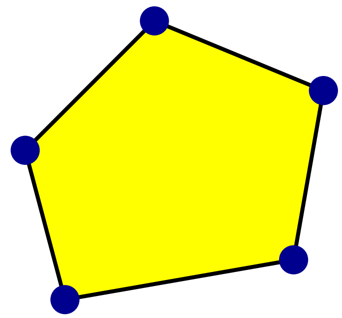
Polytope : $\mathcal{H}$-representation



$$P = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$$

P bounded

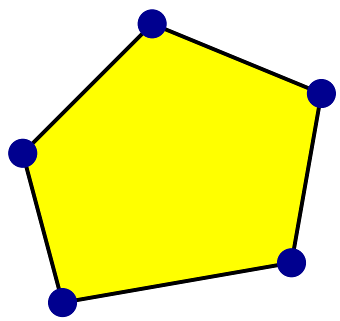
Polytope : $\mathcal{V}$-representation



$$P = conv\{v_1, \ldots, v_k\} =$$

$$\left\{ x \in \mathbb{R}^d : x = \sum_{i=1}^{k} \lambda_i v_i,\ \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0\ \forall i \right\}$$

Minkowski-Weyl Theorem



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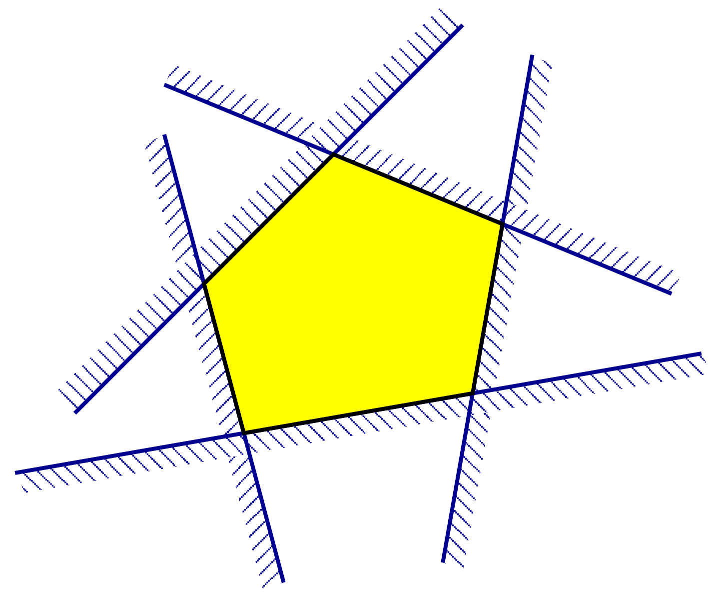
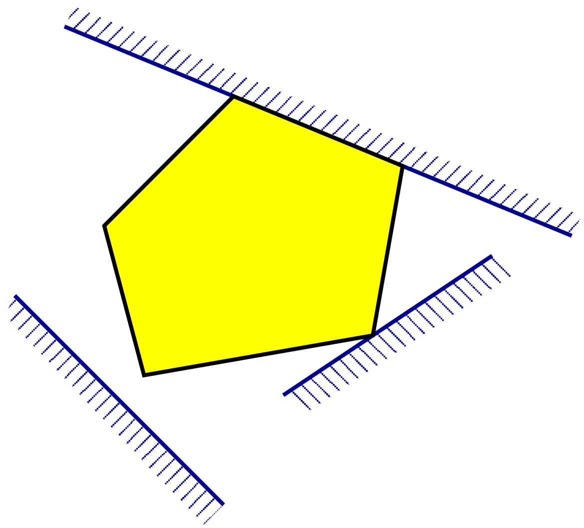


$P = conv\{v_1, \ldots, v_k\}$

$P = \left\{x \in \mathbb{R}^d : Ax \leq b\right\}$

Valid inequalities



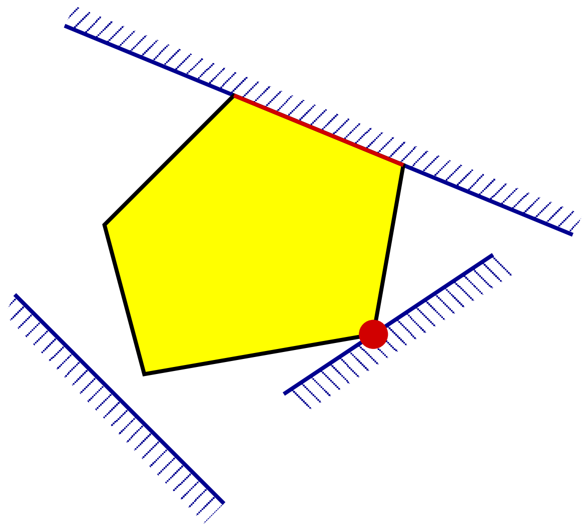
$$c \in \mathbb{R}^d \quad \beta \in \mathbb{R}$$

