Determination of Convex Bodies by Projection Functions

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The classical uniqueness result

For $1 \leq i \leq j \leq n-1$, the (i,j)-th **projection function** $\pi_{ij}(K,\cdot)$ of a **convex body** K is defined by

$$\pi_{ij}(K,\cdot) := V_i(K|L), \qquad L \in \mathcal{L}_j^n.$$

$\uparrow$

i-th intrinsic volume of orthogonal projection

A **classical result** of Alexandrov (1937) yields that a centrally symmetric body K (of dimension $\geq i+1$) is uniquely determined (up to translation) by the projection function $\pi_{ij}(K,\cdot)$.

The **proof** is based on the following steps:

• The **Cauchy-Kubota formula** shows that $\pi_{ij}(K, \cdot)$ determines $\pi_{i\,n-1}(K, \cdot)$.

• The integral representation

$$\pi_{i\,n-1}(K, u^{\perp}) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x \rangle| S_i(K; dx), \quad u \in S^{n-1},$$

and the injectivity properties of the **Cosine Transform** show that the surface area measure $S_i(K; \cdot)$ of K is determined (here the symmetry of K comes in).

• For $\dim(K) \geq i + 1$, the surface area measure $S_i(K; \cdot)$ determines K (up to translation).

Natural questions

(1) What do we know about K, if only partial knowledge of $\pi_{ij}(K, \cdot)$ or knowledge of modified functions (averages) is assumed?

(2) Are there variants of $\pi_{ij}(K, \cdot)$, which allow the determination of general (non-symmetric) bodies?

Problem by Hermann Dinges for my diploma thesis (1968, Frankfurt):

Let Z be a body with support function

$$h(Z,u) = \int_{S^{n-1}} |\langle u,x\rangle| \rho(Z;dx), \qquad u \in S^{n-1}.$$

The behavior of $h(Z,\cdot)$ on a small 'cap' $U_\epsilon(u) \subset S^{n-1}$ (with centre u) should be determined by the behavior of $\rho(Z;\cdot)$ in the orthogonal 'zone'

$$U_\epsilon(u^\perp) = \{v \in S^{n-1} : v \perp w \text{ for some } w \in U_\epsilon(u)\}.$$

If we let the $U_\epsilon(u)$ vary over a whole zone $U_\epsilon(v^\perp)$, $u \in v^\perp$, v fixed, the corresponding zones $U_\epsilon(u^\perp)$ cover S^{n-1}. Show that therefore the knowledge of $h(Z,\cdot)$ in $U_\epsilon(v^\perp)$ determines $\rho(Z;\cdot)$ uniquely (and so also Z).

R. Schneider, Zu einem Problem von Shephard über die Projektionen konvexer Körper, *Math. Z.* **101**, 71–82 (1967):

The Cosine Transform C is a continuous bijection on $C_e^\infty(S^{n-1})$.

($\Rightarrow$ Answer to above problem is no!)

Theorem (Schneider/W. 1970). *Let K, M be convex bodies in $\mathbb{R}^n$, n odd, which have a vertex with normal $u \in S^{n-1}$ and are centrally symmetric to 0. If, for some $\epsilon > 0$,*

$$V_{n-1}(K|v^\perp) = V_{n-1}(M|v^\perp), \qquad \text{for all } v \in U_\epsilon(u^\perp),$$

then $K = M$.

The theorem is wrong in even dimensions or if only one of the bodies has a vertex.

Theorem (Schneider/W. 1970). *Let K, M be convex bodies in $\mathbb{R}^n$, n odd, which have a vertex with normal $u \in S^{n-1}$ and are centrally symmetric to* 0. *If, for some $\epsilon > 0$,*

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then $K = M$.

Schneider/W. 1983: Survey on zonoids (green book)

Schneider/W. 1986: Kinematic formula for curvature measures (basic paper for translative integral geometry)

Theorem (Goodey/Schneider/W. 1995). *For $1 < i < n-1$, there are convex bodies K in $\mathbb{R}^n$, $n \geq 3$, which are not centrally symmetric and which are uniquely determined (up to translation and reflection) by $\pi_{ii}(K, \cdot)$.*

Moreover, there are also centrally symmetric bodies M which are uniquely determined (up to translations) within all convex bodies by $\pi_{ii}(K, \cdot)$.

