Università degli Studi di Firenze

Dipartimento di Matematica "Ulisse Dini"


Dottorato di Ricerca in Matematica

# Elements of Function Theory in the Unit Ball of Quaternions


Giulia Sarfatti


| **Tutor** | **Coordinatore del Dottorato** |
| --- | --- |
| Prof. Graziano Gentili | Prof. Alberto Gandolfi |


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# Introduction

The search of a definition of regularity for quaternion valued functions of a quaternionic variable interested mathematicians since about a century ago. There have been different approaches that generated different theories. The most famous one is certainly the theory of *Fueter regular* functions, introduced by Fueter himself in the 1930s (see [20, 21]). The class of Fueter regular functions, defined by means of a Cauchy-Riemann type equation, satisfies indeed many key properties of holomorphic functions of one complex variable. For example, we can mention a Cauchy Theorem, a Cauchy Integral Formula, suitable generalizations of Taylor and Laurent series (to deepen this subject see e.g. the survey [49], the book [12], and references therein). Nevertheless, the theory still presents some inconveniences that motivate the search of an alternative definition. For instance, the identity function is not regular according to the definition given by Fueter, and natural quaternionic polynomials and power series are not regular as well.

Quite recently, in 2006, Gentili and Struppa, inspired by the work done by Cullen in [14], introduced a new notion of regularity in the quaternionic setting (see [31, 32]) now known as *slice regularity*. They consider the 4-dimensional real algebra $\mathbb{H}$ of quaternions as the union of complex planes $\mathbb{R}+\mathbb{R}I$ (also called *slices*), each one identified by an imaginary unit $I$. The set of all imaginary units is a 2-dimensional sphere, denoted in the sequel by $\mathbb{S}$, lying in the 3-dimensional space of purely imaginary quaternions. Hence each quaternion $q \in \mathbb{H}$ will be written as $q = x + yI$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$.

**Definition.** *Let $\Omega$ be a domain in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is said to be* slice regular *if, for any $I \in \mathbb{S}$, the restriction $f_I$ of $f$ to $\Omega \cap (\mathbb{R} + \mathbb{R}I)$ has continuous partial derivatives and satisfies the following equation*

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) = 0$$

*for any $x + yI \in \Omega \cap (\mathbb{R} + \mathbb{R}I)$.*

The theory of slice regular functions, already quite rich, is in continuous development. Despite their diverse peculiarities, slice regular functions proved to be a good candidate to play in the quaternionic setting, the role played by holomorphic functions in the complex setting. Several basic results in analogy with the holomorphic case have been proved: we can cite for example the analogues of Cauchy Theorem, Liouville Theorem, Maximum Modulus Principle, Open Mapping Theorem, the Schwarz Lemma, and many others. Moreover the algebraic and topological structure of the zeros and of the singularities of these functions has been understood, different notions of analyticity have been introduced. The theory of slice regular functions has been extended to the more general setting of Clifford and real alternative algebras. It has important applications to the construction of a non-commutative functional calculus and, recently, to the study and classification of orthogonal complex structures in subsets of $\mathbb{H}$, see [25]. The state of the art of this subject is presented in the recent monograph [30] and in [13].

In this Thesis we turn our attention to the study of slice regular functions defined in the unit ball $\mathbb{B}$ of the quaternions,

$$\mathbb{B} = \{q \in \mathbb{H} \mid |q| < 1\},$$

as always a very interesting domain. Geometric function theory in the unit ball of the complex space $\mathbb{C}^n$, for $n \geq 1$, is a very fertile subject, that has produced many important results carrying deep meanings and significance for the the theory of holomorphic functions in general. The ball is in fact a prototype of two classes of domains that have peculiar properties: the strictly pseudoconvex domains and the bounded symmetric domains. In the case of slice regular functions of a quaternionic variable, the ball still occupies a special place: it is the simplest example of a *slice symmetric domain*. This class of domains is important since it plays, in the quaternionic setting, the same role played by domains of holomorphy in classical complex analysis. Another key fact, about slice regular functions defined on an open ball centered at the origin is the following characterization.

**Theorem.** *A function $f$ is slice regular on an open ball $B(0, R) = \{q \in \mathbb{H} \mid |q| < R\}$ if and only if it has a power series expansion of the form*

$$f(q) = \sum_{n \geq 0} q^n a_n, \qquad \text{with } \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H},$$

*converging absolutely and uniformly on compact sets of $B(0, R)$.*

We will present in the first chapter, all the preliminary results about slice regular functions that will be used in the sequel. We do not intend to be exhaustive on the subject, for a complete exposition we refer to the book [30].

The second chapter discusses some properties of the modulus of slice regular functions, in particular it studies how the modulus of a slice regular function is related with the modulus of its *regular conjugate*. The main result presented (appeared in [15]) is an analogue of the Borel-Carathéodory Theorem, an instrument useful to bound the modulus of a slice regular function by means of the modulus of its real part.

**Theorem** (Borel-Carathéodory for slice regular functions). *Let $q_0 \in \mathbb{R}$, $r > 0$, and let $f$ be a slice regular function on (a neighborhood of) the closed ball centered in $q_0$, $\overline{B(q_0, r)}$. Set*

$$A = \max_{|q-q_0|=r} |\operatorname{Re} f(q)|,$$

*and*

$$f(q_0) = \beta + \gamma I \quad \textit{for some } \beta, \gamma \in \mathbb{R} \textit{ and } I \in \mathbb{S}.$$

*If $0 < \varrho < r$, then*

$$|f(q)| \le |\gamma| + |\beta| \frac{r+\varrho}{r-\varrho} + 2A \frac{\varrho}{r-\varrho}$$

*for all $q \in \overline{B(q_0, \varrho)}$.*

The central part of the Thesis contains quaternionic analogues of some classical results regarding holomorphic functions in the complex unit disc $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. The third chapter contains the quaternionic counterpart of the Bohr Theorem concerning power series, together with a weaker version, that follows as in the complex case from the Borel-Carathéodory Theorem (both results appeared in [15]). In the classical setting, the Bohr Theorem answers to the following question, arising from the study of Dirichlet series in Diophantine approximation.

**Question.** *Is there a radius $R > 0$ such that every complex power series of the type*

$$\sum_{n \ge 0} z^n a_n,$$

*holomorphic on $\mathbb{D}$, continuous up to the closure $\overline{\mathbb{D}}$, that maps the closed unit disc to the open unit disc, satisfies*

$$\sum_{n \ge 0} |z^n a_n| < 1 \quad \textit{for any } |z| \le R \ ?$$

The Bohr Theorem establishes that we can find such a radius, and moreover that there exists a maximum radius, $R = 1/3$, for which the statement holds true. In the quaternionic context we overcome the peculiarities of the non-commutative setting and we obtain the exact analogue of the classic case.

**Theorem** (Bohr for slice regular functions). *Let $f(q) = \sum_{n \ge 0} q^n a_n$ be a slice regular function on $\mathbb{B}$, continuous on the closure $\overline{\mathbb{B}}$, such that $|f(q)| < 1$ for all $|q| \le 1$. Then*

$$\sum_{n \ge 0} |q^n a_n| < 1$$

*for all $|q| \le 1/3$. Moreover $1/3$ is the largest radius for which the statement is true.*

In the fourth chapter we prove a Bloch-Landau type theorem, showing that in some sense the image of a ball under a slice regular function can not be too much thin. The classical result says that there exists a universal constant $L$, called *Landau constant*, such that the image of the complex unit disc through a holomorphic function suitably normalized, always contains a disc of radius $L$. There are some estimates that bound $L$ from below with $1/2$ and from above with some constant depending on the Euler function $\Gamma$, but the exact value of this constant is still unknown (see [19]). In the quaternionic setting, mainly because of the fact that composition of two slice regular functions is not slice regular in general, we obtain a weaker statement. Namely, under suitable normalizations, we can find a universal open set (different from a ball) always contained in the image of the unit ball $\mathbb{B}$ through a particular transformation, called *regular translation*, of a slice regular function. The universal set has the form

$$\mathcal{O}(\varrho) = \{q \in \mathbb{H} \mid |q|^3 < \varrho |\operatorname{Re}(q)|^2\},$$

where $\varrho$ is a positive constant. The statement that we obtain (that appeared in [16]) is the following.

**Theorem** (a Bloch-Landau type theorem)**.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a slice regular function that maps $0$ to $0$ and such that its* slice derivative $\partial_c f$ *in $0$ equals $1$. Then there exists $u \in \mathbb{B}$ such that the image of the* regular translation $\tilde{f}_u$ *of $f$ contains an open set obtained by means of a rotation and a translation of $\mathcal{O}(\varrho)$, where the "radius" $\varrho$ is at least* $1/(32\sqrt{2})$.

The fifth chapter is dedicated to Landau-Toeplitz type theorems, that can be found in [26], and that study the possible *shapes* of the image of a slice regular function. We present the quaternionic analogue of the Landau-Toeplitz Theorem together with a (partial) generalization, inspired by the work done in [8] by Burckel, Marshall, Minda, Poggi-Corradini e Ransford. The classical Landau-Toeplitz Theorem can be interpreted as a generalization of the Schwarz Lemma, formulated in terms of the *diameter* of the image of the unit disc through a holomorphic function. In the quaternionic setting, again since slice regularity is not maintained by composition, we have to introduce a new notion of diameter, called *regular diameter*.

**Definition.** *Let $f$ be a slice regular function on $\mathbb{B}$ and let $f(q) = \sum_{n\geq 0} q^n a_n$ be its power series expansion. For $r \in (0,1)$, we define the* regular diameter *of the image of $r\mathbb{B}$ under $f$ as*

$$\tilde{d}_2(f(r\mathbb{B})) = \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\leq r} |f_u(q) - f_v(q)|,$$

*where*

$$f_u(q) = \sum_{n\geq 0} q^n u^n a_n, \qquad f_v(q) = \sum_{n\geq 0} q^n v^n a_n.$$

*Moreover, we define the* regular diameter *of the image of $\mathbb{B}$ under $f$ as*

$$\tilde{d}_2(f(\mathbb{B})) = \lim_{r\to 1^-} \tilde{d}_2(f(r\mathbb{B})).$$

The regular diameter can play the role of the diameter, in fact the former is finite if and only if the latter is finite. The regular diameter hence appears in the statement of the announced result.

**Theorem** (Landau-Toeplitz for slice regular functions)**.** *Let $f$ be a slice regular function on $\mathbb{B}$ such that $\tilde{d}_2(f(\mathbb{B})) = 2$ and let $\partial_c f(0)$ be its slice derivative in $0$. Then*

$$\tilde{d}_2(f(r\mathbb{B})) \leq 2r \quad \textit{for all} \quad r \in (0,1). \tag{1}$$

*and*

$$|\partial_c f(0)| \leq 1. \tag{2}$$

*Moreover, equality holds in (1) for some $r \in (0,1)$, or in (2), if and only if $f$ is an affine function, i.e. $f(q) = a + qb$, with $a, b \in \mathbb{H}$ and $|b| = 1$.*

The new version of the Landau-Toeplitz Theorem proposed in [8] concerns holomorphic functions whose image is measured with a notion of diameter more general than the classic one, the *$n$-diameter*. In the quaternionic setting, after giving an appropriate definition of *regular $n$-diameter*, we are able to obtain the inequality part of the statement, namely we prove inequalities (1) and (2) with the regular $n$-diameter replacing the regular diameter.

The last chapter is devoted to the study of quaternionic Hardy spaces. We begin by the definition of the spaces $H^p(\mathbb{B})$, for $0 < p \leq +\infty$, as follows

**Definition.** *Let $f$ be a slice regular function on $\mathbb{B}$ and let $0 < p < +\infty$. Set*

$$||f||_p = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^p d\theta \right)^{\frac{1}{p}}, \tag{3}$$

*and set*

$$||f||_\infty = \sup_{q \in \mathbb{B}} |f(q)|.$$

*Then, for any $0 < p \leq +\infty$, we define the quaternionic Hardy space $H^p(\mathbb{B})$ as*

$$H^p(\mathbb{B}) = \{f : \mathbb{B} \to \mathbb{H} \mid f \textit{ is slice regular and } ||f||_p < +\infty\}.$$

Then we prove some of the basic properties of $H^p$ functions, that hold in analogy with the classic case. The results presented are part of an ongoing project, planned in collaboration with Professor Chiara de Fabritiis.

In particular, we study the boundary behavior of functions in $H^p(\mathbb{B})$, obtaining the following result

**Theorem.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$. Then for any $I \in \mathbb{S}$, the limit*

$$\lim_{r \to 1^-} f(re^{I\theta}) = \tilde{f}_I(e^{I\theta})$$

*exists for almost every $\theta \in [-\pi, \pi)$ and it belongs to $L^p(\partial\mathbb{B}_I)$. Moreover*

$$||\tilde{f}_I||_{L^p(\partial\mathbb{B}_I)} = ||f_I||_p.$$

We introduce analogues of *inner* and *outer* functions on $\mathbb{B}$ and we investigate factorization properties of $H^p$ functions. We are able to identify the *Blaschke factor* of a function $f$ in $H^p(\mathbb{B})$, built from the zero set of $f$, and, if we consider a particular class of functions, we are able to exhibit a complete factorization. We study separately the cases of $H^2(\mathbb{B})$ (defined already in [3]) and $H^\infty(\mathbb{B})$ that, as in the complex setting, have special properties. Eventually, we introduce the Corona Problem in the quaternionic setting proving a partial statement of the Corona Theorem.

In the next future, we plan to further deepen the study of quaternionic Hardy spaces that we began in this dissertation. There are many directions to explore, as the search for more general versions of the Corona Theorem in the quaternionic setting, or the study of certain classes of operators on these function spaces.

# Chapter 1

# Preliminary results

In this chapter we give the fundamental definitions and the basic results for the class of slice regular functions. Let us state the notations precisely. The 4-dimensional real algebra of quaternions is denoted by $\mathbb{H}$. An element $q$ in $\mathbb{H}$ can be expressed in terms of the standard basis, denoted by $\{1, i, j, k\}$, as $q = x_0 + x_1 i + x_2 j + x_3 k$, where $i, j, k$ are imaginary units, $i^2 = j^2 = k^2 = -1$, related by the multiplication rule $ij = k$. The *real part* of $q$ is denoted by $\mathrm{Re}(q)$, and, in the previous decomposition, it corresponds to $\mathrm{Re}(q) = x_0$; the *imaginary part* is $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The *conjugate* of $q$ is defined as $\overline{q} = \mathrm{Re}(q) - \mathrm{Im}(q)$ and the *modulus* of $q$ is $|q| = \sqrt{q\overline{q}}$. Every non-zero quaternion has a multiplicative inverse, denoted by $q^{-1}$, that can be computed as $q^{-1} = \overline{q}/|q|^2$, hence providing $\mathbb{H}$ with the structure of skew field over $\mathbb{R}$. To every non-real quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ we can associate an imaginary unit, with the map

$$q \mapsto I_q = \frac{\mathrm{Im}(q)}{|\,\mathrm{Im}(q)|},$$

if instead $q \in \mathbb{R}$, we can set $I_q$ to be any arbitrary imaginary unit. In this way, for any $q \in \mathbb{H}$ there exist, and are unique, $x, y \in \mathbb{R}$, with $y \geq 0$ ($y = 0$ if $q \in \mathbb{R}$), such that

$$q = x + yI_q.$$

The set of all imaginary units is denoted by $\mathbb{S}$,

$$\mathbb{S} = \{q \in \mathbb{H} \,|\, q^2 = -1\},$$

and, from a topological point of view, it is a 2-dimensional sphere sitting in the 3-dimensional space of purely imaginary quaternions. This set plays a key role in all the theory. In fact to each element $I$ of $\mathbb{S}$ there corresponds a copy of the complex plane, namely $L_I = \mathbb{R} + \mathbb{R}I \simeq \mathbb{C}$. All these complex planes, also called *slices*, intersect along the real axis, and their union gives back the space of quaternions,

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R}I) = \bigcup_{I \in \mathbb{S}} L_I.$$

As proven in [32], when we multiply two elements of $\mathbb{S}$ we get the following result.

**Proposition 1.1.** *Let $I, J \in \mathbb{S}$ be two imaginary units and let us consider $\mathbb{S}$ as a subset of the 3-dimensional real space of purely imaginary quaternions. If we denote by $\langle\ ,\ \rangle$ the usual scalar product in $\mathbb{R}^3$ , and by $\times$ the usual vector product in $\mathbb{R}^3$, then the following multiplication rule holds true*

$$IJ = -\langle I, J\rangle + I \times J.$$

In particular, if $I$ and $J$ are orthogonal, then $IJ$ belongs to $\mathbb{S}$ and it is orthogonal both to $I$ and to $J$. Hence, in this case, $\{1, I, J, IJ\}$ is an orthonormal basis of $\mathbb{H}$ over $\mathbb{R}$ with the same properties of the standard basis.

## 1.1 Regular functions

Let us set just one more notation before recalling the definition of slice regularity. For any $I \in \mathbb{S}$, if $\Omega$ is a subset of $\mathbb{H}$, we will denote by $\Omega_I$ the intersection of $\Omega$ with the slice $L_I$,

$$\Omega_I = \Omega \cap L_I.$$

**Definition 1.2.** *Let $\Omega$ be a domain (open connected subset) in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is* slice regular *if for any $I \in \mathbb{S}$ the restriction of $f$ to $\Omega_I$, denoted by $f_I$, is* holomorphic *in $\Omega_I$, namely it has continuous partial derivatives and it is such that*

$$\overline{\partial}_I f_I(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) = 0$$

*for all $x + yI \in \Omega_I$.*

Since no confusion can arise, from now on, we will refer to slice regular functions simply as *regular.*

Even if the definition of regularity can be given for any domain in $\mathbb{H}$, to avoid degenerate cases, like regular functions that are not even continuous, we need a special class of domains, introduced in [11, 47].

**Definition 1.3.** *Let $\Omega$ be a domain in $\mathbb{H}$. We say that $\Omega$ is a* slice domain *if $\Omega \cap \mathbb{R} \neq \varnothing$ and if, for every $I \in \mathbb{S}$, $\Omega_I$ is a domain (open connected subset) in $L_I$.*

For regular functions we can give a natural notion of derivative.

**Definition 1.4.** *Let $\Omega$ be a slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a regular function. The* slice derivative *(or* Cullen derivative*) of $f$ at $q = x + yI \in \Omega$ is defined as*

$$\partial_c f(x + yI) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f(x + yI) = \frac{\partial}{\partial x} f(x + yI).$$

This definition is well posed because it is applied only to regular functions. We point out that, since the definition is given "slicewise", it is actually possible to compute the slice derivative of a function that is only holomorphic on a slice. Notice also that the operators $\partial_c$ and $\overline{\partial}_I$ do commute, hence the slice derivative of a regular function is still regular. Thus, we can iterate the differentiation process obtaining (see for instance [32]),

$$\partial_c^n f = \frac{\partial^n}{\partial x^n} f \quad \text{for any} \quad n \in \mathbb{N}.$$

As stated in [32], a quaternionic power series $\sum_{n\geq 0} q^n a_n$ with $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ defines a regular function in its domain of convergence, indeed the analogue of Abel Theorem holds true.

**Theorem 1.5.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{H}$ and let*

$$R = \frac{1}{\limsup_{n\in\mathbb{N}} |a_n|^{1/n}}.$$

*If $R > 0$, then the power series*

$$\sum_{n\geq 0} q^n a_n$$

*converges absolutely and uniformly on compact sets in $B(0, R)$. Moreover, its sum defines a regular function on $B(0, R)$.*

Furthermore, in [32], a very useful characterization of regular functions on open balls centered at the origin is proven.

**Theorem 1.6.** *A function $f$ is regular in $B = B(0, R)$ if and only if $f$ has a power series expansion*

$$f(q) = \sum_{n\geq 0} q^n a_n \quad \textit{with} \quad a_n = \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0)$$

*converging in $B$.*

The following result establishes the connection between the class of regular functions and the class of complex holomorphic functions of one complex variable (proved in [32] for regular functions on an open ball $B(0, R)$, but the proof applies to any open set).

**Lemma 1.7** (Splitting Lemma)**.** *If $f$ is a regular function on a slice domain $\Omega$, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, $J$ orthogonal to $I$, there exist two holomorphic functions $F, G : \Omega_I \to L_I$, such that for every $z = x + yI \in \Omega_I$, we have*

$$f_I(z) = F(z) + G(z)J.$$

The following version of the Identity Principle holds true for regular functions on a slice domain.

**Theorem 1.8** (Identity Principle)**.** *Let $f$ be a regular function on a slice domain $\Omega$. Denote by $Z_f$ the zero set of $f$,*

$$Z_f = \{q \in \Omega \mid f(q) = 0\}.$$

*If there exists $I \in \mathbb{S}$ such that $\Omega_I \cap Z_f$ has an accumulation point in $\Omega_I$, then $f$ vanishes identically on $\Omega$.*

A stronger version of the Identity Principle (that we will not use here) holds; it can be found, e.g., in [30].

### 1.1.1 Representation Formula and extension results

A second condition that is natural to require for a domain of definition of a regular function, and that appears in [10, 11, 28], is the following.

**Definition 1.9.** *A subset $A$ of $\mathbb{H}$ is said to be* axially symmetric *if for all $x + yI \in A$, with $x, y \in \mathbb{R}$, $y \neq 0$, and $I \in \mathbb{S}$, the whole 2-sphere $x + y\mathbb{S} = \{x + yJ \mid J \in \mathbb{S}\}$ is contained in $A$.*

We point out that axially symmetric sets were previously introduced in [14], under the name of intrinsic domains. For the sake of shortness we will refer to axially symmetric sets, simply as symmetric sets. As anticipated in the Introduction, open balls centered at the origin are the simplest examples of symmetric slice domains.
The following result shows how we can recover the values of a function defined on a symmetric slice domain from its values on a single slice $L_I$ (see [10, 11]).