$$c^T x \leq \beta, \forall x \in P$$

Faces

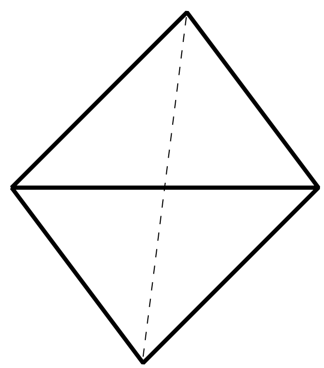
$$F = P \cap \left\{ x \in \mathbb{R}^d : c^T x = \beta \right\}$$

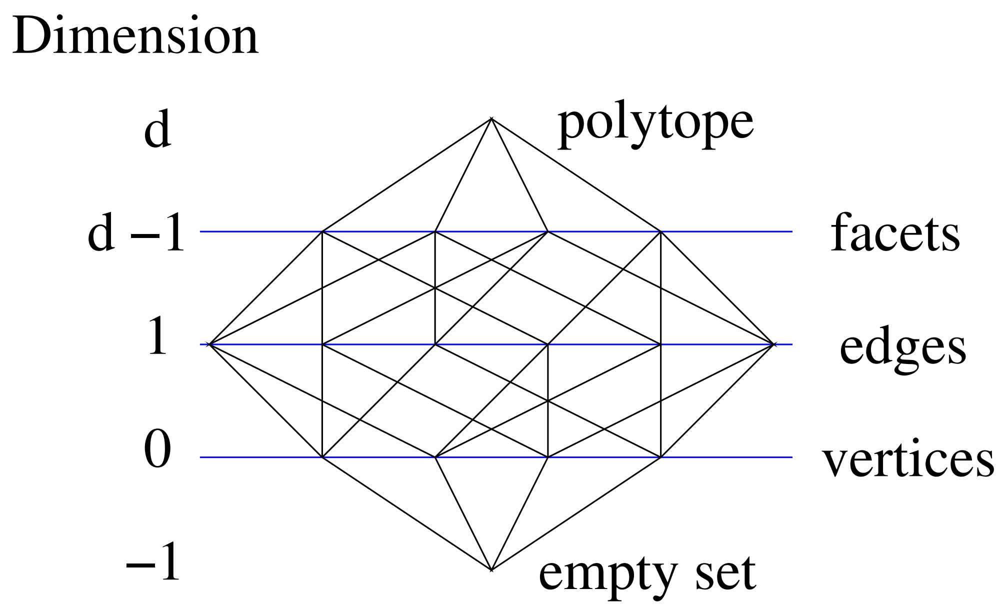
for some valid inequality $c^T x \leq \beta$