Flag representations of convex bodies

Motivation:

For **generalized zonoids** K, M, and $j = 1, ..., n-1$, we have

$$\pi_{jj}(K, \cdot) = c_{nj} \int_{\mathcal{L}_j^n} |\langle L, \cdot \rangle| \rho_j(K; dL)$$

and

$$\begin{aligned} V(K[j], M[n-j]) &= d_{nj} \int_{\mathcal{L}_{n-j}^n} \pi_{jj}(K, N^\perp) \rho_{n-j}(M; dN) \\ &= e_{nj} \int_{\mathcal{L}_j^n} \int_{\mathcal{L}_{n-j}^n} |\langle L, N^\perp \rangle| \rho_{n-j}(M; dN) \rho_j(K; dL). \end{aligned}$$

For arbitrary (non-symmetric) convex bodies K, M a corresponding formula is classical for $j = 1$:

$$V(K[1], M[n-1]) = \frac{1}{n} \int_{S^{n-1}} h(K,u) S_{n-1}(M; du).$$

What about $j > 1$?

Flag measures

Schneider (1978): Integral geom. interpr. of curvature meas.
$\rightarrow$ W. (1981): Extended curvature measures on flats
$\rightarrow$ Kropp (1990): Extended area measures (flag measures)
($\rightarrow$ common generalization: Extended support measures)
For

$$\mathcal{F}_k^n := \{(u, L) \in S^{n-1} \times \mathcal{L}_k^n : u \perp L\}, \qquad 1 \leq k \leq n-1,$$

let

$$\Omega_k(K; A) := \int_{\mathcal{L}_{k+1}^n} S_k'(K|N, A|N) dN,$$

where $A \subset \mathcal{F}_{n-1-k}^n$ runs through the Borel sets and

$$A|N := \{u \in S^{n-1} : (u, L) \in A \text{ for some } L \perp N\}.$$

$\Rightarrow S_k(K; \cdot)$ is the image of $\Omega_k(K; \cdot)$ under $(u, L) \mapsto u$.

Representation of mixed volumes
(Project by **Goodey, Hinderer, Hug, Rataj, W.**)

There exists a measurable function f_k on $\mathcal{F}^n_{n-1-k} \times \mathcal{F}^n_{k-1}$ such that

$$V(K[k], M[n-k]) \\ = \int_{\mathcal{F}^n_{n-1-k}} \int_{\mathcal{F}^n_{k-1}} f_k(u, L; v, N)\Omega_{n-k}(M; d(v, N))\Omega_k(K; d(u, L)),$$

for all convex bodies K, M.

Representation of projection functions

For $M =$ unit ball in $L^\perp$, $L \in \mathcal{L}_j^n$, one gets

$$V_j(K|L) = \int_{\mathcal{F}_{n-1-j}^n} g_j(u, N; L)\Omega_j(K; d(u, N)),$$

with some function g_j on $\mathcal{F}_{n-1-j}^n \times \mathcal{L}_j^n$ (**Hinderer, 2002**).

Directed projection functions (flag functions)

(1) First approach (**Goodey/W. 2004**):

We have

$$V_i(K) = c_{in} S_i(K; S^{n-1}), \qquad 1 \leq i < n.$$

We therefore define a **directed projection function** $v_{ij}(K; L, u)$, for $1 \leq i < j \leq n-1$, $L \in \mathcal{L}_j^n$ and almost all $u \in L \cap S^{n-1}$, by

$$v_{ij}(K; L, u) := c_{ij} S_i' \left(K|L; u^+ \cap L \right).$$

Here, $u^+ := \{v \in S^{n-1} : \langle u, v \rangle \geq 0\}$.
(**Groemer (1997)** discussed the function $v_{12}(K, \cdot)$, for $n = 3$.)
$v_{ij}(K; \cdot)$ can be interpreted as a function on $\mathcal{F}_{j-1}^n$.