**Theorem 1.10** (Representation Formula)**.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$ and let $I \in \mathbb{S}$. Then, for all $x + yJ \in \Omega$, the following equality holds*

$$f(x + yJ) = \frac{1}{2}\left[f(x + yI) + f(x - yI)\right] + \frac{JI}{2}\left[f(x - yI) - f(x + yI)\right].$$

*In particular the function $f$ is* affine *on each sphere of the form $x + y\mathbb{S}$ contained in $\Omega$, namely there exist $b, c \in \mathbb{H}$ such that $f(x + yJ) = b + Jc$ for all $J \in \mathbb{S}$ .*

Thanks to the Representation Formula it is possible to prove that we can estimate the maximum modulus of a function with its maximum modulus on each slice, (see [36]).

**Proposition 1.11.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$ and let $K$ be a symmetric compact subset of $\Omega$. If there exist an imaginary unit $I \in \mathbb{S}$, a point $p \in \mathbb{H}$ and a real number $M \in (0, +\infty)$ such that*

$$f_I(K_I) \subset B(p, M),$$

*then*

$$f(K) \subset B(p, 2M).$$

In particular, if $\Omega = B(0,R)$, we have the following result.

**Proposition 1.12.** *Let $f$ be a regular function on $B = B(0,R)$. If we set $M = \sup_{z\in B_I} |f(z)|$, then*

$$\sup_{q\in B} |f(q)| \leq 2M.$$

In the special case of a regular function $f$ that maps a slice $L_I$ to itself, we obtain that the maximum (and minimum) modulus of $f$ is actually equal to its maximum (and minimum) modulus on the preserved slice.

**Proposition 1.13.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$ such that $f(\Omega_I) \subset L_I$ for some $I \in \mathbb{S}$. Then, for any $x + y\mathbb{S} \subset \Omega$,*

$$\max_{J\in\mathbb{S}} |f(x+yJ)| = \max\{|f(x+yI)|, |f(x-yI)|\}$$

*and*

$$\min_{J\in\mathbb{S}} |f(x+yJ)| = \min\{|f(x+yI)|, |f(x-yI)|\}.$$

*Proof.* Representation Formula 1.10 yields that for any $q = x + yJ \in \Omega$, if we denote by $z = x + yI \in \Omega_I$, then

$$|f(q)|^2 = \left|\frac{1}{2}\left(f(z) + f(\overline{z})\right) + \frac{JI}{2}\left(f(\overline{z}) - f(z)\right)\right|^2. \tag{1.1}$$

In vector notation, according with Proposition 1.1, we can set $\langle J, I\rangle = \cos\theta$, and $J \times I = \sin\theta L$ with $L$ orthogonal to $I$ and $\theta$ the angle between $J$ and $I$. Taking into account that $f$ maps $L_I$ to itself and that $I$ is orthogonal to $L$, we can write equation (1.1) as

$$\begin{aligned}
|f(q)|^2 &= \left|\frac{1}{2}\left(f(z) + f(\overline{z})\right) + \frac{-\cos\theta + \sin\theta L}{2}\left(f(\overline{z}) - f(z)\right)\right|^2 \\
&= \frac{1}{4}\big(\,|f(z) + f(\overline{z}) - \cos\theta\,(f(\overline{z}) - f(z))|^2 + |\sin\theta\,(f(\overline{z}) - f(z))|^2\,\big) \\
&= \frac{1}{4}\big(\,|f(z) + f(\overline{z})|^2 + \cos^2\theta\,|f(\overline{z}) - f(z)|^2 - 2\cos\theta\,\mathrm{Re}\,\big(\,(f(z) + f(\overline{z}))\,(\overline{f(\overline{z}) - f(z)})\big)\big) \\
&+ \frac{1}{4}\sin^2\theta\,|f(\overline{z}) - f(z)|^2 \\
&= \frac{1}{4}\big(\,|f(z) + f(\overline{z})|^2 + |f(\overline{z}) - f(z)|^2\,\big) - \frac{\cos\theta}{2}\,\mathrm{Re}\,\big((f(z) + f(\overline{z}))(\overline{f(\overline{z}) - f(z)})\big).
\end{aligned}$$

Therefore $|f(q)| = |f(x + yJ)|$ attains its extremal values at $\theta = 0$ (corresponding to $J = I$) and at $\theta = \pi$ (corresponding to $J = -I$), that leads to the thesis. □

The fact that a regular function on a symmetric slice domain is uniquely determined by its restriction to a slice, motivates the following definition and extension result (see [10, 11]).

**Definition 1.14.** *Let $A$ be a subset of $\mathbb{H}$. The* symmetric completion $\widetilde{A}$ *of $A$ is the smallest symmetric set containing $A$, namely*

$$\widetilde{A} = \bigcup_{x+yI \in A} x + y\mathbb{S}.$$

**Lemma 1.15** (Extension Lemma)**.** *Let $\Omega_I$ be a domain in $L_I$ symmetric with respect to the real axis. If $f_I : \Omega_I \to \mathbb{H}$ is holomorphic, then setting*

$$f(x+yJ) = \frac{1}{2}[f_I(x+yI) + f_I(x-yI)] + \frac{JI}{2}[f_I(x-yI) - f_I(x+yI)]$$

*extends $f_I$ to a regular function on the symmetric completion $\widetilde{\Omega}$ of $\Omega_I$. Moreover $f$ is the unique regular extension of $f_I$ and it is denoted by* $\mathrm{ext}(f_I)$.

In particular it has been proved (see for instance [30]) that every regular function, defined on a slice domain $\Omega$, can be uniquely extended to the smallest symmetric domain containing $\Omega$. Hence the symmetric slice domains play the role that the domains of holomorphy play in classical complex analysis.
In the special case of functions that are regular on an open ball centered at the origin $B(0,R)$, the Extension Lemma 1.15 can be stated in terms of power series expansion.

**Remark 1.16.** If $f_I$ is a holomorphic function on a disc $B_I = B(0,R) \cap L_I$ and its power series expansion is

$$f_I(z) = \sum_{n \geq 0} z^n a_n, \qquad \text{with } \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H},$$

then the unique regular extension of $f_I$ to the whole ball $B(0,R)$ is the function defined as

$$\mathrm{ext}(f_I)(q) = \sum_{n \geq 0} q^n a_n.$$

The uniqueness is guaranteed by the Identity Principle 1.8.

### 1.1.2 Algebraic structure and zeros of regular functions

The pointwise product of two regular functions is not, in general, regular. To preserve the regularity, a new multiplication operation, the $*$-product, was introduced. On open balls centered at the origin, the $*$-product of two regular functions can be defined by means of their power series expansions (as done in [27]), extending the classical $*$-product of polynomials with coefficients in a non commutative ring (see, e.g., [41]).

**Definition 1.17.** *Let $f, g : B = B(0,R) \to \mathbb{H}$ be regular functions and let*

$$f(q) = \sum_{n \geq 0} q^n a_n, \quad g(q) = \sum_{n \geq 0} q^n b_n$$

*be their series expansions. The* regular product *(or* $*$-product*) of $f$ and $g$ is the function defined as*

$$(f * g)(q) = f * g(q) = \sum_{n \geq 0} q^n \sum_{k=0}^{n} a_k b_{n-k},$$

*regular on $B$.*

Notice that if the power series expansion of $f$ has real coefficients, then the regular product of $f$ and $g$ coincides with their usual product.
In order to define the regular product of $f$ and $g$, regular functions on a symmetric slice domain $\Omega$, take $I, J \in \mathbb{S}$, with $I$ orthogonal to $J$, and, according to the Splitting Lemma 1.7, let $F, G, H, K : \Omega_I \to L_I$ be holomorphic functions such that for all $z \in \Omega_I$

$$f_I(z) = F(z) + G(z)J \quad \text{and} \quad g_I(z) = H(z) + K(z)J.$$

Set $f_I * g_I : \Omega_I \to L_I$ to be the holomorphic function defined as

$$f_I * g_I(z) = [F(z)H(z) - G(z)\overline{K(\overline{z})}] + [F(z)K(z) + G(z)\overline{H(\overline{z})}]J.$$

The following definition is given in [11].

**Definition 1.18.** *Let $\Omega$ be a symmetric slice domain in $\mathbb{H}$, and let $f, g : \Omega \to \mathbb{H}$ be regular functions. The* regular product *(or* $*$-product*) of $f$ and $g$ is the function defined as*

$$(f * g)(q) = f * g(q) = \text{ext}(f_I * g_I)(q),$$

*regular on $\Omega$.*

Notice that the $*$-product is associative and not, in general, commutative. Moreover it is possible to prove (see for instance [30]) that, together with the addition, it gives a ring structure to the space of regular functions.

**Proposition 1.19.** *Let $\Omega$ be a symmetric slice domain. Then the set $\mathcal{R}(\Omega)$ of regular functions on $\Omega$,*

$$\mathcal{R}(\Omega) = \{f : \Omega \to \mathbb{H} \mid f \textit{ is regular}\},$$

*endowed with the operations of addition and $*$-product is a non-commutative ring.*

The $*$-product and the pointwise product of regular functions are connected by the following relation (see [30]).

**Proposition 1.20.** *Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then*

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \textit{if} \quad f(q) \neq 0 \\ 0 & \textit{if} \quad f(q) = 0 \end{cases} \tag{1.2}$$

Notice that if $q = x + yI$ (and if $f(q) \neq 0$), then $f(q)^{-1}qf(q)$ has the same modulus and same real part as $q$, hence $f(q)^{-1}qf(q)$ lies in the same 2-sphere $x + y\mathbb{S}$ as $q$. This means that a zero $x_0 + y_0 I$ of the function $g$ is not necessarily a zero of $f * g$, but an element on the same sphere $x_0 + y_0\mathbb{S}$ does.

To present a characterization of the structure of the zero set of a regular function $f$ we need to introduce the following operations.

**Definition 1.21.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$ and suppose that for $z \in \Omega_I$, the splitting of $f$ with respect to $J$ ($J \in \mathbb{S}$ orthogonal to $I$) is $f_I(z) = F(z) + G(z)J$. Consider the holomorphic function*

$$f_I^c(z) = \overline{F(\overline{z})} - G(z)J.$$

*The* regular conjugate *of $f$ is the function defined by*

$$f^c(q) = \text{ext}(f_I^c)(q).$$

*The* symmetrization *of $f$ is the function defined by*

$$f^s(q) = f * f^c(q) = f^c * f(q).$$

*Both $f^c$ and $f^s$ are regular functions on $\Omega$.*

**Remark 1.22.** In the special case of a regular function $f$ on an open ball $B(0, R)$, if the power series expansion of $f$ is

$$f(q) = \sum_{n \geq 0} q^n a_n,$$

then

$$f^c(q) = \sum_{n \geq 0} q^n \overline{a}_n \quad \text{and} \quad f^s(q) = \sum_{n \geq 0} q^n \sum_{k=0}^{n} a_k \overline{a}_{n-k}.$$

**Remark 1.23.** Thanks to Proposition 1.20 we obtain that each zero of $f$ is a zero of $f^s$ and the same holds true for $f^c$ (because $f^s = f * f^c = f^c * f$).

Indeed something more is true, let us begin by recalling the following definition.

**Definition 1.24.** *A regular function $f$ on a symmetric slice domain $\Omega$ is said to be* slice preserving *if it maps every slice to itself, namely if $f_I(\Omega_I) \subset L_I$ for any $I \in \mathbb{S}$.*

Such functions are very special, in fact on each slice they behave exactly as holomorphic functions of one complex variable. One of their peculiarities, consequence of the Representation Formula 1.10, is that if they vanish in one point of a sphere of the type $x + y\mathbb{S}$, then they vanish everywhere on that sphere.

**Proposition 1.25.** *Let $f$ be a slice preserving function on a symmetric slice domain $\Omega$. If $f(x+yJ)=0$ for some $J\in\mathbb{S}$, then $f(x+yI)=0$ for any $I\in\mathbb{S}$.*

In [11] it has been proved that the symmetrization $f^s$ of a regular function $f$ is slice preserving. (Indeed, if $f$ is regular on a ball $B(0,R)$, from Remark 1.22 it is not difficult to see that the coefficients of the power series expansion of $f^s$ are real numbers). This fact, together with Remark 1.23, leads to the following statement (for a proof see e.g. [30]).

**Proposition 1.26.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$. Then in any sphere $x+y\mathbb{S}\subset\Omega$ the zeros of $f$ and of $f^c$ are in one-to-one correspondence. Moreover, $f^s$ has a zero in $x+y\mathbb{S}$ if and only if $f^s$ vanishes identically on $x+y\mathbb{S}$.*

Before we give the complete characterization of the zero set of a regular function, let us recall another result in this setting, consequence of the Representation Formula 1.10.

**Proposition 1.27.** *Let $f$ be a regular function on a symmetric slice domain $\Omega\subset\mathbb{H}$ such that $f(\Omega_I)\subset L_I$ for some $I\in\mathbb{S}$. If $f(x+yJ)=0$ for $J\in\mathbb{S}\setminus\{\pm I\}$, then $f(x+yK)=0$ for any $K\in\mathbb{S}$.*

Using the relations of the zeros of a regular function with the zeros of its symmetrization it is possible to prove (see for instance [27]) the announced characterization of the zero set of a regular function. This characterization can also be viewed as a consequence of the Representation Formula 1.10.

**Theorem 1.28** (Zero set structure)**.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$. If $f$ does not vanish identically, then its zero set consists of the union of isolated points and isolated 2-spheres of the form $x+y\mathbb{S}$ with $x,y\in\mathbb{R}$, $y\neq 0$.*

The previous theorem suggests the following classification.

**Definition 1.29.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$. A 2-dimensional sphere $x+y\mathbb{S}\subset\Omega$ of zeros of $f$ is called a* spherical zero *of $f$. Any point $x+yI$ of such a sphere is called a* generator *of the spherical zero $x+y\mathbb{S}$. Any zero of $f$ that is not a generator of a spherical zero is called an* isolated zero *(or a* non spherical zero *or simply a* zero*) of $f$.*

**Remark 1.30.** A real zero $x$ can be interpreted as spherical zero, where the sphere $x+y\mathbb{S}$ coincides with $x$ ($y=0$).

Zeros can be extracted from a regular function in the following manner (see [30]).

**Proposition 1.31.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$. A point $p=x+yI\in\Omega$ is a zero of $f$ if and only if there exists a regular function $g:\Omega\to\mathbb{H}$ such that*

$$f(q)=(q-p)*g(q).$$

*Furthermore, $f$ vanishes identically on the sphere $x+y\mathbb{S}$ generated by $p$ if and only if there exists a regular function $h:\Omega\to\mathbb{H}$ such that*

$$f(q)=\left((q-x)^2+y^2\right)*h(q)=\left((q-x)^2+y^2\right)h(q).$$

An appropriate notion of multiplicity, introduced in [33] for the case of regular quaternionic polynomials, can be given for zeros of regular functions. To this aim, we point out that if the polynomial $(q-p_1)*(q-p_2)*\cdots*(q-p_n)$ does not have spherical zeros (i.e. $p_i\neq\overline{p}_{i+1}$ for all $i\in\{1,\dots,n-1\}$), then its only root is the quaternion $p_1$ (see [33]).

**Definition 1.32.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$ and let $x+y\mathbb{S}\subset\Omega$ with $y\neq 0$. Let $m,n\in\mathbb{N}$ and $p_1,\dots,p_n\in x+y\mathbb{S}$, with $p_i\neq\overline{p}_{i+1}$ for all $i\in\{1,\dots,n-1\}$, be such that*

$$f(q)=\left((q-x)^2+y^2\right)^m(q-p_1)*(q-p_2)*\cdots*(q-p_n)*g(q)$$

*for some regular function $g:\Omega\to\mathbb{H}$ which does not have zeros in $x+y\mathbb{S}$. Then $2m$ is called the* spherical multiplicity *of $x+y\mathbb{S}$ and $n$ is called the* isolated multiplicity *of $p_1$. On the other hand, if $x\in\mathbb{R}$, then we call* isolated multiplicity *of $f$ at $x$ the number $k\in\mathbb{N}$ such that*

$$f(q)=(q-x)^kh(q)$$

*for some regular function $h:\Omega\to\mathbb{H}$ which does not vanish at $x$.*

In [36] it is proved that is possible to write a complete factorization of the zeros of an entire function (i.e. a function that is regular on $\mathbb{H}$) and in order to do it, there are proved the following results concerning infinite products of quaternions and infinite regular products of regular functions.

**Theorem 1.33.** *Let $\{a_n\}_{n\in\mathbb{N}}\subset\mathbb{H}$. A sufficient condition for the convergence of the product*

$$\prod_{n\geq 0}(1+a_n)$$

*is the convergence of the series*

$$\sum_{n\geq 0}|a_n|.$$

**Theorem 1.34.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions defined on a symmetric slice domain $\Omega$. The infinite $*$-product*

$$\prod_{n\geq 0}^{*}(1+f_n(q))$$

*converges uniformly on compact sets of $\Omega$ if and only if the infinite product*

$$\prod_{n\geq 0}(1+f_n(q))$$

*converges uniformly on compact sets of $\Omega$.*

Moreover, where the infinite regular product does converge, it defines a regular function.

**Theorem 1.35.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular functions defined on a symmetric slice domain $\Omega$. If the infinite regular product*

$$\prod_{n\geq 0}^{*}(1+f_n(q))$$

*converges uniformly on compact sets of $\Omega$ to a function $F$, then $F$ is regular on $\Omega$.*

The knowledge of the zero set of a regular function $f$, allows us to define the inverse element of $f$ with respect to the $*$-product. Recall that $Z_{f^s}$ denotes the zero set of the symmetrization $f^s$ of $f$.

**Definition 1.36.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$. If $f$ does not vanish identically, its* regular inverse *is defined as the function*

$$f^{-*}(q) = f^s(q)^{-1}f^c(q)$$

*regular on $\Omega \setminus Z_{f^s}$.*

Thanks to the study of regular inverses of regular functions it is possible to prove that for any couple of regular functions $f, g$ we have

$$(f * g)^{-*} = g^{-*} * f^{-*}.$$

The following relation between the regular inverse and the standard inverse of a regular function is proven in [28].

**Proposition 1.37.** *Let $f : B \to \mathbb{H}$ be a regular function. If we set*

$$T_f(q) = f^c(q)^{-1}qf^c(q),$$

*then*

$$f^{-*}(q) = f(T_f(q))^{-1} \quad \textit{for every} \quad q \in B \setminus Z_{f^s}.$$

It is then possible to consider regular quotients of the form $f^{-*} * g$, defined outside the zero set of $f^s$.

Regular quotients are related to pointwise quotients by the following result, [47].

**Proposition 1.38.** *Let $\Omega$ be a symmetric slice domain in $\mathbb{H}$ and let $f$ and $g$ be regular functions on $\Omega$. If $T_f : \Omega \setminus Z_{f^s} \to \Omega \setminus Z_{f^s}$ is defined as*

$$T_f(q) = f^c(q)^{-1}qf^c(q),$$

*then*

$$(f^{-*} * g)(q) = f^{-*} * g(q) = f(T_f(q))^{-1}g(T_f(q)) \quad \textit{for every} \quad q \in B \setminus Z_{f^s}.$$

*Furthermore, $T_f$ and $T_{f^c}$ are mutual inverses so that $T_f$ is a diffeomorphism.*

Let us mention at this point the Leibniz rule, that holds in terms of the $*$-product (see [30]), and that will be used in the sequel.

**Proposition 1.39** (Leibniz rule)**.** *Let $f$ and $g$ be regular functions on a symmetric slice domain $\Omega$. Then*

$$\partial_c(f * g)(q) = \partial_c f * g(q) + f * \partial_c g(q)$$

*for every $q \in \Omega$.*

### 1.1.3 Regular composition of regular functions

One of the issues that arises in the quaternionic setting is the fact that the composition of two regular functions, say $f$ and $g$, is not regular in general. In fact, for a general $f$, the only case in which the regularity of $f \circ g$ is maintained is when $g$ is slice preserving (under the right hypotheses about the domain and range of the considered functions). If instead $g$ has no constraints, then $f$ has to be an affine function of the form $f(q) = a + qb$ for some $a, b \in \mathbb{H}$. For instance, if we restrict our attention to regular functions on open balls centered at the origin, all these considerations can be shown by means of formal composition of power series expansions of $f$ and $g$. Therefore we need to introduce a different notion of composition, that preserves regularity and that coincides with the usual one in the special cases cited before.

Let us concentrate on the case of functions defined on open balls centered at the origin $B(0, R)$. The general situation is studied in [50], and just as a quick insight, we can say that the so called *Bell Polynomials* appear in the expression of the coefficients of the power series of the regular composition.
In particular we are interested in the simplest cases, namely when $g$ is either a linear function or a translation.