P and $\emptyset$ are called *trivial faces*

Face lattices : Faces ordered by inclusion





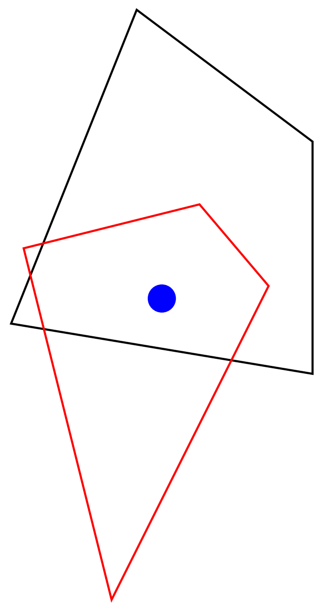
tetrahedron

lattice

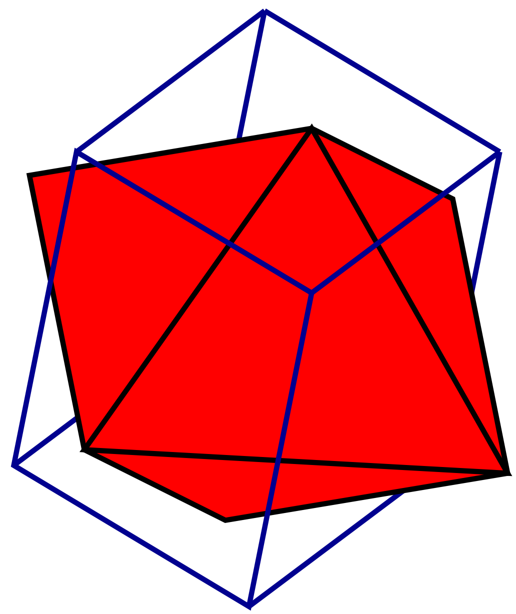
f-vector : $f_k(P) := |\{\text{Faces of } P \text{ of dimension } k\}|$

Dual polytope

$$P^* = \left\{x \in \mathbb{R} : x^T p \leq 1,\ \forall p \in P\right\}$$

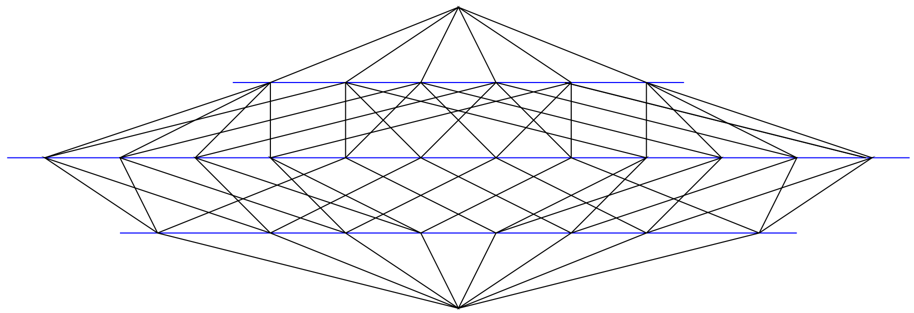


quadrangle

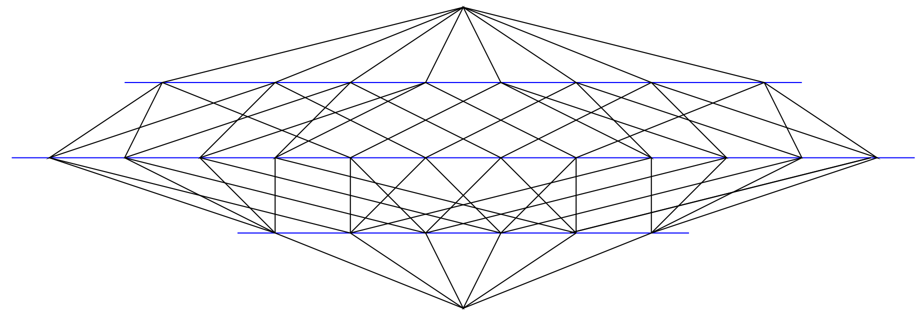


cube

Dual faces and lattices



cube



octahedron

$$F = conv\{v_1, \ldots, v_n\} \text{ face of P,}$$

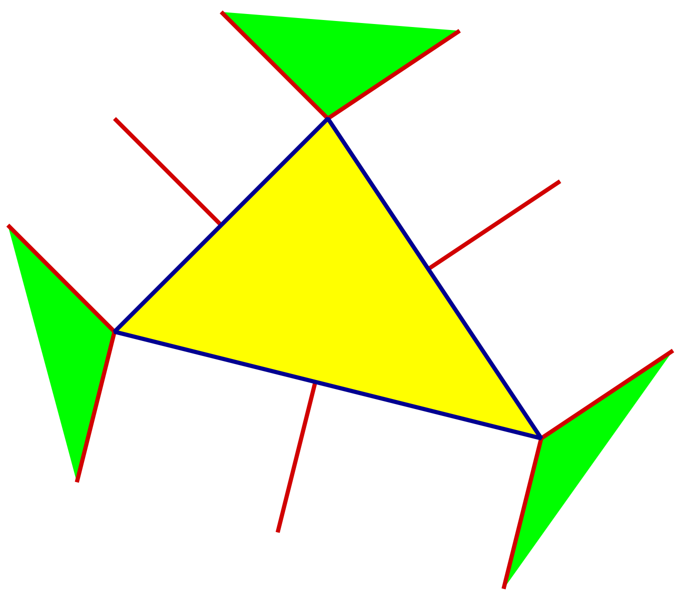
$$F' = P^* \cap \left\{x \in \mathbb{R} : x^T v_k = 1,\ k = 1, \ldots, n\right\}$$

