The classical uniqueness result

For $1 \leq i \leq j \leq n - 1$, the $(i, j)$-th projection function $\pi_{ij}(K, \cdot)$ of a convex body $K$ is defined by

$$\pi_{ij}(K, \cdot) := V_i(K|L), \quad L \in \mathcal{L}^n_{j}.$$  

A classical result of Alexandrov (1937) yields that a centrally symmetric body $K$ (of dimension $\geq i + 1$) is uniquely determined (up to translation) by the projection function $\pi_{ij}(K, \cdot)$. 
The proof is based on the following steps:

- The Cauchy-Kubota formula shows that $\pi_{ij}(K, \cdot)$ determines $\pi_{i n-1}(K, \cdot)$.

- The integral representation

$$
\pi_{i n-1}(K, u^\perp) = \frac{1}{2} \int_{S^{n-1}} |\langle u, x \rangle| S_i(K; dx), \quad u \in S^{n-1},
$$

and the injectivity properties of the Cosine Transform show that the surface area measure $S_i(K; \cdot)$ of $K$ is determined (here the symmetry of $K$ comes in).

- For $\dim(K) \geq i + 1$, the surface area measure $S_i(K; \cdot)$ determines $K$ (up to translation).
Natural questions

(1) What do we know about $K$, if only partial knowledge of $\pi_{ij}(K, \cdot)$ or knowledge of modified functions (averages) is assumed?

(2) Are there variants of $\pi_{ij}(K, \cdot)$, which allow the determination of general (non-symmetric) bodies?
**Problem** by Hermann Dinges for my diploma thesis (1968, Frankfurt):

Let $Z$ be a body with support function

$$h(Z, u) = \int_{S^{n-1}} |\langle u, x \rangle| \rho(Z; dx), \quad u \in S^{n-1}.$$ 

The behavior of $h(Z, \cdot)$ on a small ‘cap’ $U_\varepsilon(u) \subset S^{n-1}$ (with centre $u$) should be determined by the behavior of $\rho(Z; \cdot)$ in the orthogonal ‘zone’

$$U_\varepsilon(u^\perp) = \{v \in S^{n-1} : v \perp w \text{ for some } w \in U_\varepsilon(u)\}.$$ 

If we let the $U_\varepsilon(u)$ vary over a whole zone $U_\varepsilon(v^\perp)$, $u \in v^\perp$, $v$ fixed, the corresponding zones $U_\varepsilon(u^\perp)$ cover $S^{n-1}$. Show that therefore the knowledge of $h(Z, \cdot)$ in $U_\varepsilon(v^\perp)$ determines $\rho(Z; \cdot)$ uniquely (and so also $Z$).

The Cosine Transform $C$ is a continuous bijection on $C^\infty_e(S^{n-1})$.

($\Rightarrow$ Answer to above problem is no!)
Theorem (Schneider/W. 1970). Let $K, M$ be convex bodies in $\mathbb{R}^n$, $n$ odd, which have a vertex with normal $u \in S^{n-1}$ and are centrally symmetric to 0. If, for some $\epsilon > 0$,

$$V_{n-1}(K|v^\perp) = V_{n-1}(M|v^\perp),$$

for all $v \in U_\epsilon(u^\perp)$, then $K = M$.

The theorem is wrong in even dimensions or if only one of the bodies has a vertex.
Theorem (Schneider/W. 1970). Let $K, M$ be convex bodies in $\mathbb{R}^n$, $n$ odd, which have a vertex with normal $u \in S^{n-1}$ and are centrally symmetric to 0. If, for some $\epsilon > 0$,

$$V_{n-1}(K|v^\perp) = V_{n-1}(M|v^\perp), \quad \text{for all } v \in U_\epsilon(u^\perp),$$

then $K = M$.

Schneider/W. 1983: Survey on zonoids (green book)

Schneider/W. 1986: Kinematic formula for curvature measures (basic paper for translative integral geometry)
Theorem (Goodey/Schneider/W. 1995). For $1 < i < n - 1$, there are convex bodies $K$ in $\mathbb{R}^n$, $n \geq 3$, which are not centrally symmetric and which are uniquely determined (up to translation and reflection) by $\pi_{ii}(K, \cdot)$.