**Definition 1.40.** *Let $f(q) = \sum_{n\geq 0} q^n a_n$ be a regular function on $B(0, R)$, and let $u \in \mathbb{H}$. We define the* regular composition *of $f$ with the function $\mu_u : q \mapsto qu$ as*

$$f_u(q) = \sum_{n\geq 0} (qu)^{*n} a_n = \sum_{n\geq 0} q^n u^n a_n.$$

*For any $u \neq 0$, the function $f_u$ is regular on the ball $B(0, R/|u|)$. If instead $u = 0$ $f_u$ is trivially regular on $\mathbb{H}$.*

**Remark 1.41.** In terms of Extension Lemma 1.16, the function $f_u$ is the unique regular extension of the holomorphic function defined by the composition of $f$ with $\mu_u$ on the slice identified by $u$, $L_{I_u}$,

$$f_u(q) = \mathrm{ext}(f_{I_u} \circ \mu_u)(q).$$

In fact if $u$ and $z$ lie in the same plane $L_{I_u}$, then $u$ and $z$ do commute and hence $f_u(z) = f(zu)$ (where this expression makes sense). In particular, if $u \in \mathbb{R}$, then $\mu_u$ is slice preserving and $f_u(q) = f(qu)$ for every $q \in B(0, R/|u|)$ (or every $q \in \mathbb{H}$ if $u = 0$). Moreover, according to what we said previously, if $f$ is an affine function, $f(q) = a + qb$, for some quaternions $a$ and $b$, then $f_u(q)$ coincides again with the usual composition $f_u(q) = a + qub = f(qu)$ for any $u \in \mathbb{H}$.

**Definition 1.42.** *Let $f(q) = \sum_{n\geq 0} q^n a_n$ be a regular function on $B = B(0, R)$, and let $w \in B$. We define the* regular composition *of $f$ with the translation $\tau_w : q \mapsto q + w$ as*

$$\tilde{f}_w(q) = \sum_{n\geq 0} (q + w)^{*n} a_n,$$

*regular on $B(0, R - |w|)$. The function $\tilde{f}_w$ is called the* regular translation *by $w$ of $f$.*

**Remark 1.43.** As in the case of composition with a linear function, $\tilde{f}_w$ is the unique regular extension of the holomorphic function obtained composing $f$ with $\tau_w$ on (a suitable subset of) the slice $L_{I_w}$ identified by $w$,

$$\tilde{f}_w(q) = \mathrm{ext}(f_{I_w} \circ (\tau_w)_{I_w})(q).$$

Also, if $w \in \mathbb{R}$, then $\tau_w$ is slice preserving and the regular composition coincides with the usual one.

### 1.1.4 Integral representations

With no intent to be exhaustive, let us just mention some results concerning integral representations of regular functions that will be used in the sequel. In [30] it is proved the following slicewise Cauchy Integral Formula

**Theorem 1.44.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$, let $I \in \mathbb{S}$ and let $U_I$ be a bounded Jordan domain in $L_I$, such that its closure $\overline{U}_I$ is contained in $\Omega_I$. If $\partial U_I$ is rectifiable, then*

$$f(z) = \frac{1}{2\pi I} \int_{\partial U_I} \frac{d\zeta}{\zeta - z} f(\zeta)$$

*for any $z \in U_I$.*

The previous result allows the computation of the $n$-th slice derivative of a regular function as follows.

**Theorem 1.45.** *Let $f$ be a regular function on a symmetric slice domain $\Omega$, let $I \in \mathbb{S}$ and let $U_I$ be a bounded Jordan domain in $L_I$, such that its closure $\overline{U}_I$ is contained in $\Omega_I$. If $\partial U_I$ is rectifiable, then*

$$\partial_c^n f(z) = \frac{n!}{2\pi I} \int_{\partial U_I} \frac{d\zeta}{(\zeta - z)^{n+1}} f(\zeta)$$

*for any $z \in U_I$ and for any $n \in \mathbb{N}$.*

## 1.2 Maximum Modulus Principle and applications

In analogy with the complex case, a Maximum Modulus Principle holds for regular functions (see [32]).

**Theorem 1.46** (Maximum Modulus Principle)**.** *Let $f : B \to \mathbb{H}$ be a regular function. If there exists $I \in \mathbb{S}$ such that the restriction $|f_I|$ has a local maximum in $B_I$, then $f$ is constant in $B$. In particular, if $|f|$ has a local maximum in $B$, then $f$ is constant in $B$.*

Among all the consequences that the previous result yields, there is the analogue of the Open Mapping Theorem for regular functions, whose statement needs a preliminary definition.

**Definition 1.47.** *Let $\Omega$ be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a regular function. We define the* degenerate set *of $f$ as the union $D_f$ of the 2-spheres $x + y\mathbb{S}$ (with $y \neq 0$) such that $f|_{x+y\mathbb{S}}$ is constant.*

The degenerate set of a non-constant regular function has empty interior. We are now able to recall the following result (see [28, 29]).

**Theorem 1.48** (Open Mapping Theorem)**.** *Let $\Omega$ be a symmetric slice domain and let $f : \Omega \to \mathbb{H}$ be a regular function. If $D_f$ is the degenerate set of $f$, then $f : \Omega \setminus \overline{D}_f \to \mathbb{H}$ is open.*

In the particular case of a function defined on an open ball centered at the origin, the following theorem holds.

**Theorem 1.49.** *Let $f$ be a non-constant regular function on $B = B(0, R)$. Then $f(B)$ is an open set.*

As in the complex setting, using a maximum modulus argument it is possible to prove the analogue of the Schwarz Lemma. Recall that $\mathbb{B}$ denotes the open unit ball of $\mathbb{H}$,

$$\mathbb{B} = \{q \in \mathbb{H} \,|\, |q| < 1\}.$$

**Theorem 1.50** (Schwarz Lemma)**.** *Let $f : \mathbb{B} \to \mathbb{B}$ be a regular function such that $f(0) = 0$. Then*

$$|f(q)| \leq |q| \quad \text{for every} \quad q \in \mathbb{B} \tag{1.3}$$

*and*

$$|\partial_c f(0)| \leq 1. \tag{1.4}$$

*Moreover, equality holds in* (1.3) *for some $q \neq 0$, or in* (1.4)*, if and only if $f(q) = qu$, with $u \in \mathbb{H}$, $|u| = 1$.*

## 1.3 Regular fractional transformations

Let us denote by $GL(2,\mathbb{H})$ the set of all invertible $2 \times 2$ matrices with quaternionic coefficients, namely

$$GL(2,\mathbb{H}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{H},\ \det_{\mathbb{H}}(A) \neq 0 \right\},$$

where $\det_{\mathbb{H}}$ denotes the Dieudonné determinant (see [5]),

$$\det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |c|^2|b|^2 - 2\,\mathrm{Re}(c\overline{a}b\overline{d})}.$$

To every $A \in GL(2,\mathbb{H})$ there corresponds a fractional linear transformations of the Alexandroff compactification of $\mathbb{H}$, $\widehat{\mathbb{H}}$.

**Definition 1.51.** *A function $L : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ is a* fractional linear transformation *if it is of the form*

$$L(q) = (qa+b)^{-1}(qc+d)$$

*where $a,b,c,d \in \mathbb{H}$ are such that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{H}).$$

Even if they are not regular, fractional linear transformations are homeomorphism of $\widehat{\mathbb{H}}$ onto itself and have some nice properties. For instance, let $\mathcal{F}_3$ be the family of 3-dimensional spheres and 3-dimensional affine subspaces of $\mathbb{H}$. The following result is proven in [5].

**Proposition 1.52.** *Every fractional linear transformation maps the family $\mathcal{F}_3$ to itself.*

The regular version of fractional linear transformations is the following

**Definition 1.53.** *A function $T : \widehat{\mathbb{H}} \to \widehat{\mathbb{H}}$ is a* regular fractional transformation *if it is of the form*

$$T(q) = (qa+b)^{-*} * (qc+d)$$

*where $a,b,c,d \in \mathbb{H}$ are such that*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{H}).$$

Let us conclude this chapter by mentioning a special example of regular fractional transformations, namely the analogues of Moebius transformations, introduced in [48].

**Definition 1.54.** *A* regular Moebius transformation *is a regular function $M : \mathbb{B} \to \mathbb{B}$ of the form*

$$M(q) = (1 - q\overline{a})^{-*} * (q-a)u$$

*where $a \in \mathbb{B}$ and $u \in \partial\mathbb{B}$.*

Regular Moebius transformations have several interesting properties (see [48]). In particular, they are regular bijections of the open unit ball onto itself (so they only have one zero) and, by direct computation, it is possible to prove that they map the boundary of the unit ball to itself. More precisely the following result holds true.

**Proposition 1.55.** *A function $M$ is a regular bijection from $\mathbb{B}$ to itself if and only if $M$ is a regular Moebius transformation.*

# Chapter 2

# On the modulus of regular functions

The *modulus* of a regular function $f$ can be a useful tool to study the *shape* of the range of $f$. In fact if the image of $f$ is in some sense centered in $q = 0$, the modulus of $f$ represents the *radius* of the image itself. For instance, this is what happens in the case of the Schwarz Lemma 1.50. As we have seen in Chapter 1, a basic, but fundamental, result concerning the modulus of regular functions, is the Maximum Modulus Principle 1.46. In this chapter we will study some properties of the modulus and of the maximum modulus of regular functions, descending from the Maximum Modulus Principle 1.46. We will also see how these properties are related to the uniform norm. Furthermore we will prove an important result, that holds in analogy with the complex case: the Borel-Carathéodory Theorem. This result shows that it is possible to bound the modulus of a regular function with the modulus of its real part. We refer to [34], for a more detailed study of real parts of regular functions.

## 2.1 Uniform norm and regular conjugation

Let us denote by $B = B(0, R)$ the open ball

$$B = \{q \in \mathbb{H} \mid |q| < R\},$$

and by $\mathcal{R}(B)$ the real vector space

$$\mathcal{R}(B) = \{f : B \to \mathbb{H} \mid f \text{ is regular}\}.$$

We can endow $\mathcal{R}(B)$ with the standard uniform norm $||\cdot||_B$, that is, for all $f \in \mathcal{R}(B)$ we can set

$$||f||_B = \sup_{q\in B} |f(q)|.$$

Clearly we can write

$$||f||_B = \sup_{|q|<R} |f(q)|,$$

and in fact it is possible to prove that the supremum is a continuous function of the radius $r \in [0, R)$. More precisely

**Proposition 2.1.** *Let $f : B(0,R) \to \mathbb{H}$ be a regular function. Then $M : [0,R) \to \mathbb{R}$ defined as*

$$M(r) = \max_{|q|\leq r} |f(q)|$$

*is a continuous function.*

*Proof.* If $f$ is constant the statement is trivially true. Let us suppose then that $f$ is not a constant function. The Maximum Modulus Principle 1.46 yields that the function $M(r)$ is (strictly) increasing and hence for any sequence $r_n$ converging (from above or from below) to $r$ there exists the limit $\lim_{n\to\infty} M(r_n)$. To show that the limit is equal to $M(r)$, consider first the sequence $\{r + 1/n\}_{n\in\mathbb{N}}$. Since $B(0, r + 1/n)$ is relatively compact, we can find $q_n$ such that $M(r + 1/n) = |f(q_n)|$ for all $n \in \mathbb{N}$, and, up to subsequences, we can suppose that $q_n$ converges to $q_0 \in \partial B(0,r)$. Therefore we have

$$M(r) \leq \lim_{n\to\infty} M\left(r + \frac{1}{n}\right) = \lim_{n\to\infty} |f(q_n)| = |f(q_0)|,$$

where the last equality is due to the continuity of the function $|f(q)|$. Moreover, by definition of $M$ we have that $|f(q_0)| \leq M(r)$ and hence that

$$M(r) = \lim_{n\to\infty} M\left(r + \frac{1}{n}\right).$$

Now $q_0$ lies in the closure $\overline{B(0,r)}$; hence we can find a sequence whose term $p_n$ has modulus $|p_n| = r - \frac{1}{n}$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} p_n = q_0$. Then

$$M(r) \geq \lim_{n\to\infty} M\left(r - \frac{1}{n}\right) \geq \lim_{n\to\infty} |f(p_n)| = |f(q_0)| = M(r).$$

Therefore we can conclude that $M$ is continuous. $\square$

The uniform norm on an open ball centered at the origin has the important property of being the same for a regular function and for its regular conjugate. The first step to prove this fact is the following result.

**Proposition 2.2.** *Let $c$ be in $\mathbb{H}$. Then the sets $\{cI \mid I \in \mathbb{S}\}$ and $\{Ic \mid I \in \mathbb{S}\}$ do coincide.*

*Proof.* If $c = 0$ the statement is trivially true. Suppose then that $c = a + bJ$ for some $a, b \in \mathbb{R}$, $(a, b) \neq (0, 0)$, and $J \in \mathbb{S}$, and let us fix $I \in \mathbb{S}$. We want to find an imaginary unit $L$ of $\mathbb{S}$ such that $cI = Lc$, that is such that

$$aI + bJI = aL + bLJ. \tag{2.1}$$

According to Proposition 1.1, for all imaginary units $I, J \in \mathbb{S}$ the following multiplication rule holds

$$IJ = -\langle I, J\rangle + I \times J,$$

hence we can write equation (2.1) as

$$aI + b(-\langle J, I\rangle + J \times I) = aL + b(-\langle L, J\rangle + L \times J).$$

If we complete $J$ to a orthonormal basis $\{1, J, K, JK\}$ of $\mathbb{H}$ over $\mathbb{R}$, then we can decompose $I$ and $L$ as

$$I = i_1J + i_2K + i_3JK \quad \text{and} \quad L = l_1J + l_2K + l_3JK,$$

obtaining that

$$J \times I = -i_3K + i_2JK \quad \text{and} \quad L \times J = l_3K - l_2JK.$$

Hence we need an imaginary unit $L$ such that

$$\begin{aligned} &a(i_1J + i_2K + i_3JK) + b(-i_1 + -i_3K + i_2JK) \\ &= a(l_1J + l_2K + l_3JK) + b(-l_1 + l_3K - l_2JK). \end{aligned}$$

Considering the different components along $1, J, K, JK$, we get that $L$ has to satisfy the following linear system

$$\begin{cases} bi_1 = bl_1 \\ ai_1 = al_1 \\ ai_2 - bi_3 = al_2 + bl_3 \\ bi_2 + ai_3 = -bl_2 + al_3 \end{cases}$$

that, for all choice of $(a, b) \neq (0, 0)$, has a unique solution $(l_1, l_2, l_3)$ which determines $L$. □

The previous result allows us to show that the supremum and the infimum of the modulus of a regular function and of its regular conjugate do coincide on each sphere of the form $x + y\mathbb{S}$ (where $x, y \in \mathbb{R}$).

**Proposition 2.3.** *Let $f$ be a regular function on $B = B(0, R)$. For any sphere of the form $x + y\mathbb{S}$ contained in $B$, the following equalities hold true*

$$\inf_{I\in\mathbb{S}} |f(x + yI)| = \inf_{I\in\mathbb{S}} |f^c(x + yI)| \quad \textit{and} \quad \sup_{I\in\mathbb{S}} |f(x + yI)| = \sup_{I\in\mathbb{S}} |f^c(x + yI)|.$$

*Proof.* Let $q = x + yI$ be an element of $B$ (then the entire sphere $x + y\mathbb{S}$ is contained in $B$). Theorem 1.10 yields that $f$ is affine on the sphere $x + y\mathbb{S}$ and there exist $b, c \in \mathbb{H}$ such that $f(x + yI) = b + Ic$ for all $I \in \mathbb{S}$. We want to compare now the value of $f$ with the one attained by $f^c$ by means of their power series expansions. If $f$ has power series expansion $f(q) = \sum_{n\geq 0} q^n a_n$ and we set $z = x + yJ$ (with $y \geq 0$), then by the Representation Formula 1.10 we get

$$\begin{aligned}
f(q) = f(x+yI) &= \frac{1}{2}\Big(\sum_{n\geq 0} z^n a_n + \sum_{n\geq 0} \overline{z^n} a_n\Big) + \frac{IJ}{2}\Big(\sum_{n\geq 0} \overline{z^n} a_n - \sum_{n\geq 0} z^n a_n\Big) \\
&= \frac{1}{2}\sum_{n\geq 0} (z^n + \overline{z^n}) a_n + \frac{IJ}{2}\sum_{n\geq 0} (\overline{z^n} - z^n) a_n \\
&= \frac{1}{2}\sum_{n\geq 0} 2\,\mathrm{Re}(z^n) a_n + \frac{IJ}{2}\sum_{n\geq 0} -2\,\mathrm{Im}(z^n) a_n \\
&= \sum_{n\geq 0} \mathrm{Re}(z^n) a_n + I \sum_{n\geq 0} |\,\mathrm{Im}(z^n)| a_n.
\end{aligned}$$

Hence the constants $b$ and $c$ are

$$b = \sum_{n\geq 0} \mathrm{Re}(z^n) a_n \quad \text{and} \quad c = \sum_{n\geq 0} |\,\mathrm{Im}(z^n)| a_n.$$

Since the power series expansion of $f^c$ is $f^c(q) = \sum_{n\geq 0} q^n \overline{a}_n$, we obtain that for all $I \in \mathbb{S}$

$$f^c(x+yI) = \sum_{n\geq 0} \mathrm{Re}(z^n) \overline{a}_n + I \sum_{n\geq 0} |\,\mathrm{Im}(z^n)| \overline{a}_n.$$

Notice that since $\mathrm{Re}(z^n)$ and $|\,\mathrm{Im}(z^n)| \in \mathbb{R}$ for all $n \geq 0$, then, in terms of $b$ and $c$, we can write

$$f^c(x+yI) = \overline{b} + I\overline{c}$$

for all $I$ in $\mathbb{S}$. Hence

$$\sup_{I\in\mathbb{S}} |f^c(x+yI)| = \sup_{I\in\mathbb{S}} \left|\overline{b} + I\overline{c}\right| = \sup_{I\in\mathbb{S}} \left|\overline{\overline{b} + I\overline{c}}\right| = \sup_{I\in\mathbb{S}} |b - cI| = \sup_{I\in\mathbb{S}} |b + cI|.$$

By Proposition 2.2 we obtain that

$$\sup_{I\in\mathbb{S}} |f^c(x+yI)| = \sup_{I\in\mathbb{S}} |b + cI| = \sup_{I\in\mathbb{S}} |b + Ic| = \sup_{I\in\mathbb{S}} |f(x+yI)|.$$

Exactly the same arguments hold for the infimum, so we can conclude also that

$$\inf_{I\in\mathbb{S}} |f^c(x+yI)| = \inf_{I\in\mathbb{S}} |f(x+yI)|.$$

□

As a consequence we get not only that $||f||_B = ||f^c||_B$ for all $f \in \mathcal{R}(B)$, but also the following more general result.

**Corollary 2.4.** *Let $f$ be a regular function on $B = B(0, R)$. Then*

$$\sup_{q\in B} |f(q)| = \sup_{q\in B} |f^c(q)| \quad \textit{and} \quad \inf_{q\in B} |f(q)| = \inf_{q\in B} |f^c(q)|.$$

*Proof.* If $S$ is the subset of $\mathbb{R}^2$ defined as

$$S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\, x^2 + y^2 \leq R^2\},$$

then we can cover the entire ball $B$ with spheres of the form $x + y\mathbb{S}$ as

$$B = \bigcup_{(x,y)\in S} x + y\mathbb{S} = \bigcup_{(x,y)\in S} \bigcup_{I\in\mathbb{S}} x + yI.$$

By Proposition 2.3 we get

$$\sup_{q\in B} |f(q)| = \sup_{(x,y)\in S} \sup_{I\in\mathbb{S}} |f(x + yI)| = \sup_{(x,y)\in S} \sup_{I\in\mathbb{S}} |f^c(x + yI)| = \sup_{q\in B} |f^c(q)|$$

and the same holds for the infimum, hence we have also

$$\inf_{q\in B} |f(q)| = \inf_{q\in B} |f^c(q)|.$$

□

## 2.2 The Borel-Carathéodory Theorem for regular functions

This section is devoted to prove an analogue of the Borel-Carathéodory Theorem for regular functions. As anticipated at the beginning of the chapter, this result will show how we can control the modulus of a regular function by means of the modulus of its real part. In addition to its independent importance, we recall that, historically, the complex Borel-Carathéodory Theorem has been used as a fundamental tool to prove a weak version of the Bohr Theorem, see [6]. We will see in Chapter 3 the analogous application of Theorem 2.5 that leads to a weak version of the Bohr Theorem in the case of regular functions.

**Theorem 2.5** (Borel-Carathéodory for regular functions)**.** *Let $q_0 \in \mathbb{R}$, $r > 0$, and let $f$ be a regular function on (a neighborhood of) the closed ball $\overline{B(q_0, r)}$. Set*

$$A = \max_{|q-q_0|=r} |\operatorname{Re} f(q)|,$$

*and*

$$f(q_0) = \beta + \gamma I \quad \text{for some } \beta, \gamma \in \mathbb{R} \text{ and } I \in \mathbb{S}.$$

*If* $\varrho \in (0, r)$, *then*

$$|f(q)| \leq |\gamma| + |\beta|\frac{r+\varrho}{r-\varrho} + 2A\frac{\varrho}{r-\varrho}$$

*for all* $q \in \overline{B(q_0, \varrho)}$.