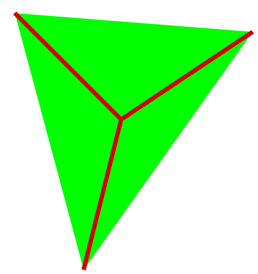
$$F \subseteq G \Leftrightarrow F' \supseteq G'$$

Normal cone

F Face of the polytope P

$$\mathcal{N}(F) := \{ y : y^T x < y^T f, \forall x \in P \setminus F, \forall f \in F \}$$

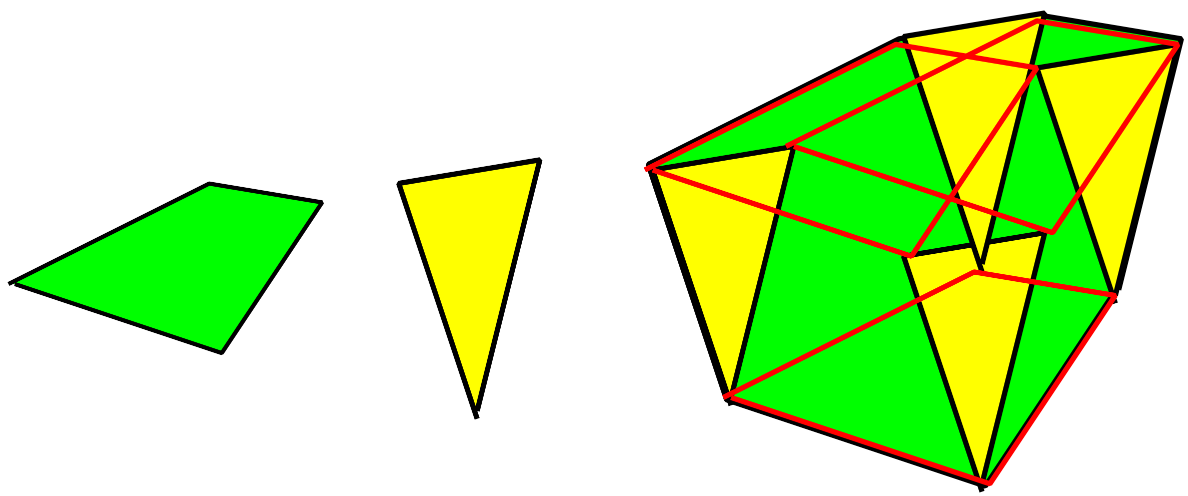




Normal fan

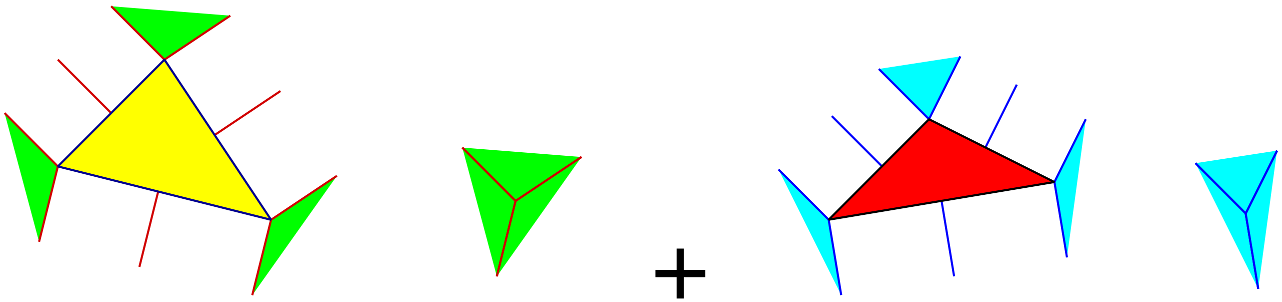
Minkowski Sums

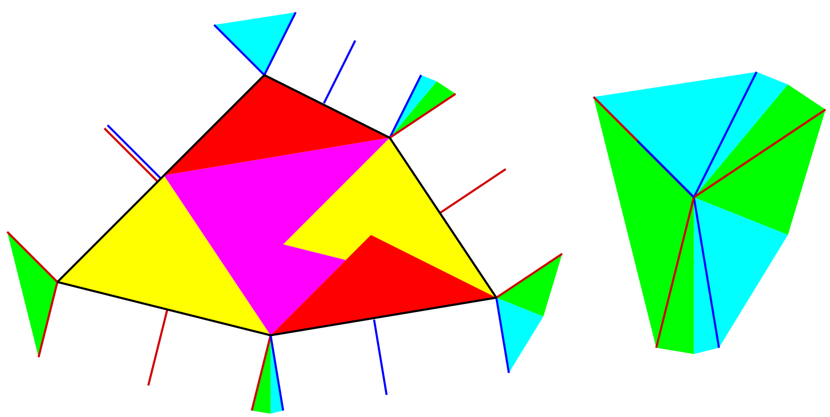
$$P_1 + P_2 := \{x_1 + x_2 : x_1 \in P_1,\ x_2 \in P_2\}$$