Theorem. *If* $1 \leq i < j \leq n-1$ *and* $K,\ M$ *are convex bodies of dimension at least* $i+1$ *with*

$$v_{ij}(K;\cdot) = v_{ij}(M;\cdot),$$

then K *and* M *are translates.*

There is also a corresponding **stability result**.

Disadvantages: (a) Determination only up to translation.
(b) Approach does not work for $i = j$.
(c) K may be 'over-determined' by such flag functions.

(2) Second approach (**Goodey/W. 2005**):

We have

$$V_i(K) = c_{in} \int_{\mathrm{Nor}(K)} \langle x, u\rangle \langle u, v\rangle \Theta_{i-1}(K; d(x,v)) \\ + d_{in} \int_{S^{n-1}} \langle u, v\rangle^2 S_i(K; dv)\,, \qquad 1 \le i \le n.$$

(Here, $u \in S^{n-1}$ is arbitrary and $\Theta_j(K;\cdot)$ is the j-th **support measure** on the **normal bundle** $\mathrm{Nor}(K)$ of K.)

We therefore define a (second kind of) **directed projection function**

$$p_{ij}(K; L, u) = c_{ij} \int_{\mathrm{Nor}(K|L)} \mathbf{1}_{u^+}(v) \langle x, u\rangle \langle u, v\rangle \Theta'_{i-1}(K|L; d(x,v)) \\ + d_{ij} \int_{u^+ \cap L} \langle u, v\rangle^2 S'_i(K|L; dv)\,,$$

(for $1 \le i \le j \le n-1$, $L \in \mathcal{L}_j^n$ and $u \in L \cap S^{n-1}$).

Theorem. *If $1 \leq i \leq j \leq n-1$ and K, M are convex bodies of dimension at least $i+1$ with*

$$p_{ij}(K;\cdot) = p_{ij}(M;\cdot),$$

then $K = M$.

There is no corresponding **stability result** yet.

Again, K may be over-determined by $p_{ij}(K;\cdot)$.

For $i = 1$, $v_{1j}(K;L,\cdot)$ is the **hemispherical transform** of the first surface area measure $S_1'(K|L;\cdot)$, whereas $p_{1j}(K;L,\cdot)$ is the hemispherical transform of the support function $h(K|L,\cdot)$.

Projection averages

Let $\overline{v}_j(K,u)$ and $\overline{p}_j(K,u)$ be the averages of $v_{1j}(K;L,u)$ resp. $p_{1j}(K;L,u)$ over all $L \in \mathcal{L}_j^n$ which contain u (we only have results for this linear case $i=1$).

Theorem. *Let $K, M \subset \mathbb{R}^n$ be convex bodies and $2 \le j < \frac{2n-3}{5}$ or $\frac{n-2}{2} \le j \le n-1$. If*

$$\overline{v}_j(K,\cdot) = \overline{v}_j(M,\cdot),$$

then K, M are translates.
On the other hand, for $i = 1, 2, \ldots,$ there are convex bodies K, M in $\mathbb{R}^{5i+4}$, which are not translates, with

$$\overline{v}_{2i+1}(K,\cdot) = \overline{v}_{2i+1}(M,\cdot).$$

Theorem. *Let* $K, M \subset \mathbb{R}^n$ *be convex bodies and* $1 \leq j \leq \frac{n}{2}+1$ *or* $\frac{2n+1}{3} < j \leq n-1$, *for* $n \neq 4$ *(resp.* $j = 2$ *if* $n = 4$*). If*

$$\overline{p}_j(K, \cdot) = \overline{p}_j(M, \cdot),$$

then $K = M$.
On the other hand, for $i = 1, 2, \ldots,$ *there are convex bodies* $K \neq M$ *in* $\mathbb{R}^{3i+1}$ *with*

$$\overline{p}_{2i+1}(K, \cdot) = \overline{p}_{2i+1}(M, \cdot).$$