*Proof.* The statement is trivially true if the function $f$ is constant. If this is not the case then its real part $\operatorname{Re} f(q)$ is not constant. Otherwise the image of $B(q_0, r)$ under $f$ would be contained in a 3-dimensional space, in contradiction with Corollary 1.49. Let $J \in \mathbb{S}$ be such that $\operatorname{Re} f(q)$ is not constant on $L_J$. By definition, $\beta = \operatorname{Re} f(q_0) \leq A$, and moreover equality can not hold. In fact, suppose that $\beta = A$. By the Splitting Lemma 1.7 there exist $F, G : B_J(q_0, r) = B(q_0, r) \cap L_J \to L_J$ holomorphic functions, and $K \in \mathbb{S}$, $K$ orthogonal to $J$, such that

$$f(z) = F(z) + G(z)K$$

for every $z \in B_J(q_0, r)$. Then $\operatorname{Re} F(q_0) = \operatorname{Re} f(q_0) = A$. Since $F$ is holomorphic then $\operatorname{Re} F(z)$ is harmonic and hence, if it attains its maximum at an interior point, it must be constant (together with $F$). Consequently $\operatorname{Re} f(q)$ would be constant on $L_J$, a contradiction with the choice of $J$. Hence $\beta < A$.
We will now define an auxiliary function $H : B(q_0, r) \to \mathbb{H}$ and, as a first step, prove that $H(B(q_0, r)) \subseteq \mathbb{B}$. Let

$$H(q) = (f(q) - A + \overline{f(q_0) - A})^{-*} * (f(q) - A - (f(q_0) - A)). \tag{2.2}$$

The function $H$ is regular for $|q - q_0| \leq r$. Indeed if the symmetrization $(f(q) - A + \overline{f(q_0) - A})^s$ vanishes somewhere in $B(q_0, r)$, then also $f(q) - A + \overline{f(q_0) - A} = 0$ for some $q \in B(q_0, r)$. In particular its real part should vanish, but

$$\operatorname{Re}(f(q)) - A + \operatorname{Re}(\overline{f(q_0)}) - A \leq A - A + \beta - A = \beta - A < 0. \tag{2.3}$$

Then $(f(q) - A + \overline{f(q_0) - A})^s$ (as well as $f(q) - A + \overline{f(q_0) - A}$) never vanishes in $B(q_0, r)$. By Proposition 1.38 we can express $H(q)$ in terms of the linear fractional transformation

$$g(q) = (q + \overline{f(q_0) - A})^{-1}(q - f(q_0) + A),$$

and of the transformation

$$T(q) = ((f(q) - A + \overline{f(q_0) - A})^c)^{-1} q (f(q) - A + \overline{f(q_0) - A})^c. \tag{2.4}$$

Namely

$$H(q) = g(f(T(q)) - A).$$

From this expression we easily get the value of $H$ in $q = q_0$. In fact $q_0 \in \mathbb{R}$ yields that $T(q_0) = q_0$ and hence $H(q_0) = g(f(q_0) - A) = 0$. Moreover we know that the linear fractional transformation $g$ sends the family $\mathcal{F}_3$ of 3-spheres and affine 3-subspaces in itself, (see 1.52). In particular, since for every purely imaginary quaternion $w$, $|g(w)| = 1$, we have that $g$ maps the 3-space $\{q \in \mathbb{H} \mid \mathrm{Re}(q) = 0\}$ onto the unit sphere $\mathbb{S}^3 = \partial\mathbb{B}$. Recalling that $H(q_0) = 0$ and $\mathrm{Re}(f(T(q) - A) \leq 0$ for all $q \in B(q_0, r)$ (see equation (2.3)), we get that $H$ maps $B(q_0, r)$ inside the unit ball $\mathbb{B}$, namely

$$|H(q)| \leq 1 \quad \text{for all} \quad q \in B(q_0, r). \tag{2.5}$$

Furthermore, if we choose $\varrho \in (0, r)$, for all $q$ such that $0 < |q - q_0| \leq \varrho < r$, we can improve the estimate of the modulus $|H(q)|$ as follows. Consider the function defined by $q \mapsto (q - q_0)^{-*} * H(q)$. Since $H(q_0) = 0$ and $q_0 \in \mathbb{R}$, this is a regular function and $(q - q_0)^{-*} * H(q) = (q - q_0)^{-1} H(q)$. Then, for all $q \in B$, the Maximum Modulus Principle 1.46 and inequality (2.5) yield

$$|(q - q_0)^{-*} * H(q)| = \frac{|H(q)|}{|q - q_0|} \leq \frac{1}{r}. \tag{2.6}$$

Moreover, for all $q \in B(q_o, \varrho)$,

$$\frac{|H(q)|}{|q - q_0|} \leq \max_{|q - q_0| = \varrho} \frac{|H(q)|}{|q - q_0|} = \max_{|q - q_0| = \varrho} \frac{|H(q)|}{\varrho} \leq \frac{1}{r} \tag{2.7}$$

where the last inequality is due to (2.6) (that holds for all $q \in B$). Therefore for all $q \in B(q_o, \varrho)$

$$|H(q)| \leq \max_{|q - q_0| = \varrho} |H(q)| \leq \frac{\varrho}{r}. \tag{2.8}$$

Now, we want to use this inequality to estimate the modulus of the function $f$. Recall that

$$H(q) = (f(T(q)) - A + \overline{f(q_0) - A})^{-1}(f(T(q)) - A - (f(q_0) - A)).$$

Let us prove that in fact we can switch the order of the factors in the previous expression of $H$. Set

$$z = f(T(q)) - A, \quad \text{and} \quad w = f(q_0) - A.$$

Under the assumption $z + \overline{w} \neq 0$, we have that

$$(z + \overline{w})^{-1}(z - w) = (z - w)(z + \overline{w})^{-1}$$

if, and only if,

$$(z - w)(z + \overline{w}) = (z + \overline{w})(z - w),$$

that is

$$z^2 + z\overline{w} - wz - |w|^2 = z^2 - zw + \overline{w}z - |w|^2$$

and this is equivalent to require

$$z2\,\mathrm{Re}(w) = 2\,\mathrm{Re}(w)z$$

that clearly holds true. Therefore we can write

$$H(q) = (f(T(q)) - A - (f(q_0) - A))(f(T(q)) - A + \overline{f(q_0) - A})^{-1},$$

and hence

$$H(q)(f(T(q)) - 2A + \overline{f(q_0)}) = f(T(q)) - f(q_0),$$

that yields

$$(H(q) - 1)f(T(q)) = H(q)(2A - \overline{f(q_0)}) - f(q_0).$$

Therefore, we get

$$\begin{aligned}
|f(T(q))| &= \left|(H(q)-1)^{-1}\left(H(q)(2A - \overline{f(q_0)}) - f(q_0)\right)\right| \\
&= \left|(H(q)-1)^{-1}\left((H(q)-1)f(q_0) + H(q)(2A - \overline{f(q_0)} - f(q_0))\right)\right| \\
&= \left|\beta + \gamma I + (H(q)-1)^{-1}H(q)(-\beta + \gamma I + 2A - \beta - \gamma I)\right| \\
&= \left|\beta + \gamma I + (H(q)-1)^{-1}H(q)2(A - \beta)\right|.
\end{aligned}$$

Using the triangle inequality and inequality (2.8) we have that for $0 \le |q - q_0| \le \varrho \le r$

$$\begin{aligned}
|f(T(q))| &\le |\beta| + |\gamma| + 2\frac{(A + |\beta|)|H(q)|}{|H(q) - 1|} \le |\beta| + |\gamma| + 2\frac{(A + |\beta|)\frac{\varrho}{r}}{1 - \frac{\varrho}{r}} \\
&= |\gamma| + |\beta|\left(1 + 2\frac{\frac{\varrho}{r}}{1 - \frac{\varrho}{r}}\right) + 2\frac{A\frac{\varrho}{r}}{1 - \frac{\varrho}{r}} = |\gamma| + |\beta|\frac{r + \varrho}{r - \varrho} + A\frac{2\varrho}{r - \varrho}.
\end{aligned} \tag{2.9}$$

The fact that $f(q) - A + \overline{f(q_0) - A} \neq 0$ for all $q \in B(q_0, r)$, guarantees that the transformation $T$ defined by equation (2.4) is a diffeomorphism of $B(q_0, r)$ onto itself. Consequently inequality (2.9) leads to

$$|f(q)| \le |\gamma| + |\beta|\frac{r + \varrho}{r - \varrho} + 2A\frac{\varrho}{r - \varrho}$$

for all $q$ such that $|q - q_0| \le \varrho$, and hence to the conclusion of the proof.

□

# Chapter 3

# The Bohr Theorem for regular functions

One of the classical results in the study of the image of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

under holomorphic functions is due to Bohr, who, while studying problems in the field of Diophantine approximation, investigated the following question.

**Problem 3.1.** *Let $x \in (0, 1)$ be a real number. Establish whether it is possible to find a power series $\sum_{n\geq 0} a_n z^n$, with $a_n \in \mathbb{C}$, such that*

1. *$f(z) = \sum_{n\geq 0} a_n z^n$ is holomorphic for $|z| < 1$ and continuous for $|z| \leq 1$;*
2. *$|f(z)| < 1$ for $|z| \leq 1$;*
3. *$\sum_{n\geq 0} x^n |a_n| > 1$.*

Bohr himself presented in [6] first a partial solution of Problem 3.1 (whose proof follows from the complex Borel-Carathéodory Theorem) and then the complete result, known as Bohr Theorem, relating the proof due to Wiener.

**Theorem 3.2** (Bohr)**.** *Let*

$$f(z) = \sum_{n\geq 0} a_n z^n$$

*be holomorphic for $z \in \mathbb{D}$, continuous for $z \in \overline{\mathbb{D}}$, and let $|f(z)| < 1$ for all $|z| \leq 1$. Then*

$$\sum_{n\geq 0} |a_n z^n| < 1$$

*for $|z| \leq 1/3$. Moreover $1/3$ is the largest radius for which the statement is true.*

For different reasons, recent studies of various authors are dedicated to generalize the Bohr Theorem to new settings. For instance, we can mention the case of holomorphic functions of several complex variables, see e.g. [2], and the case of classical monogenic functions, treated, e.g., in [37, 38].

This chapter is dedicated to prove the Bohr Theorem in the quaternionic setting. Furthermore we will present, as an application of the (quaternionic) Borel-Carathéodory Theorem 2.5, also a weaker version of the statement.

## 3.1 The sharp version

**Theorem 3.3** (Bohr for regular functions)**.** *Let $f(q) = \sum_{n\geq 0} q^n a_n$ be a regular function on $\mathbb{B}$, continuous on the closure $\overline{\mathbb{B}}$, such that $|f(q)| < 1$ for all $|q| \leq 1$. Then*

$$\sum_{n\geq 0} |q^n a_n| < 1$$

*for all $|q| \leq 1/3$. Moreover $1/3$ is the largest radius for which the statement is true.*

*Proof.* Up to right-multiplying $f$ by the unitary constant $\frac{\overline{a_0}}{|a_0|}$, we can suppose $a_0 \in [0, 1)$. We want to bound the modulus of each coefficient $a_n$, showing that $|a_n| < 1 - a_0^2$ for all $n \geq 1$. Let us first treat the case $n = 1$. Consider the function defined as

$$H(q) = (1 - f(q)a_0)^{-*} * (q^{-1}(f(q) - a_0)).$$

Since $f : \mathbb{B} \to \mathbb{H}$ is continuous up to the boundary, $f(0) = a_0 < 1$ and $|f(q)| < 1$ on $\overline{\mathbb{B}}$, we get that $H$ is regular on $\mathbb{B}$ and continuous up to the boundary. Moreover, if we set $T(q) = ((1 - f(q)a_0)^c)^{-1} q(1 - f(q)a_0)^c$, by Proposition 1.38, we can write

$$\begin{aligned} H(q) &= (1 - f(q)a_0)^{-*} * (q^{-1} \sum_{n\geq 1} q^n a_n) = (1 - f(q)a_0)^{-*} * \sum_{n\geq 1} q^{n-1} a_n \\ &= (1 - f(T(q))a_0)^{-1} \sum_{n\geq 1} T(q)^{n-1} a_n. \end{aligned}$$

Furthermore, $T(0) = 0$ implies that $H(0) = (1 - a_0^2)^{-1} a_1$, and, since $|T(q)| = |q|$, by the Maximum Modulus Principle 1.46 we get that

$$\begin{aligned} |H(q)| &\leq \max_{|q|=1} |H(q)| = \max_{|q|=1} |1 - f(T(q))a_0|^{-1} |T(q)|^{-1} |f(T(q)) - a_0| \\ &= \max_{|q|=1} |1 - f(T(q))a_0|^{-1} |f(T(q)) - a_0|. \end{aligned} \tag{3.1}$$

Notice that $(1 - qa_0)^{-1}(q - a_0)$ is a fractional linear transformation that maps $\mathbb{B}$ into $\mathbb{B}$. Hence, since $|f(T(q))| < 1$, by (3.1) we get that $|H(q)| < 1$ for all $q \in \mathbb{B}$. In particular we have then $|H(0)| < 1$, that is

$$a_1 < 1 - {a_0}^2.$$

For the case $n > 1$ we want to build a function with the same properties as $f$, whose first degree coefficient is $a_n$. Therefore, with the same argument used for $n = 1$, we would obtain that $|a_n| < 1 - {a_0}^2$. Let $\omega$ be a quaternionic primitive $n$-th root of unity (see [35]) and let $I = I_\omega$ be the imaginary unit identified by $\omega$. Consider the function defined on $\mathbb{B}_I = L_I \cap \mathbb{B}$ as

$$g_I(z) = f_I(z) + f_I(z\omega) + \cdots + f_I(z\omega^{n-1})$$

where $f_I$ is the restriction of $f$ to the plane $L_I$. The function $g_I$ is holomorphic on $\mathbb{B}_I$, continuous on $\overline{\mathbb{B}}_I$, and its modulus

$$|g_I(z)| \leq |f_I(z)| + |f_I(z\omega)| + \cdots + |f_I(z\omega^{n-1})| < n.$$

Its power series expansion is

$$g_I(z) = \sum_{m\geq 0} z^m a_m + \sum_{m\geq 0} z^m \omega^m a_m + \cdots + \sum_{m\geq 0} z^m \omega^{(n-1)m} a_m = \sum_{m\geq 0} z^m \sum_{k=0}^{n-1} \omega^{km} a_m.$$

Now if $n$ divides $m$, then $\omega^m = 1$ and

$$\sum_{k=0}^{n-1} \omega^{km} = n.$$

Otherwise $\omega^m$ is a $n$-th root of unity different from 1, and hence a root of the polynomial $z^{n-1} + \cdots + z + 1$, that yields

$$\sum_{k=0}^{n-1} \omega^{km} = 0.$$

Therefore

$$g_I(z) = \sum_{m\geq 0} z^{nm} n a_{nm}.$$

Notice that the coefficients of $g_I$ do not depend on $\omega$, and hence neither on $I$. This implies that for all $J \in \mathbb{S}$, if $\omega_J$ is an $n$-th root of unity in $L_J$, the function $g_J(u)$ defined on $\mathbb{B}_J$ as

$$g_J(u) = f_J(u) + f_J(u\omega_J) + \cdots + f_J(u\omega_J^{n-1})$$

has the same regular extension (see Lemma 1.15) to the unit ball $\mathbb{B}$ as $g_I$, namely

$$g(q) = \text{ext}(g_I)(q) = \text{ext}(g_J)(q) = \sum_{m\geq 0} q^{nm} n a_{nm}.$$

Hence we have that $|g(q)| < n$ for all $q \in \mathbb{B}$. Let us set $w = q^n$ and consider the function $\Phi = \frac{1}{n} g$,

$$\Phi(w) = \sum_{m\geq 0} w^m a_{mn} = a_0 + w a_n + w^2 a_{2n} + \dots\ ,$$

then we obtain that $\Phi$ is regular on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$ and $|\Phi(w)| < 1$ for all $q \in \mathbb{B}$. Since the coefficient of the first degree term of $\Phi$ is $a_n$, with the same argument used to prove that $|a_1| < 1 - a_0^2$, we can conclude that $|a_n| < 1 - a_0^2$. Since $0 \leq a_0 < 1$, we get that

$$|a_n| < (1 + a_0)(1 - a_0) < 2(1 - a_0).$$

Therefore, for $|q| = \frac{1}{3}$,

$$\begin{aligned}\sum_{n\geq 0} |q^n a_n| &= \sum_{n\geq 0} \frac{1}{3^n}|a_n| < a_0 + 2(1 - a_0)\sum_{n\geq 1}\frac{1}{3^n} \\ &= a_0 + 2(1 - a_0)\frac{1}{2} = a_0 + 1 - a_0 = 1.\end{aligned}$$

To show that the statement does not hold for any radius larger than $\frac{1}{3}$, we will proceed as follows. For any point $q_0 \in \mathbb{B}$ such that $|q_0| > \frac{1}{3}$, we will find a regular function $g_{q_0} : \mathbb{B} \to \mathbb{B}$ continuous up to the boundary of $\mathbb{B}$, such that $g_{q_0}(q) = \sum_{n\geq 0} q^n b_n$ and $\sum_{n\geq 0} |q_0^n b_n| > 1$.
To begin with, take $a \in (0, 1)$ and consider the function

$$\varphi(q) = (1 - qa)^{-*} * (1 - q) = (1 - qa)^{-1}(1 - q).$$

Since $a < 1$, then $\varphi$ is regular on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$ and since it has real coefficients, $\varphi$ is slice preserving. By the Maximum Modulus Principle 1.46

$$\max_{q\leq 1} |\varphi(q)| = \max_{|q|=1} |\varphi(q)|,$$

and for all $I \in \mathbb{S}$,

$$\max_{|q|=1} |\varphi(q)| = \max_{|z|=1} |\varphi_I(z)| = \frac{2}{1+a} > 1.$$

If the power series expansion of $\varphi$ is $\sum_{n\geq 0} q^n b_n$, with $b_n \in \mathbb{R}$, then

$$\sum_{n\geq 0} q^n b_n = (1 - qa)^{-1}(1 - q) = (\sum_{n\geq 0} q^n a^n)(1 - q) = 1 + q(a - 1)\sum_{n\geq 0} q^n a^n$$

and hence

$$\sum_{n\geq 0} |q^n b_n| = 1 + |q|(1 - a)\sum_{n\geq 0} |q|^n a^n = 1 + \frac{|q|(1 - a)}{1 - |q|a}.$$

In particular

$$\sum_{n\geq 0} |q^n b_n| > \frac{2}{1+a} \quad \text{if and only if} \quad 1 + \frac{|q|(1-a)}{1-|q|a} > \frac{2}{1+a},$$

that holds if and only if

$$|q| > \frac{1}{1+2a}.$$

Fix now $q_0 \in \mathbb{B}$ such that $|q_0| > \frac{1}{3}$. Then we can take $a \in (0,1)$ such that $|q_0| > \frac{1}{1+2a} > \frac{1}{3}$. Therefore the correspondent $\varphi$ is such that

$$\sum_{n\geq 0} |q_0|^n |b_n| > \frac{2}{1+a}$$

Let us consider the function $\varphi_c$, defined as

$$\varphi_c(q) = c\varphi(q) = c(1-qa)^{-1}(1-q),$$

where $c \in (0,1)$. Then $\varphi_c$ is regular on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$, and its maximum modulus is

$$\max_{|q|=1} |\varphi_c(q)| = \frac{2c}{1+a}.$$

Moreover, its power series expansion is obtained multiplying by $c$ the one of $\varphi$, so with the same calculation that we have done for $\varphi$, we get that

$$c\sum_{n\geq 0} |q^n b_n| > \frac{2c}{1+a}$$

if and only if

$$|q| > \frac{1}{1+2a}.$$

Hence

$$c\sum_{n\geq 0} |q_0^n b_n| > \frac{2c}{1+a}.$$

To conclude, notice that we can choose $c \in (0,1)$ such that

$$c\sum_{n\geq 0} |q_0^n b_n| > 1 > \frac{2c}{1+a}$$

and set

$$g_{q_0} = \varphi_c.$$

$\square$

## 3.2 A weak version of the Bohr Theorem

In analogy with the historical approach used by Bohr himself in [6], we exhibit here also a weak version of Theorem 3.3, that follows from the Borel-Carathéodory Theorem 2.5.

**Theorem 3.4** (Bohr, weak version). *Let $f(q) = \sum_{n\geq 0} q^n a_n$ be a regular function on the unit ball $\mathbb{B}$, continuous on the closure $\overline{\mathbb{B}}$, such that $|f(q)| < 1$ for all $|q| \leq 1$. Then*

$$\sum_{n\geq 0} |q^n a_n| < 1$$

*for all $|q| \leq \frac{1}{6}$.*

*Proof.* As in the previous case, we can suppose $a_0 \in [0,1)$. Consider the function $g(q) = f(q) - a_0$, regular on $\mathbb{B}$, continuous on $\overline{\mathbb{B}}$. Set

$$A = \max_{|q|=1} \operatorname{Re} g(q)$$

and

$$m = \max_{|q|=\frac{1}{2}} |g(q)|.$$

By the Borel-Carathéodory Theorem 2.5 we get then that for all $|q| \leq \frac{1}{2}$

$$|g(q)| \leq 2A$$

and hence also that

$$m = \max_{|q|=\frac{1}{2}} |g(q)| \leq 2A.$$

Moreover, since $a_0$ and $A$ are non-negative,

$$a_0 + A = a_0 + \max_{|q|=1} \operatorname{Re} g(q) = \max_{|q|=1} \operatorname{Re}(a_0 + g(q)) \leq \max_{|q|=1} |a_0 + g(q)| = \max_{|q|=1} |f(q)| < 1.$$

Therefore $A < 1 - a_0$ and $m < 2(1-a_0)$. Now we want to show that $|a_n| < 2^{n+1}(1-a_0)$ for all $n \geq 1$. Let $J \in \mathbb{S}$, and set

$$\Delta_J = \{z \in L_J : |z| < \frac{1}{2}\}.$$

Thanks to Theorem 1.45, we get the integral representation of the coefficients $a_n$, for all $n \geq 1$, as

$$a_n = \frac{1}{2\pi J} \int_{\partial \Delta_J} \frac{dz}{z^{n+1}} g(z).$$

Hence

$$|a_n| \leq \frac{1}{2\pi} \int_{\partial \Delta_J} \frac{|g(z)|}{|z^{n+1}|} dz \leq m 2^n < 2^{n+1}(1-a_0)$$

for all $n \geq 1$. Therefore

$$\begin{aligned}\sum_{n\geq 0} |q^n a_n| &= a_0 + \sum_{n\geq 1} |q^n a_n| < a_0 + \sum_{n\geq 1} |q|^n 2^{n+1}(1-a_0) \\ &= a_0 + 2(1-a_0) \sum_{n\geq 1} |q|^n 2^n = a_0 + \frac{4(1-a_0)|q|}{1-2|q|}.\end{aligned}$$

An easy computation shows that

$$a_0 + \frac{4(1-a_0)|q|}{1-2|q|} < 1 \quad \text{if and only if} \quad |q| \leq \frac{1}{6}.$$

Thus we can conclude that $\sum_{n\geq 0} |q^n a_n| < 1$ for all $|q| \leq \frac{1}{6}$.