P_1 P_2 $P_1 + P_2$

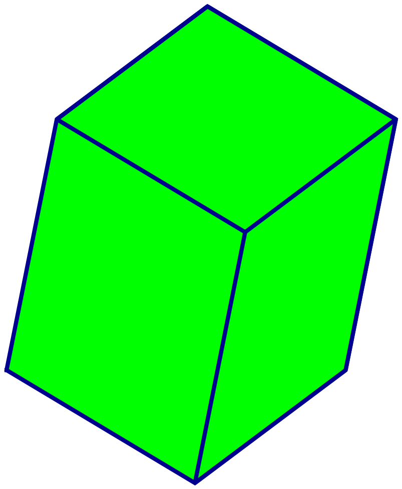
Normal fan of sums



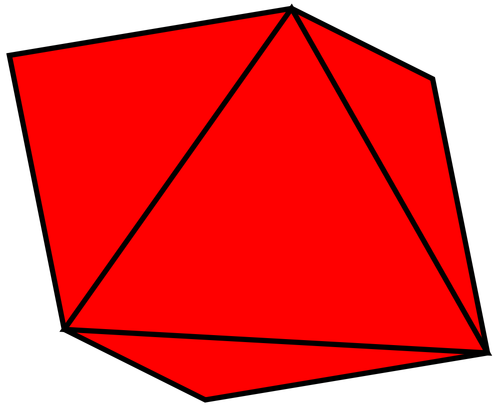


Intersection of normal fans

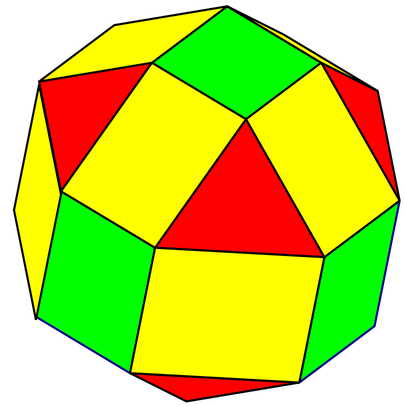
Mixed faces



Cube



Octahedron



Sum

Applications

- Tolerance computation in mechanics (Petit, 2004)
- Gröbner bases recognition (Sturmfels, 1996)
- Tropical fan computing (Sturmfels)