Moreover, there are also centrally symmetric bodies $M$ which are uniquely determined (up to translations) within all convex bodies by $\pi_{ii}(K, \cdot)$. 
Flag representations of convex bodies

Motivation:

For generalized zonoids $K, M$, and $j = 1, \ldots, n - 1$, we have

$$\pi_{jj}(K, \cdot) = c_{nj} \int_{\mathcal{L}_j^n} |\langle L, \cdot \rangle| \rho_j(K; dL)$$

and

$$V(K[j], M[n - j]) = d_{nj} \int_{\mathcal{L}_{n-j}^n} \pi_{jj}(K, N^\bot) \rho_{n-j}(M; dN)$$

$$= e_{nj} \int_{\mathcal{L}_j^n} \int_{\mathcal{L}_{n-j}^n} |\langle L, N^\bot \rangle| \rho_{n-j}(M; dN) \rho_j(K; dL).$$
For arbitrary (non-symmetric) convex bodies $K, M$ a corresponding formula is classical for $j = 1$:

$$V(K[1], M[n - 1]) = \frac{1}{n} \int_{S^{n-1}} h(K, u) S_{n-1}(M; du).$$

What about $j > 1$?
Flag measures

→ W. (1981): Extended curvature measures on flats
→ Kropp (1990): Extended area measures (flag measures)
(→ common generalization: Extended support measures)

For $F_n^k := \{(u, L) \in S^{n-1} \times \mathcal{L}_k^n : u \perp L\}, \quad 1 \leq k \leq n-1,$
let

$$\Omega_k(K; A) := \int_{\mathcal{L}_{k+1}^n} S'_k(K \mid N, A \mid N) dN,$$

where $A \subset \mathcal{F}_{n-1-k}^n$ runs through the Borel sets and

$$A \mid N := \{u \in S^{n-1} : (u, L) \in A \text{ for some } L \perp N\}.$$

$\Rightarrow S_k(K; \cdot)$ is the image of $\Omega_k(K; \cdot)$ under $(u, L) \mapsto u.$
Representation of mixed volumes
(Project by Goodey, Hinderer, Hug, Rataj, W.)

There exists a measurable function $f_k$ on $\mathcal{F}_{n-1-k}^n \times \mathcal{F}_{k-1}^n$ such that

$$V(K[k], M[n-k]) = \int_{\mathcal{F}_{n-1-k}^n} \int_{\mathcal{F}_{k-1}^n} f_k(u, L; v, N) \Omega_{n-k}(M; d(v, N)) \Omega_{k}(K; d(u, L)),$$

for all convex bodies $K, M$. 
Representation of projection functions

For $M = \text{unit ball in } L_\perp$, $L \in \mathcal{L}_j^n$, one gets

$$V_j(K|L) = \int_{\mathcal{F}_{n-1-j}^n} g_j(u, N; L) \Omega_j(K; d(u, N)),$$

with some function $g_j$ on $\mathcal{F}_{n-1-j}^n \times \mathcal{L}_j^n$ (Hinderer, 2002).
Directed projection functions (flag functions)

(1) First approach (Goodey/W. 2004):

We have

\[ V_i(K) = c_{in}S_i(K; S^{n-1}), \quad 1 \leq i < n. \]

We therefore define a directed projection function \( v_{ij}(K; L, u) \), for \( 1 \leq i < j \leq n-1 \), \( L \in \mathcal{L}_j^n \) and almost all \( u \in L \cap S^{n-1} \), by

\[ v_{ij}(K; L, u) := c_{ij}S'_i(K|L; u^+ \cap L). \]

Here, \( u^+ := \{ v \in S^{n-1} : \langle u, v \rangle \geq 0 \} \).