□

# Chapter 4

# A Bloch-Landau type theorem

The classical Bloch-Landau Theorem establishes an important property of the range of holomorphic functions on the complex open unit disc $\mathbb{D}$. It states that the image of $\mathbb{D}$ under a holomorphic function cannot be too much thin. In fact, under certain normalizations, there exists a universal radius $L > 0$, called the *Landau constant*, such that $f(\mathbb{D})$ always contains a disc of radius $L$. One of the first lower bounds of $L$ was proven to be $1/16$ by Landau in [42]. The same author gave also better estimates, see for instance [43], but the exact value of the Landau constant is still unknown (see [19] for more details). The purpose of this chapter is to prove an analog of the Bloch-Landau Theorem for regular functions, following the ideas that led to the first estimate. We will show the existence of a universal open set, different from a ball, always contained in the image of the open unit ball $\mathbb{B}$ under some *regular translation* (see Definition 1.42) of any (suitably normalized) regular function $f$.

## 4.1 A norm for a mean value theorem

The first step towards the proof of a Bloch Landau type result, is a technical lemma, stated in terms of a new norm, defined on the space $\mathcal{R}(B)$ of regular functions on $B = B(0, R)$. To be precise, what we define is a norm-like function when not restricted to the linear subspace of $\mathcal{R}(B)$ where it is finite (the same happens for the classic uniform norm). Since this does not affect our results and no confusion can arise, we will refer to it as an actual norm. This new norm, that turns out to be equivalent to the uniform one, is a key tool to prove a mean value theorem.

Let $f : B \to \mathbb{H}$ be a regular function. Take $I, J \in \mathbb{S}$, $I$ orthogonal to $J$ and, according to the Splitting Lemma 1.7, let $F, G : B_I \to L_I$ be the holomorphic functions such that the restriction of $f$ to $B_I$ is

$$f_I(z) = F(z) + G(z)J.$$

Let $\Omega$ be a subset of the ball $B$, and let $||\cdot||_\Omega$ denote the uniform norm on $\Omega \subseteq B$,

$$||\cdot||_\Omega = \sup_\Omega |\cdot|$$

For any $I \in \mathbb{S}$, let us indicate with $||\cdot||_I$ the function

$$||\cdot||_I : \mathcal{R}(B) \longrightarrow [0, +\infty]$$

defined by

$$||f||_I^2 = ||F||_{B_I}^2 + ||G||_{B_I}^2.$$

**Remark 4.1.** For all $I \in \mathbb{S}$, the function $||\cdot||_I$ does not depend on the choice of $J$. In fact, if we choose another imaginary unit $K \in \mathbb{S}$, orthogonal to $I$, then the splitting of $f$ on $L_I$ is

$$f_I(z) = \widetilde{F}(z) + \widetilde{G}(z)K,$$

where $\widetilde{F}(z) = F(z)$, because $I$ and $K$ are orthogonal, and hence $\widetilde{G}(z) = G(z)JK^{-1}$. Then

$$|\widetilde{G}(z)| = |G(z)JK^{-1}| = |G(z)|$$

for all $z$ in $B_I$, and hence $||f||_I$ does not change.

Consider now the function

$$||\cdot|| : \mathcal{R}(B) \longrightarrow [0, +\infty]$$

defined by

$$||f|| = \sup_{I \in \mathbb{S}} ||f||_I.$$

**Proposition 4.2.** *The function $||\cdot||$ is a norm on the real vector space $\mathcal{R}(B)$.*

*Proof.* Let $f \in \mathcal{R}(B)$, $I \in \mathbb{S}$, and take $J \in \mathbb{S}$ orthogonal to $I$. Let $F$, and $G$ be the holomorphic functions on $L_I$, such that $f_I(z) = F(z) + G(z)J$ for all $z \in B_I$. Then:

- $||f|| = 0$ if and only if, for all $I \in \mathbb{S}$,

  $$0 = ||f||_I^2 = ||F||_{B_I}^2 + ||G||_{B_I}^2,$$

  and hence if and only if $F = G = 0$ for all $I \in \mathbb{S}$. The fact that the component $F$ of $f$ along $L_I$ vanishes for any $I \in \mathbb{S}$ clearly implies that $f$ must vanish as well. Hence $||f|| = 0$ if and only if $f = 0$.

- Let $c \in \mathbb{R}$. Then the splitting of the function $q \mapsto f(q)c$ on $L_I$, with respect to $J$, is $f_I(z)c = F(z)c + G(z)cJ$. Hence, using the homogeneity of the uniform norm, we have

  $$\begin{aligned} ||fc||^2 &= \sup_{I \in \mathbb{S}} ||fc||_I^2 = \sup_{I \in \mathbb{S}} \left(||F||_{B_I}^2 |c|^2 + ||G||_{B_I}^2 |c|^2\right) = \sup_{I \in \mathbb{S}} \left(||F||_{B_I}^2 + ||G||_{B_I}^2\right)|c|^2 \\ &= |c|^2 ||f||^2. \end{aligned}$$

- If $F_j, G_j$ are the splitting functions of regular functions $f_j$, with respect to $I$ and $J$, for $j = 1, 2$, then

$$\begin{aligned}(||f_1||_I + ||f_2||_I)^2 &= \left(\sqrt{||F_1||_{B_I}^2 + ||G_1||_{B_I}^2} + \sqrt{||F_2||_{B_I}^2 + ||G_2||_{B_I}^2}\right)^2 \\ &= ||F_1||_{B_I}^2 + ||G_1||_{B_I}^2 + ||F_2||_{B_I}^2 + ||G_2||_{B_I}^2 \\ &\quad + 2\sqrt{(||F_1||_{B_I}^2 + ||G_1||_{B_I}^2)(||F_2||_{B_I}^2 + ||G_2||_{B_I}^2)},\end{aligned}$$

and

$$\begin{aligned}&||f_1 + f_2||_I^2 = ||F_1 + F_2||_{B_I}^2 + ||G_1 + G_2||_{B_I}^2 = ||(F_1 + F_2)^2||_{B_I} + ||(G_1 + G_2)^2||_{B_I} \\ &= ||F_1^2 + F_2^2 + 2F_1F_2||_{B_I} + ||G_1^2 + G_2^2 + 2G_1G_2||_{B_I} \\ &\leq ||F_1^2||_{B_I} + ||F_2^2||_{B_I} + 2||F_1||_{B_I}||F_2||_{B_I} + ||G_1^2||_{B_I} + ||G_2^2||_{B_I} + 2||G_1||_{B_I}||G_2||_{B_I}.\end{aligned} \tag{4.1}$$

The last quantity in inequality (4.1) is less or equal than $(||f_1||_I + ||f_2||_I)^2$ if and only if

$$||F_1||_{B_I}||F_2||_{B_I} + ||G_1||_{B_I}||G_2||_{B_I} \leq \sqrt{(||F_1||_{B_I}^2 + ||G_1||_{B_I}^2)(||F_2||_{B_I}^2 + ||G_2||_{B_I}^2)},$$

that is, if and only if

$$\langle(||F_1||_{B_I}, ||G_1||_{B_I}), (||F_2||_{B_I}, ||G_2||_{B_I})\rangle \leq \sqrt{||F_1||_{B_I}^2 + ||G_1||_{B_I}^2}\sqrt{||F_2||_{B_I}^2 + ||G_2||_{B_I}^2},$$

that holds thanks to Cauchy-Schwarz Inequality for the scalar product on $\mathbb{R}^2$.

Thus the function $|| \cdot ||$ is a norm on $\mathcal{R}(B)$. □

Let us now show that the norms $|| \cdot ||$ and $|| \cdot ||_B$, defined on $\mathcal{R}(B)$, are equivalent.

**Proposition 4.3.** *Let $f \in \mathcal{R}(B)$. Then*

$$\frac{\sqrt{2}}{2}||f|| \leq ||f||_B \leq ||f||.$$

*Proof.* Let $I, J \in \mathbb{S}$, $I$ orthogonal to $J$, and let $F, G$ be holomorphic functions on $L_I$, such that $f_I(z) = F(z) + G(z)J$ for all $z \in B_I$. Then we have

$$\begin{aligned}||f||_B^2 &= \sup_{q\in B}|f(q)|^2 = \sup_{I\in\mathbb{S}}\sup_{z\in B_I}|f_I(z)|^2 = \sup_{I\in\mathbb{S}}\sup_{z\in B_I}|F(z) + G(z)J|^2 \\ &= \sup_{I\in\mathbb{S}}\sup_{z\in B_I}\left(|F(z)|^2 + |G(z)|^2\right) \leq \sup_{I\in\mathbb{S}}\left(\sup_{z\in B_I}|F(z)|^2 + \sup_{z\in B_I}|G(z)|^2\right) \\ &= \sup_{I\in\mathbb{S}}\left(||F||_{B_I}^2 + ||G||_{B_I}^2\right) = ||f||^2.\end{aligned}$$

Conversely

$$||f||^2 = \sup_{I\in\mathbb{S}} ||f||_I^2 = \sup_{I\in\mathbb{S}} \left(||F||_{B_I}^2 + ||G||_{B_I}^2\right) \leq \sup_{I\in\mathbb{S}} \left(||f_I||_{B_I}^2 + ||f_I||_{B_I}^2\right) = 2||f||_B^2.$$

□

As we anticipated at the beginning of this section, we can prove a mean value theorem by means of the norm $||\cdot||$ .

**Theorem 4.4** (Mean value). *Let $f$ be a regular function on $B = B(0,R)$ such that $f(0) = 0$. Then*

$$|q^{-1}f(q)| \leq ||\partial_c f||$$

*for all $q \in B \setminus \{0\}$.*

*Proof.* Let $I \in \mathbb{S}$ and take $J \in \mathbb{S}$ orthogonal to $I$. By the Splitting Lemma 1.7 there exist $F, G : B_I \to L_I$ holomorphic functions, such that $f_I(z) = F(z) + G(z)J$ for all $z \in B_I$. By the analogue of the Fundamental Theorem of Calculus for line integrals (see for instance Theorem 2.1, Chapter III, in book [45]), for all $z \in B_I \setminus \{0\}$, we easily get that

$$|z^{-1}F(z)| \leq ||F'||_{B_I} \quad \text{and} \quad |z^{-1}G(z)| \leq ||G'||_{B_I},$$

where $F'$ and $G'$ denote the complex derivatives of $F$ and $G$. Hence

$$\begin{aligned} |z^{-1}f(z)|^2 &= |z^{-1}F(z) + z^{-1}G(z)J|^2 = |z^{-1}F(z)|^2 + |z^{-1}G(z)|^2 \\ &\leq ||F'||_{B_I}^2 + ||G'||_{B_I}^2 = ||\partial_c f||_I^2 \\ &\leq \sup_{I\in\mathbb{S}} ||\partial_c f||_I^2 = ||\partial_c f||^2. \end{aligned} \tag{4.2}$$

Since $||\partial_c f||$ does not depend on $I$, we have that inequality (4.2) holds for every $q \in B \setminus \{0\}$. □

**Remark 4.5.** As a consequence of Theorem 4.4 and of the Maximum Modulus Principle 1.46, we get also that if $f$ is regular on $B(0,R)$ and $f(0) = 0$, then for all $r \in (0,R)$

$$\max_{|q|\leq r} |f(q)| \leq r||\partial_c f||. \tag{4.3}$$

We conclude this section by showing that, as the uniform norm, the norm $||\cdot||$ takes the same value on a regular function and on its regular conjugate.

**Proposition 4.6.** *Let $f$ be a regular function on $B = B(0,R)$. Then $||f|| = ||f^c||$.*

*Proof.* Let $I \in \mathbb{S}$. By Definition 1.21, if the splitting of $f_I$ is $f_I(z) = F(z) + G(z)J$ for all $z \in B_I$, then the regular conjugate of $f$ splits as

$$f_I^c(z) = \overline{F(\overline{z})} - G(z)J.$$

Since the (complex) conjugation is a bijection of $B_I$ and the modulus $|\overline{F(z)}|$ equals $|F(z)|$ for all $z \in B_I$, we get that

$$||f^c||_I^2 = ||\overline{F(\overline{z})}||_{B_I}^2 + || - G||_{B_I}^2 = ||F||_{B_I}^2 + ||G||_{B_I}^2 = ||f||_I^2. \tag{4.4}$$

Since equality (4.4) holds for every $I$ in $\mathbb{S}$, we can conclude.

□

## 4.2 The Bloch Landau type theorem

### 4.2.1 A preliminary lemma

For $\varrho > 0$, let us denote by $\mathcal{O}(\varrho)$ the open set

$$\mathcal{O}(\varrho) = \{q \in \mathbb{H} \mid |q|^3 < \varrho|\operatorname{Re}(q)|^2\}.$$

Notice that the intersection of $\mathcal{O}(\varrho)$ with a slice $L_I$ is the interior of an eight shaped curve with equation

$$(x^2 + y^2)^{\frac{3}{2}} = \varrho x^2.$$

We remark that this curve always contains two open discs with positive radius depending just on $\varrho$, Hence the open set $\mathcal{O}(\varrho)$ always contains two open (4-dimensional) balls of positive radius (depending just on $\varrho$ as well).

**Lemma 4.7.** *Let $f : B = B(0, R) \to \mathbb{H}$ be a (non constant) regular function such that $f(0) = 0$ and that $\partial_c f(0) \in \mathbb{R}$. Then the image of $B$ under $f$ contains an open set of the form $\mathcal{O}(\varrho)$ where*

$$\varrho = \frac{R|\partial_c f(0)|^2}{4||\partial_c f||}.$$

*Proof.* Since $f(0) = 0$, if $\partial_c f(0) = 0$ there is nothing to prove. Suppose then that $\partial_c f(0) \neq 0$. Consider a point $c$ outside the image of $B$ under $f$, then $c \neq 0$. We want to show that $c$ does not belong to $\mathcal{O}(\varrho)$.
For all $q \in B$, define $g(q)$ to be

$$g(q) = (1 - f(q)c^{-1})^s.$$

The function $g$ is regular on $B$ and we can estimate its modulus in the following manner: let $\tau(q)$ be the transformation defined by

$$\tau(q) = (1 - f(q)c^{-1})^{-1} q (1 - f(q)c^{-1}).$$

Then, according to Proposition 1.20 and recalling that $|\tau(q)| = |q|$ for all $q$, we can write

$$\begin{aligned} |g(q)| &= |(1 - f(q)c^{-1}) * (1 - f(q)c^{-1})^c| = |(1 - f(q)c^{-1})||(1 - f(\tau(q))c^{-1})^c| \\ &\leq \sup_{|q|<R} |(1 - f(q)c^{-1})| \sup_{|q|<R} |(1 - f(q)c^{-1})^c|. \end{aligned}$$

Hence, using Proposition 2.3,

$$|g(q)| \leq \big( \sup_{|q|<R} |(1 - f(q)c^{-1})|\big)^2,$$

that is

$$|g(q)|^{\frac{1}{2}} \leq \sup_{|q|<R} |(1 - f(q)c^{-1})|.$$

By the properties of the uniform norm and by Remark 4.5 we get then

$$|g(q)|^{\frac{1}{2}} \leq 1 + \sup_{|q|<R} |f(q)c^{-1}| \leq 1 + |c|^{-1} \sup_{|q|<R} |f(q)| \leq 1 + |c|^{-1} ||\partial_c f|| R. \tag{4.5}$$

The next step is to estimate from below the quantity $|g(q)|^{\frac{1}{2}}$. Notice that $g$ is slice preserving, since it is the symmetrization of a regular function. Moreover $g(q)$ is never zero. For all $I \in \mathbb{S}$, the map $z \mapsto z^4$ from $L_I \setminus \{0\} \to L_I \setminus \{0\}$ is a covering map. Since $B_I$ is simply connected, we can lift the function $g$ obtaining a holomorphic function $\Psi_I : B_I \to L_I \setminus \{0\}$ such that

$$\Psi_I(z)^4 = (1 - f(z)c^{-1})^s$$

and $\Psi_I(0) = 1$ (since $g(0) = 1$). Let $\Psi$ be the (unique) regular extension to $B$ of $\Psi_I$. Now $\Psi(0) = 1$,

$$\Psi(q)^4 = (1 - f(q)c^{-1})^s = g(q)$$

for all $q \in B$ and in particular

$$|\Psi(q)|^2 = |g(q)|^{\frac{1}{2}} \quad \text{for all} \quad q \in B. \tag{4.6}$$

We want to use the power series expansion of $\Psi$ to find a lower bound of $|g|^{\frac{1}{2}}$. In particular we need to compute its slice derivative. Using the Leibniz rule 1.39 we can calculate

$$\begin{aligned}\partial_c g(q) &= \partial_c[(1 - f(q)c^{-1})^s] = \partial_c[(1 - f(q)c^{-1}) * (1 - f(q)c^{-1})^c] \\ &= -\partial_c f(q)c^{-1} * (1 - (f^c(q)c^{-1})^c) - (1 - f(q)c^{-1}) * \partial_c(f(q)c^{-1})^c.\end{aligned}$$

Since $q = 0$ is a real zero of $f$ (and hence of $f^c$), and since the operators of slice differentiation and regular conjugation do commute, if the power series expansion of $f$ is $f(q) = \sum_{n\geq 0} q^n a_n$, we obtain that

$$\big(\partial_c f(q)c^{-1}\big)^c = \sum_{n\geq 1} q^{n-1}\overline{n a_n c^{-1}}$$

and hence that

$$\partial_c g(0) = -\partial_c f(0)c^{-1} - \overline{\partial_c f(0)c^{-1}} = -2\,\mathrm{Re}(\partial_c f(0)c^{-1}).$$

Moreover, since $g$ (and $\Psi$) is slice preserving, $g(0)=1$, and $\partial_c f(0)$ is real, we have

$$\partial_c\Psi(0)=\frac{1}{4}g(0)^{\frac{1}{4}-1}\partial_c g(0)=-\frac{1}{2}\operatorname{Re}(\partial_c f(0)c^{-1})=-\frac{1}{2}\partial_c f(0)\operatorname{Re}(c^{-1}). \tag{4.7}$$

Let us set

$$M=1+|c|^{-1}||\partial_c f||R.$$

Fix $r\in(0,R)$ and $I\in\mathbb{S}$. Let $q\in\partial B_I(0,r)$, $q=re^{I\theta}$ for some $\theta\in[0,2\pi)$. By equations (4.5) and (4.6) we obtain that for every $\theta\in[0,2\pi)$

$$M\geq|\Psi(re^{I\theta})|^2. \tag{4.8}$$

Using the series expansion of $\Psi$ we can write

$$\begin{aligned}\left|\Psi(re^{I\theta})\right|^2&=\overline{\Psi(re^{I\theta})}\Psi(re^{I\theta})=\Big(\sum_{m\geq 0}\frac{1}{m!}r^m\overline{\Psi^{(m)}(0)}e^{-Im\theta}\Big)\Big(\sum_{n\geq 0}\frac{1}{n!}r^n e^{In\theta}\Psi^{(n)}(0)\Big)\\&=\sum_{m,n\geq 0}\frac{1}{m!n!}r^{m+n}\overline{\Psi^{(m)}(0)}e^{I(n-m)\theta}\Psi^{(n)}(0).\end{aligned}$$

If we integrate in $\theta$, then we get

$$\frac{1}{2\pi}\int_0^{2\pi}|\Psi(re^{I\theta})|^2d\theta=\frac{1}{2\pi}\int_0^{2\pi}\sum_{m,n\geq 0}\frac{1}{m!n!}r^{m+n}\overline{\Psi^{(m)}(0)}e^{I(n-m)\theta}\Psi^{(n)}(0)d\theta.$$

Thanks to the uniform convergence on compact sets of the series expansion, we can exchange the order of integration and summation. Then, since $\int_0^{2\pi}e^{Is\theta}d\theta=0$ if $s\in\mathbb{Z}, s\neq 0$ and equals $2\pi$ otherwise, just the terms where $n=m$ do survive and hence we get

$$\frac{1}{2\pi}\int_0^{2\pi}|\Psi(re^{I\theta})|^2d\theta=\sum_{m=0}^{\infty}\frac{r^{2m}}{(m!)^2}\overline{\Psi^{(m)}(0)}\Psi^{(m)}(0)=\sum_{m=0}^{\infty}\frac{r^{2m}}{(m!)^2}\left|\Psi^{(m)}(0)\right|^2.$$

By inequality (4.8) and since $M$ is a constant, we obtain

$$M=\frac{1}{2\pi}\int_0^{2\pi}Md\theta\geq\sum_{m=0}^{\infty}\frac{r^{2m}}{(m!)^2}\left|\Psi^{(m)}(0)\right|^2.$$

Considering just the first two terms of the series expansion and using equation (4.7), we get

$$M\geq 1+r^2|\partial_c\Psi(0)|^2=1+\frac{r^2|\partial_c f(0)|^2|\operatorname{Re}(c^{-1})|^2}{4}=1+\frac{r^2|\partial_c f(0)|^2|\operatorname{Re}(c)|^2}{4|c|^4}.$$

Recalling the expression of $M$ we have then

$$1+|c|^{-1}||\partial_c f||R \geq 1+\frac{r^2|\partial_c f(0)|^2|\operatorname{Re}(c)|^2}{4|c|^4},$$

that is

$$|c|^3 \geq \frac{r^2|\partial_c f(0)|^2|\operatorname{Re}(c)|^2}{4||\partial_c f||R}.$$

Since $r$ was arbitrarily chosen in $(0,R)$, the last inequality holds true for all $r \in (0,R)$. Then, taking the limit as $r$ approaches $R$, we obtain that if a point $c$ is outside the image $f(B)$, then it is such that

$$|c|^3 \geq \frac{R|\partial_c f(0)|^2|\operatorname{Re}(c)|^2}{4||\partial_c f||}.$$

That is equivalent to say that the image of $B$ under $f$ contains the open set

$$\mathcal{O}(\varrho)=\{q \in \mathbb{H} \mid |q|^3 < \varrho|\operatorname{Re}(q)|^2\}.$$

where

$$\varrho = \frac{R|\partial_c f(0)|^2}{4||\partial_c f||}.$$

□

Recall that for any $w \in B(0,R)$, $\tilde{f}_w$ denotes the regular translation by $w$ of the regular function $f$, namely the regular composition of $f$ with the translation $q \mapsto q+w$, defined in 1.42. In the proof of the Bloch Landau type Theorem, we need the following result about the convergence of sequences of regular translations of a regular function $f$.