TEST: Maximal sum of two tetrahedrons

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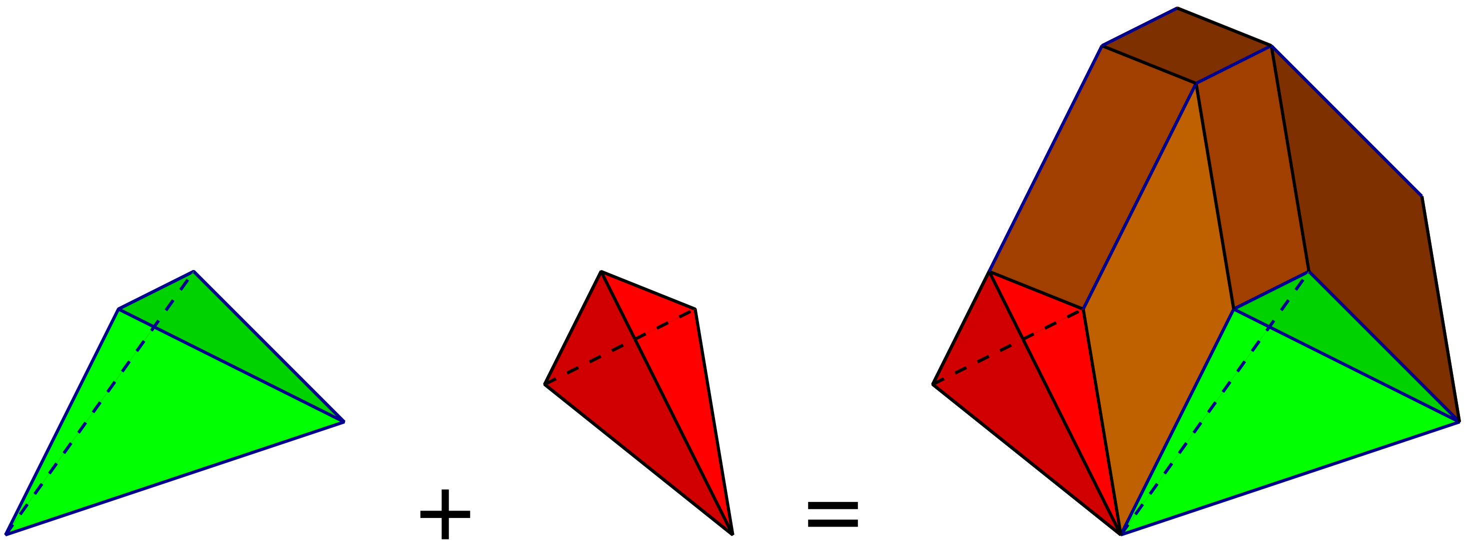
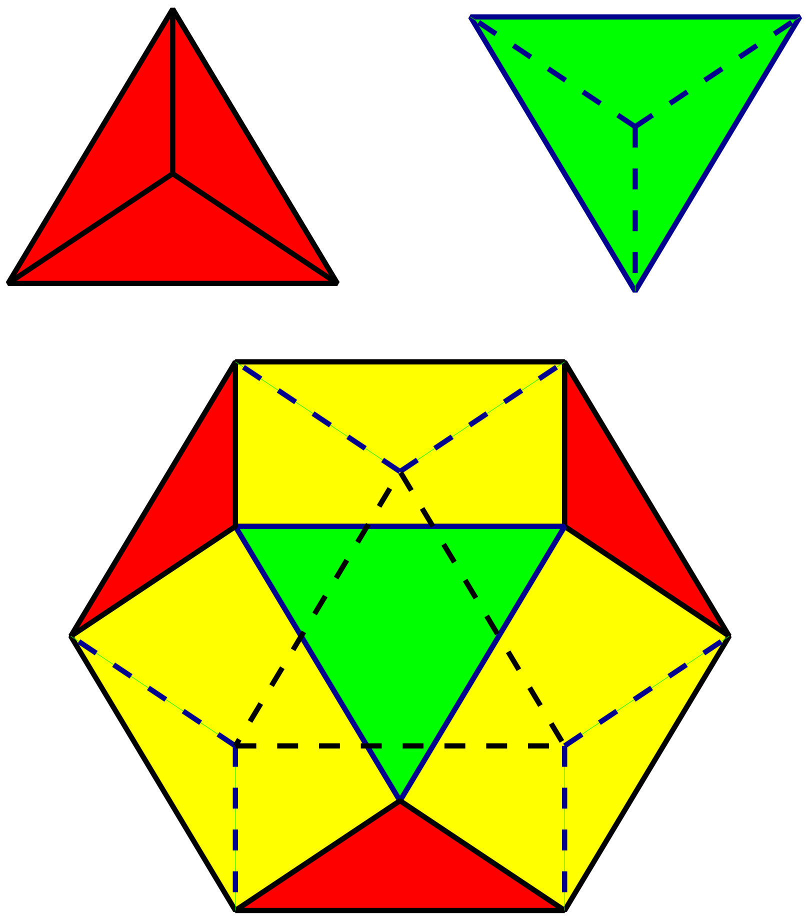
TEST: Maximal sum of two tetrahedrons



$4 \cdot 4 = 16$ vertices, $4 \cdot 2 + 10 = 18$ facets.

Sum of a regular tetrahedron with its own dual.