(Groemer (1997) discussed the function \( v_{12}(K, \cdot) \), for \( n = 3 \).) \( v_{ij}(K; \cdot) \) can be interpreted as a function on \( \mathcal{F}_{j-1}^n \).
**Theorem.** If \( 1 \leq i < j \leq n - 1 \) and \( K, M \) are convex bodies of dimension at least \( i + 1 \) with
\[
v_{ij}(K; \cdot) = v_{ij}(M; \cdot),
\]
then \( K \) and \( M \) are translates.

There is also a corresponding **stability result**.

Disadvantages: (a) Determination only up to translation. (b) Approach does not work for \( i = j \). (c) \( K \) may be ‘over-determined’ by such flag functions.
(2) Second approach (Goodey/W. 2005):

We have
\[ V_i(K) = c_{in} \int_{\text{Nor}(K)} \langle x, u \rangle \langle u, v \rangle \Theta_{i-1}(K; d(x, v)) \]
\[ + d_{in} \int_{S^{n-1}} \langle u, v \rangle^2 S_i(K; dv), \quad 1 \leq i \leq n. \]

(Here, \( u \in S^{n-1} \) is arbitrary and \( \Theta_j(K; \cdot) \) is the \( j \)-th support measure on the normal bundle \( \text{Nor}(K) \) of \( K \).)

We therefore define a (second kind of) directed projection function
\[ p_{ij}(K; L, u) = c_{ij} \int_{\text{Nor}(K \mid L)} 1_{u^+}(v) \langle x, u \rangle \langle u, v \rangle \Theta'_{i-1}(K \mid L; d(x, v)) \]
\[ + d_{ij} \int_{u^+ \cap L} \langle u, v \rangle^2 S'_i(K \mid L; dv), \]
(for \( 1 \leq i \leq j \leq n - 1, \ L \in \mathcal{L}_j^n \) and \( u \in L \cap S^{n-1} \)).
**Theorem.** If $1 \leq i \leq j \leq n - 1$ and $K$, $M$ are convex bodies of dimension at least $i + 1$ with

$$p_{ij}(K; \cdot) = p_{ij}(M; \cdot),$$

then $K = M$.

There is no corresponding **stability result** yet.

Again, $K$ may be over-determined by $p_{ij}(K; \cdot)$.

For $i = 1$, $v_{1j}(K; L, \cdot)$ is the **hemispherical transform** of the first surface area measure $S_1'(K|L; \cdot)$, whereas $p_{1j}(K; L, \cdot)$ is the hemispherical transform of the support function $h(K|L, \cdot)$. 
Projection averages

Let $\bar{v}_j(K, u)$ and $\bar{p}_j(K, u)$ be the averages of $v_{1j}(K; L, u)$ resp. $p_{1j}(K; L, u)$ over all $L \in \mathcal{L}_j^n$ which contain $u$ (we only have results for this linear case $i = 1$).

**Theorem.** Let $K, M \subset \mathbb{R}^n$ be convex bodies and $2 \leq j < \frac{2n-3}{5}$ or $\frac{n-2}{2} \leq j \leq n - 1$. If

$$\bar{v}_j(K, \cdot) = \bar{v}_j(M, \cdot),$$

then $K, M$ are translates.

On the other hand, for $i = 1, 2, \ldots$, there are convex bodies $K, M$ in $\mathbb{R}^{5i+4}$, which are not translates, with

$$\bar{v}_{2i+1}(K, \cdot) = \bar{v}_{2i+1}(M, \cdot).$$
Theorem. Let $K, M \subset \mathbb{R}^n$ be convex bodies and $1 \leq j \leq \frac{n}{2} + 1$ or 
\[ \frac{2n+1}{3} < j \leq n - 1, \text{ for } n \neq 4 \text{ (resp. } j = 2 \text{ if } n = 4). \] If 
\[ \overline{p}_j(K, \cdot) = \overline{p}_j(M, \cdot), \]
then $K = M$.

On the other hand, for $i = 1, 2, \ldots$, there are convex bodies $K \neq M$ in $\mathbb{R}^{3i+1}$ with 
\[ \overline{p}_{2i+1}(K, \cdot) = \overline{p}_{2i+1}(M, \cdot). \]