**Proposition 4.8.** *Let $f$ be a regular function on $B=B(0,R)$ and let $w_n$ be a convergent sequence in $B$, such that $\lim_{n\to\infty} w_n = w \in B$. Set*

$$m = \max\{\{|w_n|, n \in \mathbb{N}\} \cup \{|w|\}\}$$

*Then the sequence of regular translations $\tilde{f}_{w_n}$ converges to the regular translation $\tilde{f}_w$ uniformly on compact subsets of $B(0,R-m)$.*

*Proof.* First of all notice that the maximum $m$ exists because of the convergence of the sequence $\{w_n\}$. Clearly the sequence $\tau_{w_n}$ converges (uniformly on compact sets) to $\tau_w$. Moreover, if $w_n \notin \mathbb{R}$ for infinitely many $n$, then (up to a subsequence) we define

$$I_n = \frac{\operatorname{Im}(w_n)}{|\operatorname{Im}(w_n)|} \quad \text{and} \quad \lim_{n\to\infty} I_n = I.$$

If, instead, there exists a natural number $n_0$ such that $w_n$ is real for all $n > n_0$ then we choose any $I \in \mathbb{S}$ and set $I_n = I$ for all $n > n_0$ in what follows. In both cases, $I \in \mathbb{S}$ is such that $w \in L_I$.

By the Representation Formula 1.10 we can write for all $q = x + yJ \in B(0, R - m)$

$$\begin{aligned}
&\tilde{f}_{w_n}(x + yJ) \\
&= \frac{1}{2}\left(\tilde{f}_{w_n}(x + yI_n) + \tilde{f}_{w_n}(x - yI_n)\right) + \frac{JI_n}{2}\left(\tilde{f}_{w_n}(x - yI_n) - \tilde{f}_{w_n}(x + yI_n)\right) \\
&= \frac{1}{2}\left(f_{I_n}(x + yI_n + w_n) + f_{I_n}(x - yI_n + w_n)\right) \\
&\qquad + \frac{JI_n}{2}\left(f_{I_n}(x - yI_n + w_n) - f_{I_n}(x + yI_n + w_n)\right) \\
&= \frac{1}{2}\left(f(x + yI_n + w_n) + f(x - yI_n + w_n)\right) \\
&\qquad + \frac{JI_n}{2}\left(f(x - yI_n + w_n) - f(x + yI_n + w_n)\right).
\end{aligned} \tag{4.9}$$

Since $f$ is a continuous function, thanks again to the Representation Formula 1.10, we can conclude that (uniformly on compact sets)

$$\lim_{n\to\infty} \tilde{f}_{w_n}(q) = \tilde{f}_w(q)$$

for all $q \in B(0, R - m)$. □

### 4.2.2 The main result

Finally we have all the tools to prove the announced Bloch-Landau type theorem for regular functions.

**Theorem 4.9** (a Bloch-Landau type theorem)**.** *Let $f$ be a regular function on the open unit ball $\mathbb{B}$ of $\mathbb{H}$ such that $f(0) = 0$ and $\partial_c f(0) = 1$. Then there exists $u \in \mathbb{B}$ such that the image of the regular translation $\tilde{f}_u$ of $f$ contains an open set obtained by means of a rotation and a translation of $\mathcal{O}(\varrho)$, where the "radius" $\varrho$ is at least $\frac{1}{32\sqrt{2}}$.*

*Proof.* Let $M(t)$ be the function defined on $[0, 1)$ by

$$M(t) = \max_{|q|\le t} |\partial_c f(q)|,$$

fix $r$ in $(0, 1)$, and consider the function

$$\mu(s) = sM(r - s),$$

defined for $s \in [0, r]$. By Proposition 2.1, $\mu$ is a continuous function, $\mu(0) = 0$, $\mu(s) \ge 0$ for all $s \in [0, r]$, and $\mu(r) = r$. Set

$$R = \frac{1}{2}\min\{s \mid \mu(s) = r\},$$

then $0 < 2R \leq r$. Let $w \in \partial B(0, r-2R)$ be such that $|\partial_c f(w)| = M(r-2R)$, i.e. by definition of $R$, such that $|\partial_c f(w)| = \frac{r}{2R}$. Let us restrict our attention to the slice $L_I$ containing $w$. Consider the function $\varphi_I : B(0, 2R) \cap L_I \to \mathbb{H}$, defined by

$$\varphi_I(z) = (f(z+w) - f(w)) \frac{\overline{\partial_c f(w)}}{|\partial_c f(w)|}.$$

The function $\varphi_I$ is holomorphic on $B(0, 2R) \cap L_I$, because

$$|q + w| \leq |q| + |w| \leq 2R + (r - 2R) = r.$$

Let $\varphi$ be the (unique) regular extension to the entire ball $B(0, 2R)$ of $\varphi_I$. Then $\varphi(0) = 0$ and $\partial_c \varphi(0) = |\partial_c f(w)| = \frac{r}{2R}$, hence $\varphi$ satisfies the hypotheses of Lemma 4.7.
For $z \in B(0, R) \cap L_I$ we have that

$$|\partial_c \varphi_I(z)| = |\partial_c f(z+w)| \leq M(|z+w|) \leq M(r-R) = \frac{\mu(R)}{R}.$$

Since $\mu$ is continuous, $\mu(0) = 0$ and $\mu(r) = r$, then

$$\frac{\mu(R)}{R} < \frac{r}{R},$$

otherwise there would exists $s < 2R$ such that $\mu(s) = r$, a contradiction with the definition of $R$. Therefore

$$\partial_c \varphi_I(B(0, R) \cap L_I) \subset B\left(0, \frac{r}{R}\right).$$

Proposition 1.12 implies then that

$$\partial_c \varphi(B(0, R)) \subset B\left(0, \frac{2r}{R}\right).$$

Considering the uniform norm we obtain

$$||\partial_c \varphi||_{B(0,R)} \leq \frac{2r}{R},$$

and hence, by Proposition 4.3,

$$||\partial_c \varphi|| \leq \frac{2\sqrt{2}r}{R} \tag{4.10}$$

on $B(0, R)$. Lemma 4.7 yields then that $\varphi(B(0, R))$ contains an open set $\mathcal{O}(\varrho)$ where

$$\varrho = \frac{R|\partial_c \varphi(0)|^2}{4||\partial_c \varphi||} \geq \frac{R\left(\frac{r}{2R}\right)^2}{4\frac{2\sqrt{2}r}{R}} = \frac{r}{32\sqrt{2}}.$$

Recalling the definition of $\varphi$, we get then

$$\tilde{f}_w(B(0,R)) - f(w) \supset \mathcal{O}(\varrho(r))\frac{\partial_c\varphi(0)}{|\partial_c\varphi(0)|},$$

that yields

$$f(w) + \mathcal{O}(\varrho(r))\frac{\partial_c\varphi(0)}{|\partial_c\varphi(0)|} \subset \tilde{f}_w(B(0,R))$$

Therefore for all $r < 1$ there exist a radius $R_r > 0$ and a point $w_r$, with modulus $|w_r| = r - 2R_r$, such that the image of $B(0,R_r)$ through $\tilde{f}_{w_r}$ contains the open set

$$f(w_r) + \mathcal{O}(\varrho(r))\frac{\partial_c\varphi(0)}{|\partial_c\varphi(0)|}.$$

When $r$ approaches 1, by compactness, we can find subsequences of radii $\{R_n\}_{n\in\mathbb{N}}$ and points $\{w_n\}_{n\in\mathbb{N}}$, that converge respectively to $R_0 > 0$ and to $w_0 \in \mathbb{B}$ (in fact $R_0 = 0$ would imply that $\mu(0) = 1$ which is not). Thanks to Proposition 4.8 we have then that $\tilde{f}_{w_n}$ converges (uniformly on compact sets) to $\tilde{f}_{w_0}$, and hence we get that the image of $\tilde{f}_{w_0}$ contains the open set

$$\mathcal{O}(\varrho)\frac{\partial_c\varphi(0)}{|\partial_c\varphi(0)|} + f(w_0)$$

whose "radius" is at least

$$\varrho = \lim_{r\to 1}\varrho(r) \geq \lim_{r\to 1}\frac{r}{32\sqrt{2}} = \frac{1}{32\sqrt{2}}.$$

□

It is easy to prove that if the regular translation $\tilde{f}_u$ that appears in the statement of Theorem 4.9 is a real translation (i.e. if $u$ is real), then the universal set $\mathcal{O}\left(\frac{1}{32\sqrt{2}}\right)$ is contained in $f(\mathbb{B})$.

In general, we were not able to show the existence of a universal open set directly contained in the image $f(\mathbb{B})$ of a (suitably normalized) regular function $f$. This might be connected with the fact that, as proven in [18], the Bloch-Landau Theorem does not hold in $\mathbb{C}^2$.

# Chapter 5

# Landau-Toeplitz theorems

The classical Landau-Toeplitz Theorem, first appeared in [44], concerns the study of the possible *shapes* of the image of the complex unit disc under a holomorphic function and it is formulated in terms of the *diameter* of the image set.

**Theorem 5.1** (Landau-Toeplitz). *Let $f$ be holomorphic in $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ and such that the diameter* $\operatorname{diam} f(\mathbb{D})$ *of $f(\mathbb{D})$ equals 2. Then*

$$\operatorname{diam} f(r\mathbb{D}) \leq 2r \quad \textit{for all} \quad r \in (0,1) \tag{5.1}$$

*and*

$$|f'(0)| \leq 1. \tag{5.2}$$

*Moreover equality holds in* (5.1) *for some $r \in (0,1)$, or in* (5.2)*, if and only if $f$ is of the form $f(z) = a + zb$ with $a, b \in \mathbb{C}$ and $|b| = 1$.*

It is interesting that this result was proven independently in 1907, in the same year of the first publication of the nowadays established proof of the classical Schwarz Lemma, [9]. Theorem 5.1 can be rather interpreted as a generalization of the Schwarz Lemma, in which the diameter of the image set takes over the role of the maximum modulus of the function. Indeed, there exist infinitely many subsets of the plane with constant diameter that are different from a disc; the Releaux Polygons are a well known example of such sets, [23, 40].

This chapter is devoted to prove a quaternionic analogue of the classical Landau-Toeplitz Theorem, together with a (partial) generalization, inspired by [8].

## 5.1 The Landau-Toeplitz Theorem for regular functions

### 5.1.1 The *regular diameter*

In the original approach, the computation of the diameter of the image of $\mathbb{D}$ under a holomorphic function $f$ is based on the composition of the considered function with rotations of the unit disc. In the new quaternionic setting we have to take into account that in general the composition of regular functions is not regular. This fact motivates the introduction of a new measure tool for the images of regular functions that relies upon the definition of regular composition 1.40. Recall that $f_u$ denotes the regular composition of a regular function $f$ with the function that maps $q$ to $qu$.

**Definition 5.2.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function. For $r \in (0,1)$, we define the* regular diameter *of the image of $r\mathbb{B}$ under $f$ as*

$$\tilde{d}_2(f(r\mathbb{B})) = \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\le r} |f_u(q) - f_v(q)|.$$

*Moreover we define the* regular diameter *of the image of $\mathbb{B}$ under $f$ as*

$$\tilde{d}_2(f(\mathbb{B})) = \lim_{r\to 1^-} \tilde{d}_2(f(r\mathbb{B})). \tag{5.3}$$

**Remark 5.3.** Since $q \mapsto f_u(q) - f_v(q)$ is a regular function on $\mathbb{B}$, the Maximum Modulus Principle 1.46 yields that the function $r \mapsto \max_{|q|\le r} |f_u(q) - f_v(q)|$ is increasing for $r \in (0,1)$, so, if $s, t \in (0,1)$ are such that $s \le t$, then, for any $u, v \in \overline{\mathbb{B}}$,

$$\max_{|q|\le s} |f_u(q) - f_v(q)| \le \max_{|q|\le t} |f_u(q) - f_v(q)|.$$

Thus

$$\max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\le s} |f_u(q) - f_v(q)| \le \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\le t} |f_u(q) - f_v(q)|.$$

Namely, if $0 < s \le t < 1$,

$$\tilde{d}_2(f(s\mathbb{B})) \le \tilde{d}_2(f(t\mathbb{B})),$$

that means that $\tilde{d}_2(f(r\mathbb{B}))$ is an increasing function of $r$. Therefore the limit (5.3) always exists (finite or infinite) and $\tilde{d}_2(f(\mathbb{B}))$ is well defined.

**Remark 5.4.** We point out that, for any $r \in (0,1)$,

$$\max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\le r} |f_u(q) - f_v(q)| = \max_{\substack{u,v\in\overline{\mathbb{B}}\\ |q|\le r}} |f_u(q) - f_v(q)| = \max_{|q|\le r} \max_{u,v\in\overline{\mathbb{B}}} |f_u(q) - f_v(q)|.$$

In fact, the function $(q,u,v) \mapsto |f_u(q) - f_v(q)|$ is continuous with respect to all the variables $q, u, v$, and the domain $r\overline{\mathbb{B}} \times \overline{\mathbb{B}} \times \overline{\mathbb{B}}$ is compact. Hence there exist $(q_0, u_0, v_0)$, $(q_1, u_1, v_1)$, and $(q_2, u_2, v_2)$ in $r\overline{\mathbb{B}} \times \overline{\mathbb{B}} \times \overline{\mathbb{B}}$ such that

$$\max_{\substack{u,v\in\overline{\mathbb{B}}\\ |q|\le r}} |f_u(q) - f_v(q)| = |f_{u_0}(q_0) - f_{v_0}(q_0)|,$$

$$\max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\le r} |f_u(q) - f_v(q)| = |f_{u_1}(q_1) - f_{v_1}(q_1)|$$

and

$$\max_{|q|\le r}\max_{u,v\in\overline{\mathbb{B}}}|f_u(q)-f_v(q)| = |f_{u_2}(q_2)-f_{v_2}(q_2)|.$$

Thus we obtain

$$\max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|f_u(q)-f_v(q)| = |f_{u_1}(q_1)-f_{v_1}(q_1)| \le \max_{\substack{u,v\in\overline{\mathbb{B}}\\|q|\le r}}|f_u(q)-f_v(q)|,$$

$$\max_{|q|\le r}\max_{u,v\in\overline{\mathbb{B}}}|f_u(q)-f_v(q)| = |f_{u_2}(q_2)-f_{v_2}(q_2)| \le \max_{\substack{u,v\in\overline{\mathbb{B}}\\|q|\le r}}|f_u(q)-f_v(q)|,$$

and also the opposite inequalities

$$\max_{\substack{u,v\in\overline{\mathbb{B}}\\|q|\le r}}|f_u(q)-f_v(q)| = |f_{u_0}(q_0)-f_{v_0}(q_0)| \le \max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|f_u(q)-f_v(q)|,$$

$$\max_{\substack{u,v\in\overline{\mathbb{B}}\\|q|\le r}}|f_u(q)-f_v(q)| = |f_{u_0}(q_0)-f_{v_0}(q_0)| \le \max_{|q|\le r}\max_{u,v\in\overline{\mathbb{B}}}|f_u(q)-f_v(q)|.$$

A question that naturally arises is how the new geometric notion of regular diameter is related to the classical notion of diameter. Let $E$ be a subset of $\mathbb{H}$. We will denote by

$$\operatorname{diam}(E) = \sup_{q,w\in E}|q-w|$$

the usual diameter of $E$.

**Remark 5.5.** Clearly if $E_1 \subset E_2$ then $\operatorname{diam}(E_1) \le \operatorname{diam}(E_2)$, and in particular, if $f$ is a regular function on $\mathbb{B}$, the map

$$r \mapsto \operatorname{diam}(f(r\mathbb{B}))$$

is increasing for $r \in (0,1)$. This yields that the limit

$$\operatorname{diam}(f(\mathbb{B})) = \lim_{r\to 1^-}\operatorname{diam}(f(r\mathbb{B}))$$

always exists (finite or infinite).

**Proposition 5.6.** *Let $f$ be a regular function on $\mathbb{B}$. Then the following inequalities hold*

$$\operatorname{diam}(f(\mathbb{B})) \le \tilde{d}_2(f(\mathbb{B})) \le 2\operatorname{diam}(f(\mathbb{B})).$$

*Proof.* In order to prove the first one, let $r \in (0,1)$ and consider $q, w \in r\overline{\mathbb{B}}$. We want to bound the quantity $|f(q) - f(w)|$. Without loss of generality we can suppose that $|w| \geq |q|$. If $q \neq 0$ we have

$$|f(q) - f(w)| = \left| f\left(q\frac{|w|}{|w|}\right) - f\left(w\frac{|w|}{|w|}\right)\right| = \left| f_{\frac{q}{|w|}}(|w|) - f_{\frac{w}{|w|}}(|w|)\right|$$

where the last equality is due to the fact that $|w|$ is real and hence it commutes with both $q/|w|$ and $w/|w|$. Moreover, since $q/|w| \in \overline{\mathbb{B}}$ and $w/|w| \in \partial\mathbb{B}$ we have that

$$|f(q) - f(w)| = \left| f_{\frac{q}{|w|}}(|w|) - f_{\frac{w}{|w|}}(|w|)\right| \leq \max_{u,v\in\overline{\mathbb{B}}} |f_u(|w|) - f_v(|w|)|$$

and therefore that

$$|f(q) - f(w)| \leq \max_{u,v\in\overline{\mathbb{B}}} |f_u(|w|) - f_v(|w|)| \leq \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\leq r} |f_u(q) - f_v(q)| = \tilde{d}_2(f(r\mathbb{B})).$$

We have to consider now the case $q = 0$. It is not difficult to see that

$$|f(0) - f(w)| = |f_0(w) - f_1(w)|$$

and hence that also

$$|f(0) - f(w)| \leq \max_{u,v\in\overline{\mathbb{B}}} |f_u(w) - f_v(w)| \leq \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\leq r} |f_u(q) - f_v(q)| = \tilde{d}_2(f(r\mathbb{B})).$$

This implies that

$$\operatorname{diam}(f(r\overline{\mathbb{B}})) \leq \tilde{d}_2(f(r\mathbb{B})).$$

Since the previous inequality holds for any $r \in (0,1)$, recalling Remark 5.5 we obtain that

$$\operatorname{diam}(f(\mathbb{B})) = \lim_{r\to 1^-} \operatorname{diam}(f(r\overline{\mathbb{B}})) \leq \lim_{r\to 1^-} \tilde{d}_2(f(r\mathbb{B})) = \tilde{d}_2(f(\mathbb{B})).$$

To show the other inequality, let $u, v \in \overline{\mathbb{B}}$, $r \in (0,1)$, and let $J, K$ be elements of $\mathbb{S}$ such that $u \in L_J$ and $v \in L_K$. By the Representation Formula 1.10 we get that for all $q = x + yI \in \overline{r\mathbb{B}}$

$$\begin{aligned} f_u(q) &= \frac{1}{2}\left(f_u(x+yJ) + f_u(x-yJ)\right) + \frac{IJ}{2}\left(f_u(x-yJ) - f_u(x+yJ)\right) \\ &= \frac{1}{2}\left(f((x+yJ)u) + f((x-yJ)u)\right) + \frac{IJ}{2}\left(f((x-yJ)u) - f((x+yJ)u)\right) \end{aligned}$$

and

$$\begin{aligned} f_v(q) &= \frac{1}{2}\left(f_v(x+yK) + f_v(x-yK)\right) + \frac{IK}{2}\left(f_v(x-yK) - f_v(x+yK)\right) \\ &= \frac{1}{2}\left(f((x+yK)v) + f((x-yK)v)\right) + \frac{IK}{2}\left(f((x-yK)v) - f((x+yK)v)\right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&|f_u(q) - f_v(q)| \\
&= \frac{1}{2}|(f((x+yJ)u) - (f(x+yK)v)) + (f((x-yJ)u) - f((x-yK)v)) \\
&+ IJ(f((x-yJ)u) - f((x+yJ)u)) - IK(f((x-yK)v) - f((x+yK)v))| \\
&\leq \frac{1}{2}|f((x+yJ)u) - (f(x+yK)v)| + \frac{1}{2}|f((x-yJ)u) - f((x-yK)v)| \\
&+ \frac{1}{2}|f((x-yJ)u) - f((x+yJ)u)| + \frac{1}{2}|f((x-yK)v) - f((x+yK)v)| \\
&\leq \frac{1}{2}\left(4 \operatorname{diam} f(r\mathbb{B})\right) = 2 \operatorname{diam} f(r\mathbb{B}).
\end{aligned} \tag{5.4}$$

Since inequality (5.4) holds for every $u, v \in \overline{\mathbb{B}}$ and for every $q \in \overline{r\mathbb{B}}$, we get

$$\tilde{d}_2(f(r\mathbb{B})) = \max_{u,v\in\overline{\mathbb{B}}} \max_{|q|\leq r} |f_u(q) - f_v(q)| \leq 2 \operatorname{diam} f(r\mathbb{B}). \tag{5.5}$$

Moreover, since inequality (5.5) holds for every $r \in (0,1)$, we obtain

$$\tilde{d}_2(f(\mathbb{B})) = \lim_{r\to 1^-} \tilde{d}_2(f(r\mathbb{B})) \leq \lim_{r\to 1^-} 2 \operatorname{diam}(f(r\mathbb{B})) = 2 \operatorname{diam}(f(\mathbb{B})).$$

□

Notice that if $f$ is an affine function, that is if $f(q) = a + qb$ for $a, b \in \mathbb{H}$, then

$$\tilde{d}_2(f(r\mathbb{B})) = |b| \operatorname{diam}(r\mathbb{B}) = |b| r \operatorname{diam}(\mathbb{B}) = 2|b|r \quad \text{for every} \quad r \in (0,1),$$

and in particular, if $f$ is constant, then $\tilde{d}_2(f(r\mathbb{B})) = 0$. Moreover, the regular diameter $\tilde{d}_2(f(r\mathbb{B}))$ is invariant under translations, if $g(q) = f(q) - f(0)$, then $\tilde{d}_2(g(r\mathbb{B})) = \tilde{d}_2(f(r\mathbb{B}))$ for every $r \in (0,1)$.