Nesterov Rounding

$$\mathcal{B} = \{x \in \mathbb{R} : ||x|| \leq 1\}$$

For any set Q bounded closed convex,

$$r(Q) = \max_r \{r : r\mathcal{B} \subseteq Q\} \qquad R(Q) = \min_r \{r : r\mathcal{B} \supseteq Q\}$$

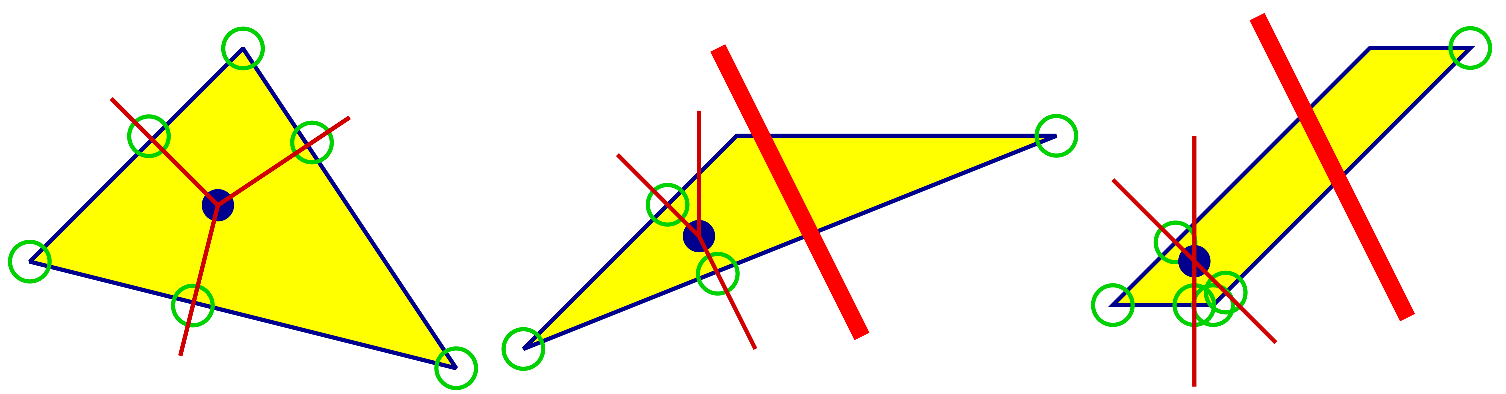
$$\gamma(Q) = \frac{R(Q)}{r(Q)} \geq 1 \quad \text{sphericity}$$

$$\exists \alpha, \beta : \gamma(\alpha Q + \beta Q^*) \leq \sqrt{\frac{1+\gamma(Q)}{2}}$$

Perfectly Centered Polytopes

P centered : $0 \in P$

P perfectly centered : $\mathcal{N}(F) \cap F \neq \emptyset, \forall F$ face of P



Main result

If P is perfectly centered, then

Faces of $P+P^*$ can be caracterized as

$$\left\{F+G' : F,\ G \text{ non-trivial faces of } P,\ F \subseteq G\right\}$$

Face inclusion

If P is perfectly centered, and

$F_1 + G'_1$ and $F_2 + G'_2$ are faces of $P + P'$, then

$$F_1 + G'_1 \subseteq F_2 + G'_2 \Leftrightarrow F_1 \subseteq F_2,\ G_2 \subseteq G_1$$

Facet adjacency

If P is perfectly centered, and

$F_1 + F_1'$ and $F_2 + F_2'$ are facets of $P + P'$, then

$F_1 + F_1'$ and $F_2 + F_2'$ are adjacent iff:

$F_1 \subset F_2 \quad \& \quad F_1 \subset G \subset F_2$ for no face G of P

or

$F_2 \subset F_1 \quad \& \quad F_2 \subset G \subset F_1$ for no face G of P

Examples

- P the Nesterov rounding of the d-simplex

$$f_k = \binom{d+1}{k+2}\left(2^{k+2}-2\right)$$

- P the Nesterov rounding of the d-cube

$$f_k = \binom{d}{k+1} 2^{d-k-1}\left(3^{k+1}-1\right)$$

Examples

- P_n the nth Nesterov rounding of a 3-dim polytope

$$f_0(P_n) = 4^{n-1} f_0(P_1) \quad f_1(P_n) = 2 f_0(P_n)$$

$$f_2(P_n) = f_0(P_n) + f_2(P_1) - f_0(P_1)$$

$$\frac{f_2(P_n)}{f_0(P_n)} \longrightarrow 1$$