### 5.1.2 Two technical lemmas

Let us begin by defining the imaginary component of a generic quaternion $w$ along the imaginary unit $I \in \mathbb{S}$. Denote by $\langle\ ,\ \rangle$ the scalar product of $\mathbb{R}^4$, and by $\times$ the vector product of $\mathbb{R}^3$. Recall that for all purely imaginary quaternions $u, v$ the customary equality $uv = -\langle u, v\rangle + u \times v$ holds. If $w = x + yL \in \mathbb{H}$, then for all $I \in \mathbb{S}$,

$$\langle w, I\rangle = \langle yL, I\rangle = -\operatorname{Re}(yLI) = -\operatorname{Re}(wI).$$

**Definition 5.7.** *Let $I \in \mathbb{S}$. For any $w \in \mathbb{H}$ we define the imaginary component of $w$ along $I$ as* $\operatorname{Im}_I(w) = \langle w, I\rangle = -\operatorname{Re}(wI)$*.*

We point out that this quantity is a real number, differently from the *imaginary part* of a quaternion $q$, $\operatorname{Im}(q)$, introduced at the beginning of Chapter 1.

We are now ready to prove a first preliminary result.

**Lemma 5.8.** *Let $w \in B = B(0,R)$, $0 < |w| = r < R$, and let $g$ be a holomorphic function on $B_{I_w} = B \cap L_{I_w}$. If*

$$g(w) = w \quad \textit{and} \quad \max_{z \in r\overline{B}_{I_w}} |g(z)| = r, \tag{5.6}$$

*then* $\operatorname{Im}_{I_w}(\partial_c g(w)) = 0$.

*Proof.* To simplify the notation, let us set $I = I_w$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ to be the function

$$\varphi(\theta) = |g(we^{I\theta})|^2.$$

The Splitting Lemma 1.7 implies that for every $J \in \mathbb{S}$, orthogonal to $I$, there exist $F, G : B_I \to L_I$, holomorphic functions, such that

$$g(z) = F(z) + G(z)J \quad \text{for every} \quad z \in B_I. \tag{5.7}$$

Taking into account that for every $I, J \in \mathbb{S}$, orthogonal to each other, $IJ = -JI$, we can write

$$\begin{aligned}\varphi(\theta) &= g(we^{I\theta})\overline{g(we^{I\theta})} = (F(we^{I\theta}) + G(we^{I\theta})J)(\overline{F(we^{I\theta}) + G(we^{I\theta})J}) \\ &= F(we^{I\theta})\overline{F(we^{I\theta})} + G(we^{I\theta})\overline{G(we^{I\theta})}.\end{aligned}$$

Since $F(z)$ and $G(z)$ are holomorphic in $B_I$, then $\overline{F(\bar{z})}$ and $\overline{G(\bar{z})}$ are holomorphic in $B_I$ as well. Therefore we can compute the derivative of $\varphi$ as follows.

$$\begin{aligned}\varphi'(\theta) &= Iwe^{I\theta}F'(we^{I\theta})\overline{F(we^{I\theta})} - I\bar{w}e^{-I\theta}F(we^{I\theta})\overline{F'(we^{I\theta})} \\ &\quad + Iwe^{I\theta}G'(we^{I\theta})\overline{G(we^{I\theta})} - I\bar{w}e^{-I\theta}G(we^{I\theta})\overline{G'(we^{I\theta})} \\ &= -2\operatorname{Im}_I\left(we^{I\theta}F'(we^{I\theta})\overline{F(we^{I\theta})}\right) - 2\operatorname{Im}_I\left(we^{I\theta}G'(we^{I\theta})\overline{G(we^{I\theta})}\right) \\ &= -2\operatorname{Im}_I\left(we^{I\theta}\left(F'(we^{I\theta})\overline{F(we^{I\theta})} + G'(we^{I\theta})\overline{G(we^{I\theta})}\right)\right),\end{aligned}$$

where $F'$ and $G'$ denote, respectively, the complex derivatives of $F$ and $G$ in $B_I$. Since, by hypothesis, $\theta = 0$ is a maximum for $\varphi$, it turns out that

$$0 = \varphi'(0) = -2\operatorname{Im}_I\left(w\left(F'(w)\overline{F(w)} + G'(w)\overline{G(w)}\right)\right). \tag{5.8}$$

Moreover

$$w = g(w) = F(w) + G(w)J,$$

that yields

$$F(w) = w \quad \text{and} \quad G(w) = 0.$$

Inserting these values in equation (5.8), we get

$$0 = -2\operatorname{Im}_I\left(wF'(w)\overline{w}\right) = -2|w|^2\operatorname{Im}_I\left(F'(w)\right)$$

that is

$$\mathrm{Im}_I\big(F'(w)\big) = 0.$$

Finally, recalling the definition of slice derivative 1.4, we have

$$\partial_c g(w) = \partial_I\left(F(z) + G(z)J\right)\Big|_{z=w} = F'(w) + G'(w)J,$$

and, by Definition 5.7, we conclude that

$$\mathrm{Im}_I\left(\partial_c g(w)\right) = \mathrm{Im}_I(F'(w)) = 0.$$

□

**Remark 5.9.** We point out that the previous result has a geometric interpretation as a consequence of the Julia-Wolff-Carathéodory Theorem (see for instance [1,7]). In fact hypotheses (5.6) yield that

$$g : r\mathbb{B}_I \to r\mathbb{B}$$

and that $w$ is a boundary fixed point for the restriction of $g$ to $r\mathbb{B}_I$. Hence, if we consider the splitting (5.7) of the function $g$, for $z \in r\mathbb{B}_I$,

$$g(z) = F(z) + G(z)J,$$

we have that

$$F : r\mathbb{B}_I \to r\mathbb{B}_I$$

and $w$ is a boundary fixed point for the restriction of $F$ to $r\mathbb{B}_I$, namely $w$ is a Wolff point for $F$ restricted to $r\mathbb{B}_I$.

The following corollary, whose statement has an exact equivalent in the complex setting, [8], is a direct consequence of Lemma 5.8.

**Corollary 5.10.** *Let $g : \mathbb{B} \to \mathbb{H}$ be a regular function and let $w \in \mathbb{B}$, $0 < |w| = r < 1$, be such that*

$$g(w) = w \quad \textit{and} \quad \max_{|q|\le r} |g(q)| = r.$$

*Then* $\mathrm{Im}_{I_w}(\partial_c g(w)) = 0$.

The proof of the classical Landau-Toeplitz Theorem in the setting of holomorphic maps, [8], relies upon an analogue of Proposition 5.8, which is not sufficient for our purposes. In fact, since in the quaternionic setting there are more that just one imaginary unit, we need also the following result.

**Lemma 5.11.** *Let $g : \mathbb{B} \to \mathbb{H}$ be a regular function such that for every $q \in \mathbb{B}$*

$$\mathrm{Im}_{I_q}\, g(q) = 0. \tag{5.9}$$

*Then $g$ is a real constant function.*

*Proof.* Let
$$g(q) = \sum_{n \geq 0} q^n a_n$$
be the power series expansion of $g$ on $\mathbb{B}$. For any $I \in \mathbb{S}$ we split the coefficient $a_n$ as $b_n + c_n J$ with $b_n, c_n \in L_I$ and $J \in \mathbb{S}$ orthogonal to $I$. By hypothesis we have
$$0 = \mathrm{Im}_I(g(z)) = \mathrm{Im}_I\Big(\sum_{n\geq 0} z^n(b_n + c_n J)\Big) = \mathrm{Im}_I\Big(\sum_{n\geq 0} z^n b_n\Big)$$
for all $z \in \mathbb{B}_I$. As a consequence of the Open Mapping Theorem the holomorphic map $\sum_{n\geq 0} z^n b_n$ is constant, i.e., $b_n = 0$ for all $n > 0$. Therefore the component of each $a_n$ along $L_I$ vanishes for all $n > 0$. Since $I \in \mathbb{S}$ is arbitrary, we get that $a_n = 0$ for all $n > 0$. Finally, hypothesis (5.9) yields that $a_0 \in \mathbb{R}$. □

### 5.1.3 The main result

**Theorem 5.12** (Landau-Toeplitz for regular functions). *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function such that $\tilde{d}_2(f(\mathbb{B})) = \mathrm{diam}(\mathbb{B}) = 2$. Then*
$$\tilde{d}_2(f(r\mathbb{B})) \leq 2r \quad \textit{for every} \quad r \in (0,1). \tag{5.10}$$
*and*
$$|\partial_c f(0)| \leq 1. \tag{5.11}$$
*Moreover, equality holds in* (5.10) *for some $r \in (0,1)$, or in* (5.11)*, if and only if $f$ is an affine function, $f(q) = a + qb$, with $a, b \in \mathbb{H}$ and $|b| = 1$.*

*Proof.* To prove the first inequality, take $u, v \in \overline{\mathbb{B}}$, and consider the auxiliary function
$$g_{u,v}(q) = \frac{1}{2} q^{-1}(f_u(q) - f_v(q)).$$
This function is regular on $\mathbb{B}$. Indeed, if the power series expansion of $f$ in $\mathbb{B}$ is
$$\sum_{n\geq 0} q^n a_n,$$
then it turns out that
$$\begin{aligned} g_{u,v}(q) &= \frac{1}{2} q^{-1}\Big(\sum_{n\geq 0} q^n u^n a_n - \sum_{n\geq 0} q^n v^n a_n\Big) = \frac{1}{2} q^{-1} \sum_{n\geq 1}(q^n u^n - q^n v^n) a_n \\ &= \frac{1}{2} q^{-1} \sum_{n\geq 1} q^n (u^n - v^n) a_n = \frac{1}{2} \sum_{n\geq 1} q^{n-1}(u^n - v^n) a_n \\ &= \frac{1}{2} \sum_{n\geq 0} q^n (u^{n+1} - v^{n+1}) a_{n+1}. \end{aligned}$$

By this expression of $g_{u,v}$ we can recover its value in $q = 0$:

$$g_{u,v}(0) = \frac{1}{2}(u-v)a_1 = \frac{1}{2}(u-v)\partial_c f(0). \tag{5.12}$$

Since $g_{u,v}$ is regular, by the Maximum Modulus Principle 1.46, we get that $|g_{u,v}(q)|$ attains its maximum on the boundary of its domain of definition and hence that $\max_{|q|\le r}|g_{u,v}(q)|$ is an increasing function of $r$. Using the same kind of argument that appears in Remark 5.3, we get that also the function

$$r \mapsto \max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|g_{u,v}(q)|$$

is increasing. Moreover, the regularity of the function $q \mapsto f_u(q) - f_v(q)$ yields that, for any fixed $r \in (0,1)$, we can write

$$\begin{aligned}\max_{|q|\le r}|g_{u,v}(q)| &= \max_{|q|\le r}\frac{|f_u(q)-f_v(q)|}{2|q|} = \max_{|q|=r}\frac{|f_u(q)-f_v(q)|}{2|q|} = \max_{|q|=r}\frac{|f_u(q)-f_v(q)|}{2r}\\ &= \frac{\max_{|q|=r}|f_u(q)-f_v(q)|}{2r} = \frac{\max_{|q|\le r}|f_u(q)-f_v(q)|}{2r},\end{aligned}$$

which leads to

$$\begin{aligned}\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} &= \frac{\max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|f_u(q)-f_v(q)|}{2r} = \max_{u,v\in\overline{\mathbb{B}}}\frac{\max_{|q|\le r}|f_u(q)-f_v(q)|}{2r}\\ &= \max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|g_{u,v}(q)|.\end{aligned} \tag{5.13}$$

Therefore $\tilde{d}_2(f(r\mathbb{B}))/2r$ is an increasing function of $r$ and hence it is always less or equal then the limit

$$\lim_{r\to 1^-}\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = \frac{\tilde{d}_2(f(\mathbb{B}))}{2} = 1.$$

This means that

$$\tilde{d}_2(f(r\mathbb{B})) \le 2r \quad \text{for every} \quad r \in (0,1), \tag{5.14}$$

proving hence inequality (5.10) of the statement. To prove the second inequality, consider the odd part of $f$,

$$f_{odd}(q) = \frac{f(q)-f(-q)}{2}.$$

Notice that $f_{odd}$ satisfies the hypotheses of the Schwarz Lemma for regular functions 1.50. Indeed $f_{odd}$ is a regular function on $\mathbb{B}$, $f_{odd}(0) = 0$, and

$$|f_{odd}(q)| = \frac{|f(q)-f(-q)|}{2} \le \frac{\tilde{d}_2(f(\mathbb{B}))}{2} = 1$$

for every $q \in \mathbb{B}$. Hence

$$1 \geq |\partial_c f_{odd}(0)| = \frac{|\partial_c f(q) - \partial_c(f(-q))|}{2}\Big|_{q=0} = \frac{|\partial_c f(q) + \partial_c f(-q)|}{2}\Big|_{q=0} = |\partial_c f(0)|.$$

We will now prove the second part of the statement. To begin with, notice that if $f(q) = a + qb$ with $a, b \in \mathbb{H}$ and $|b| = 1$, then equality holds both in (5.10) and in (5.11). Conversely, suppose that equality holds in (5.11), namely that $|\partial_c f(0)| = 1$. In this case we have $|\partial_c f_{odd}(0)| = 1$ and therefore, by the Schwarz Lemma 1.50,

$$f_{odd}(q) = q\partial_c f(0). \tag{5.15}$$

We want to show that in this case $\tilde{d}_2(f(r\mathbb{B})) = 2r$ for every $r \in (0,1)$. In fact, from (5.12) and (5.13) it follows

$$\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} \geq \max_{u,v \in \overline{\mathbb{B}}} |g_{u,v}(0)| = \max_{u,v \in \overline{\mathbb{B}}} \frac{1}{2}|(u-v)\partial_c f(0)| = 1 \quad \text{for every} \quad r \in (0,1).$$

Comparing the last inequality with (5.14) we get

$$\tilde{d}_2(f(r\mathbb{B})) = 2r \quad \text{for every} \quad r \in (0,1). \tag{5.16}$$

Consider now a new auxiliary function. Take $w \in \mathbb{B}$, with $0 < |w| = r < 1$ and set

$$h_w(q) = \frac{1}{2}(f(q) - f(-w))(\partial_c f(0))^{-1}.$$

The function $h_w$ is regular on $\mathbb{B}$ and fixes $w$; indeed

$$h_w(w) = \frac{1}{2}(f(w) - f(-w))\partial_c f(0)^{-1} = f_{odd}(w)\partial_c f(0)^{-1} = w$$

where the last equality is due to (5.15). We need now to restrict our attention to what happens in $L_{I_w}$ (isomorphic to $\mathbb{C}$).
By the Maximum Modulus Principle 1.46, we are able to find $z_0 \in L_{I_w}$, with $|z_0| = r$, such that, for $z \in L_{I_w}$,

$$\max_{|z| \leq r} |h_w(z)| = \frac{1}{2} \max_{|z| \leq r} |f(z) - f(-w)| = \frac{1}{2}|f(z_0) - f(-w)|.$$

Let $\hat{u} \in L_{I_w}$ with $|\hat{u}| = 1$ be such that $-w = z_0\hat{u}$. Then, again for $z \in L_{I_w}$, due to the fact that $z_0$ and $\hat{u}$ commute, we can write

$$\max_{|z| \leq r} |h_w(z)| = \frac{1}{2}|f(z_0) - f(z_0\hat{u})| = \frac{1}{2}|f(z_0) - f_{\hat{u}}(z_0)| \leq \frac{1}{2} \max_{u,v \in \overline{\mathbb{B}}} \max_{|z| \leq r} |f_u(z) - f_v(z)|.$$

Recalling (5.16), for $z \in L_{I_w}$ and $q \in \mathbb{H}$ we obtain

$$\max_{|z| \leq r} |h_w(z)| \leq \frac{1}{2} \max_{u,v \in \overline{\mathbb{B}}} \max_{|q| \leq r} |f_u(q) - f_v(q)| = \frac{1}{2}\tilde{d}_2(f(r\mathbb{B})) = r = |h_w(w)|.$$

The function $h_w$ satisfies then the hypotheses of Lemma 5.8, and hence

$$\mathrm{Im}_{I_w}\left(\partial_c h_w(q)_{|_{q=w}}\right) = 0.$$

If we compute the slice derivative of $h_w(q)$, with respect to the variable $q$, at the point $w$, then we get

$$\mathrm{Im}_{I_w}\left(\frac{1}{2}\partial_c f(w)\partial_c f(0)^{-1}\right) = 0.$$

Recall now that $w$ is an arbitrary element of $\mathbb{B}\setminus\{0\}$. By continuity, we get then that the function $w \mapsto \frac{1}{2}\partial_c f(w)\partial_c f(0)^{-1}$, regular on $\mathbb{B}$, satisfies the hypotheses of Lemma 5.11. Consequently $\frac{1}{2}\partial_c f(w)\partial_c f(0)^{-1}$ is a real constant function and hence $\partial_c f(w)$ is constant as well. Therefore $f$ has the required form $f(q) = f(0) + q\partial_c f(0)$.

We will show now how equality in (5.10) for some $s \in (0,1)$ implies equality in (5.11). Thus, using the preceding step of the proof, we will conclude. Suppose that there exists $s \in (0,1)$ such that $\tilde{d}_2(f(s\mathbb{B}))/2s = 1$. By (5.10) and since $\tilde{d}_2(f(r\mathbb{B}))/2r$ is increasing in $r$ (see (5.13)), it turns out

$$\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = 1 \quad \text{for every} \quad r \in [s,1).$$

Let us prove that this condition also leads to

$$\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = 1 \quad \text{for every} \quad r \in (0,1).$$

Let $\hat{u}, \hat{v} \in \overline{\mathbb{B}}$, be such that

$$\frac{\tilde{d}_2(f(s\mathbb{B}))}{2s} = \max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le s}|g_{u,v}(q)| = \max_{|q|\le s}|g_{\hat{u},\hat{v}}(q)| \tag{5.17}$$

(where the first equality follows from equation (5.13)). Let $r > s$. By the choice of $\hat{u}, \hat{v} \in \overline{\mathbb{B}}$, we get

$$1 = \frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = \max_{u,v\in\overline{\mathbb{B}}}\max_{|q|\le r}|g_{u,v}(q)| \ge \max_{|q|\le r}|g_{\hat{u},\hat{v}}(q)|. \tag{5.18}$$

If $g_{\hat{u},\hat{v}}(q)$ were not constant, the Maximum Modulus Principle 1.46 would imply

$$\max_{|q|\le r}|g_{\hat{u},\hat{v}}(q)| > \max_{|q|\le s}|g_{\hat{u},\hat{v}}(q)| = 1 \tag{5.19}$$

that, compared with (5.18), would give a contradiction. So the function $g_{\hat{u},\hat{v}}(q)$ must be constant. Hence

$$\max_{|q|\le r}|g_{\hat{u},\hat{v}}(q)| \equiv 1 \quad \text{for every} \quad r \in (0,1).$$

Consider now $t \in (0, s)$. Then

$$1 \geq \frac{\tilde{d}_2(f(t\mathbb{B}))}{2t} = \max_{u,v \in \overline{\mathbb{B}}} \max_{|q| \leq t} |g_{u,v}(q)| \geq \max_{|q| \leq t} |g_{\hat{u},\hat{v}}(q)| = 1,$$

that implies

$$\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = 1 \quad \text{for every} \quad r \in (0,1). \tag{5.20}$$

The claim is now that $|\partial_c f(0)| = 1$. By continuity, we get

$$\lim_{r \to 0^+} \frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = 1 \tag{5.21}$$

and hence, by (5.11), we obtain

$$\lim_{r \to 0^+} \frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} \geq |\partial_c f(0)|. \tag{5.22}$$

Since for every $r \in (0, 1)$

$$\frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = \max_{u,v \in \overline{\mathbb{B}}} \max_{|q| \leq r} |g_{u,v}(q)|,$$

then

$$\lim_{r \to 0^+} \frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = \lim_{r \to 0^+} \max_{u,v \in \overline{\mathbb{B}}} \max_{|q| \leq r} |g_{u,v}(q)|.$$

Hence, recalling Remark 5.4, for every $n \in \mathbb{N}$ there exist $u_n, v_n \in \overline{\mathbb{B}}$ and $q_n$, with $|q_n| = \frac{1}{n}$, such that

$$\lim_{n \to \infty} \frac{\tilde{d}_2(f(\frac{1}{n}\mathbb{B}))}{2\frac{1}{n}} = \lim_{n \to \infty} |g_{u_n,v_n}(q_n)|.$$

Since $\{u_n\}_{n \in \mathbb{N}}$ and $\{v_n\}_{n \in \mathbb{N}}$ lie in a compact set, up to subsequences, there exist $\tilde{u}, \tilde{v} \in \overline{\mathbb{B}}$, such that

$$\lim_{n \to \infty} |g_{u_n,v_n}(q_n)| = |g_{\tilde{u},\tilde{v}}(0)|.$$

Hence, using (5.12), we get

$$\lim_{n \to \infty} \frac{\tilde{d}_2(f(\frac{1}{n}\mathbb{B}))}{2\frac{1}{n}} = |g_{\tilde{u},\tilde{v}}(0)| \leq \max_{u,v \in \overline{\mathbb{B}}} |g_{u,v}(0)| = |\partial_c f(0)|.$$

By limit (5.21) and comparing with (5.22), we obtain

$$1 = \lim_{r \to 0^+} \frac{\tilde{d}_2(f(r\mathbb{B}))}{2r} = |\partial_c f(0)|$$

that implies

$$|\partial_c f(0)| = 1. \tag{5.23}$$

Since we have already shown that (5.23) implies the assertion, the proof is complete. □

## 5.2 The $n$-diameter case

### 5.2.1 The *regular* $n$-*diameter*

In [8], Burckel, Marshall, Minda, Poggi-Corradini and Ransford prove a generalization of the classical Landau-Toeplitz Theorem that involves a new geometric object, the $n$-*diameter*. To formulate a $n$-diameter version of the Landau-Toeplitz Theorem for regular functions we begin by giving the definition of $n$-diameter of a subset of $\mathbb{H}$.

**Definition 5.13.** *Let $E \subset \mathbb{H}$. For every $n \in \mathbb{N}$, $n \geq 2$, the* $n$-diameter *of $E$ is defined as*

$$d_n(E) = \sup_{w_1,\dots,w_n \in E} \left( \prod_{1\leq j<k\leq n} |w_k - w_j| \right)^{\frac{2}{n(n-1)}}$$

If the set $E$ contains less then $n$ points, then the $n$-diameter of $E$ vanishes. Notice that $d_2(E) = \mathrm{diam}(E)$ and that diameter and $n$-diameter are related by the following inequality.

**Proposition 5.14.** *Let $n \in \mathbb{N}, n \geq 2$ and let $E \subset \mathbb{H}$. Then $d_n(E) \leq d_2(E)$. Moreover $d_n(E)$ is finite if and only if $d_2(E)$ is finite.*

*Proof.*

$$\begin{aligned} d_n(E) &= \sup_{w_1,\dots,w_n \in E} \left( \prod_{1\leq j<k\leq n} |w_k - w_j| \right)^{\frac{2}{n(n-1)}} \\ &\leq \sup_{w_1,\dots,w_n \in E} \left( \prod_{1\leq j<k\leq n} d_2(E) \right)^{\frac{2}{n(n-1)}} = d_2(E), \end{aligned}$$

which immediately entails that if $d_2(E) < +\infty$ then also $d_n(E) < +\infty$. For the converse, suppose that $d_2(E)$ is not finite. Then $E$ is not bounded. So there exists a sequence

$$\{u_m\}_{m\in\mathbb{N}} \subset E \quad \text{such that} \quad \lim_{m\to\infty} |u_m| = +\infty.$$

Let us choose $n-1$ distinct points $v_1, ..., v_{n-1} \in E$. Then for every $m \in \mathbb{N}$

$$\begin{aligned} d_n(E) &= \sup_{w_1,\dots,w_n \in E} \left( \prod_{1\leq j<k\leq n} |w_k - w_j| \right)^{\frac{2}{n(n-1)}} \\ &\geq \left( \left( \prod_{1\leq j<k\leq n-1} |v_k - v_j| \right) \left( \prod_{i=1}^{n-1} |u_m - v_i| \right) \right)^{\frac{2}{n(n-1)}}. \end{aligned}$$

Hence, setting $L = \min\{|v_k - v_j| : 1 \leq j < k \leq n-1\}$, we have

$$d_n(E) \geq \lim_{m\to+\infty} \left( \left( \prod_{1\leq j<k\leq n-1} |v_k - v_j| \right) \left( \prod_{i=1}^{n-1} |u_m - v_i| \right) \right)^{\frac{2}{n(n-1)}}$$
$$\geq \lim_{m\to+\infty} \left( L^{\frac{(n-1)(n-2)}{2}} \prod_{i=1}^{n-1} |u_m - v_i| \right)^{\frac{2}{n(n-1)}} = +\infty.$$

□

As we did in Section 5.1 in the case of the classical diameter $d_2$, we will adopt a specific definition for the $n$-diameter of the image of a subset of $\mathbb{H}$ under a regular function. We will always consider images of open balls of the form $r\mathbb{B}$.

**Definition 5.15.** *Let $n \geq 2$ and let $f$ be a regular function on $\mathbb{B}$. For $r \in (0,1)$, we define, in terms of the $*$-product, the* regular $n$-diameter *of the image of $r\mathbb{B}$ under $f$ as*

$$\tilde{d}_n(f(r\mathbb{B})) = \max_{w_1,\ldots,w_n\in\overline{\mathbb{B}}} \max_{|q|\leq r} \left| \prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q)) \right|^{\frac{2}{n(n-1)}}.$$

*Moreover, we define the* regular $n$-diameter *of the image of $\mathbb{B}$ under $f$ as*

$$\tilde{d}_n(f(\mathbb{B})) = \lim_{r\to 1^-} \tilde{d}_n(f(r\mathbb{B})).$$

The same argument used for the regular diameter in Remark 5.3, guarantees that $\tilde{d}_n(f(\mathbb{B}))$ is well defined. Notice that, because of the non-commutativity of quaternions, the order of the factors of a $*$-product has its importance. We can choose any order we like, but it has to be fixed once for ever. In what follows, when we write $1 \leq j < k \leq n$ we always mean to order the couples $(j,k)$ with the lexicographic order,

$$(1,2) < \ldots < (1,n) < (2,3) < \ldots < (2,n) < \ldots < (n-1,n).$$

To simplify the notation, we will sometimes write $j < k$ meaning $1 \leq j < k \leq n$.
The first step in the direction of understanding the relation between the $n$-diameter and the regular $n$-diameter is the following result.

**Proposition 5.16.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function, and let $n \geq 2$. Then $\tilde{d}_n(f(\mathbb{B})) \leq \tilde{d}_2(f(\mathbb{B}))$.*

*Proof.* Let $r \in (0,1)$. By definition, the regular $n$-diameter of $f(r\mathbb{B})$ is

$$\tilde{d}_n(f(r\mathbb{B})) = \max_{w_1,\ldots,w_n\in\overline{\mathbb{B}}} \max_{|q|\leq r} \left| \prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q)) \right|^{\frac{2}{n(n-1)}}.$$

We can turn the $*$-product into an usual product with an iteration of Proposition 1.20. Since we are interested in maximizing the quantity

$$\Big|\prod^{*}_{1\leq j<k\leq n}(f_{w_k}(q)-f_{w_j}(q))\Big|^{\frac{2}{n(n-1)}},$$

we can discard the cases where some factor $f_{w_k}(q)-f_{w_j}(q)$ does vanish, and hence we can set

$$\prod^{*}_{1\leq j<k\leq n}(f_{w_k}(q)-f_{w_j}(q))=\prod_{1\leq j<k\leq n}(f_{w_k}(T_{k,j}(q))-f_{w_j}(T_{k,j}(q))),$$

where

$$\begin{aligned}
T_{2,1}(q)&=q,\\
T_{3,1}(q)&=\big((f_{w_2}-f_{w_1})(q)\big)^{-1}q\big((f_{w_2}-f_{w_1})(q)\big),\\
T_{4,1}(q)&=\big((f_{w_3}-f_{w_1})(T_{3,1}(q))\big)^{-1}T_{3,1}(q)\big((f_{w_3}-f_{w_1})(T_{3,1}(q))\big),\\
&\ \ \vdots\\
T_{n,n-1}(q)&=\big((f_{w_n}-f_{w_{n-2}})(T_{n,n-2}(q))\big)^{-1}T_{n,n-2}(q)\big((f_{w_n}-f_{w_{n-2}})(T_{n,n-2}(q))\big).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\tilde{d}_n(f(r\mathbb{B}))&=\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|\leq r}\Big|\prod_{j<k}(f_{w_k}(T_{k,j}(q))-f_{w_j}(T_{k,j}(q)))\Big|^{\frac{2}{n(n-1)}}\\
&=\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|\leq r}\prod_{j<k}\big|f_{w_k}(T_{k,j}(q))-f_{w_j}(T_{k,j}(q))\big|^{\frac{2}{n(n-1)}}\\
&\leq\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\prod_{j<k}\max_{|q|\leq r}\big|f_{w_k}(T_{k,j}(q))-f_{w_j}(T_{k,j}(q))\big|^{\frac{2}{n(n-1)}}.
\end{aligned}$$

Since $|T_{k,j}(q)|=|q|$ for all $1\leq j<k\leq n$, we get

$$\begin{aligned}
\tilde{d}_n(f(r\mathbb{B}))&\leq\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\prod_{j<k}\max_{|q|\leq r}\big|f_{w_k}(q)-f_{w_j}(q)\big|^{\frac{2}{n(n-1)}}\\
&\leq\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\prod_{j<k}\tilde{d}_2(f(r\mathbb{B}))^{\frac{2}{n(n-1)}}=\tilde{d}_2(f(r\mathbb{B})).
\end{aligned}$$

Since the last inequality holds for every $r\in(0,1)$, we have

$$\tilde{d}_n(f(\mathbb{B}))=\lim_{r\to1^-}\tilde{d}_n(f(r\mathbb{B}))\leq\lim_{r\to1^-}\tilde{d}_2(f(r\mathbb{B}))=\tilde{d}_2(f(\mathbb{B})).$$

□

Notice that Proposition 5.14 and Proposition 5.16 imply that if $d_n(f(\mathbb{B}))$ is finite then $\tilde{d}_n(f(\mathbb{B}))$ is finite as well (for any regular function $f$ and $n \geq 2$).
Let us make some simple remarks about the definition of regular $n$-diameter. As for the case $n = 2$, the regular $n$-diameter is invariant under translation: in fact if $f$ is a regular function on $\mathbb{B}$ and $g$ is defined as $g(q) = f(q) - f(0)$, then $\tilde{d}_n(g(r\mathbb{B})) = \tilde{d}_n(f(r\mathbb{B}))$. Moreover, if $f(q) = qb$ with $b \in \mathbb{H}$, then $\tilde{d}_n(f(r\mathbb{B})) = |b|d_n(r\mathbb{B})$, in particular, if $f$ is constant, then $\tilde{d}_n(f(r\mathbb{B})) = 0$. Hence if $f$ is of the form $f(q) = a + qb$, for some quaternions $a$ and $b$, then the regular $n$-diameter of $f(r\mathbb{B})$ coincides with its $n$-diameter.

### 5.2.2 The inequality statements

In order to obtain analogues of inequalities (5.10) and (5.11), in the $n$-diameter case, we study the ratio between the regular $n$-diameter of the image of $r\mathbb{B}$ under a regular function $f$ and the $n$-diameter of the domain of $f$, $r\mathbb{B}$. The first analogue follows from the next statement.

**Lemma 5.17.** *Let $f$ be a regular function on $\mathbb{B}$ and let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\varphi_n(r) = \frac{\tilde{d}_n(f(r\mathbb{B}))}{d_n(r\mathbb{B})} = \frac{\tilde{d}_n(f(r\mathbb{B}))}{d_n(\mathbb{B})r}$$

*is an increasing function of $r$ on the open interval $(0, 1)$.*

*Proof.* Suppose that $f(0) = 0$. If $f$ is a constant or an affine function, then $\varphi_n(r)$ is a constant function. So let $f$ be neither constant nor affine. Fix $w_1, ..., w_n \in \overline{\mathbb{B}}$ and consider the auxiliary function

$$g_{w_1,...,w_n}(q) = d_n(\mathbb{B})^{-\frac{n(n-1)}{2}} q^{-\frac{n(n-1)}{2}} \prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q)).$$

Since

$$f_{w_j}(0) = f(0) = 0 \quad \text{for every} \quad j = 1, \dots, n,$$

we have that $q = 0$ is a zero of the regular function

$$\prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q))$$

with multiplicity at least $\frac{n(n-1)}{2}$. Hence we get that $g_{w_1,...,w_n}$ is regular on $\mathbb{B}$.
Moreover, using the Maximum Modulus Principle 1.46, we can write

$$\begin{aligned}\max_{|q|\leq r} |g_{w_1,...,w_n}(q)| &= \max_{|q|=r} \left| d_n(\mathbb{B})^{-\frac{n(n-1)}{2}} q^{-\frac{n(n-1)}{2}} \prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q))\right| \\ &= d_n(\mathbb{B})^{-\frac{n(n-1)}{2}} r^{-\frac{n(n-1)}{2}} \max_{|q|=r} \left| \prod^{*}_{1\leq j<k\leq n} (f_{w_k}(q) - f_{w_j}(q))\right|\end{aligned} \tag{5.24}$$

that implies

$$\begin{aligned}
\varphi_n(r)^{\frac{n(n-1)}{2}} &= d_n(\mathbb{B})^{-\frac{n(n-1)}{2}} r^{-\frac{n(n-1)}{2}} \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} \max_{|q|\le r} \Big| \prod^{*}_{1\le j<k\le n} (f_{w_k}(q) - f_{w_j}(q)) \Big| \\
&= \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} d_n(\mathbb{B})^{-\frac{n(n-1)}{2}} r^{-\frac{n(n-1)}{2}} \max_{|q|\le r} \Big| \prod^{*}_{1\le j<k\le n} (f_{w_k}(q) - f_{w_j}(q)) \Big| \qquad (5.25) \\
&= \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} \max_{|q|\le r} |g_{w_1,\dots,w_n}(q)|.
\end{aligned}$$

With the same kind of reasoning as in Remark 5.3 we can conclude that $\varphi_n(r)$ is increasing in $r$.

□

In turn, to prove the analogue of inequality (5.11) in the $n$-diameter case, we need the following result.

**Lemma 5.18.** *Let $f$ be a regular function on $\mathbb{B}$, and let $\varphi_n$ be the ratio defined in Lemma 5.17. Then*

$$\lim_{r\to 0^+} \varphi_n(r) = |\partial_c f(0)|.$$

*Proof.*

$$\begin{aligned}
\lim_{r\to 0^+} \varphi_n(r) &= \lim_{r\to 0^+} \frac{\tilde{d}_n(f(r\mathbb{B}))}{d_n(\mathbb{B})r} \\
&= \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} r^{-1} \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} \max_{|q|\le r} \Big| \prod^{*}_{1\le j<k\le n} (f_{w_k}(q) - f_{w_j}(q)) \Big|^{\frac{2}{n(n-1)}} \\
&= \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} r^{-1} \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} \max_{|q|= r} \Big| \prod^{*}_{1\le j<k\le n} (f_{w_k}(q) - f_{w_j}(q)) \Big|^{\frac{2}{n(n-1)}}.
\end{aligned}$$

Where the last equality is due to the Maximum Modulus Principle 1.46 applied to the function

$$\prod^{*}_{1\le j<k\le n} (f_{w_k}(q) - f_{w_j}(q)),$$

that is regular on $\mathbb{B}$. We can turn the $*$-product into an usual product using the transformations $T_{k,j}$, for $1 \le j < k \le n$, defined in the proof of Proposition 5.16, again omitting the points where some factor $f_{w_k}(q) - f_{w_j}(q)$ vanishes, thus obtaining

$$\lim_{r\to 0^+} \varphi_n(r) = \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} r^{-1} \max_{w_1,\dots,w_n\in\overline{\mathbb{B}}} \max_{|q|=r} \Big| \prod_{j<k} (f_{w_k}(T_{j,k}(q)) - f_{w_j}(T_{j,k}(q))) \Big|^{\frac{2}{n(n-1)}}.$$

Recalling now that for every $1 \leq j < k \leq n$ it results $|T_{k,j}(q)| = |q|$, if $|q| = r$ we can write

$$\begin{aligned}
\lim_{r\to 0^+} \varphi_n(r) &= \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} r^{-1} \max_{w_1,\ldots,w_n \in \overline{\mathbb{B}}} \max_{|q|=r} \prod_{j<k} \left|(f_{w_k}(T_{j,k}(q)) - f_{w_j}(T_{j,k}(q)))\right|^{\frac{2}{n(n-1)}} \\
&= \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} \max_{w_1,\ldots,w_n \in \overline{\mathbb{B}}} \max_{|q|=r} \prod_{j<k} \Big(|T_{j,k}(q)|^{-1} \left|\left(f_{w_k}(T_{k,j}(q)) - f_{w_j}(T_{k,j}(q))\right)\right|\Big)^{\frac{2}{n(n-1)}} \\
&= \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} \max_{w_1,\ldots,w_n \in \overline{\mathbb{B}}} \max_{|q|=r} \prod_{j<k} \left|(T_{k,j}(q))^{-1} \left(f_{w_k}(T_{k,j}(q)) - f_{w_j}(T_{k,j}(q))\right)\right|^{\frac{2}{n(n-1)}} .
\end{aligned} \tag{5.26}$$

Using now the series expansion $f(q) = \sum_{n\geq 0} q^n a_n$, which converges uniformly on compact subsets of $\mathbb{B}$, we get

$$\begin{aligned}
&\left|(T_{k,j}(q))^{-1} \left(f_{w_k}(T_{k,j}(q)) - f_{w_j}(T_{k,j}(q))\right)\right| \\
&= \Big|(T_{k,j}(q))^{-1} \Big( \sum_{n\geq 0} (T_{k,j}(q))^n w_k^n a_n - \sum_{n\geq 0} (T_{k,j}(q))^n w_j^n a_n \Big)\Big| \\
&= \Big|(T_{k,j}(q))^{-1} \sum_{n\geq 1} (T_{k,j}(q))^n (w_k^n - w_j^n) a_n \Big| = \Big| \sum_{n\geq 1} (T_{k,j}(q))^{n-1} (w_k^n - w_j^n) a_n \Big|.
\end{aligned} \tag{5.27}$$

Hence, equations (5.26) and (5.27) yield

$$\lim_{r\to 0^+} \varphi_n(r) = \lim_{r\to 0^+} d_n(\mathbb{B})^{-1} \max_{w_1,\ldots,w_n \in \overline{\mathbb{B}}} \max_{|q|=r} \prod_{j<k} \Big| \sum_{n\geq 1} (T_{k,j}(q))^{n-1} (w_k^n - w_j^n) a_n \Big|^{\frac{2}{n(n-1)}} .$$

Since $\varphi_n(r)$ is bounded below by 0 and it is increasing in $r$, then the limit of $\varphi_n(r)$, as $r$ goes to 0, always exists. Proceeding as in the proof of Theorem 5.12, recalling Remark 5.4, we can find a sequence of points $\{q_m\}_{m\in\mathbb{N}}$, such that $|q_m| = \frac{1}{m}$ for any $m \in \mathbb{N}$, and a sequence of $n$-tuples $\{(w_{1,m}, ..., w_{n,m})\}_{m\in\mathbb{N}} \subset \overline{\mathbb{B}}^n$, such that

$$\lim_{m\to\infty} \varphi_n\left(\frac{1}{m}\right) = \lim_{m\to\infty} d_n(\mathbb{B})^{-1} \prod_{j<k} \Big| \sum_{n\geq 1} (T_{k,j}(q_m))^{n-1} (w_{k,m}^n - w_{j,m}^n) a_n \Big|^{\frac{2}{n(n-1)}} . \tag{5.28}$$

Since the sequence $\{(w_{1,m}, ..., w_{n,m})\}_{m\in\mathbb{N}}$ lies in a compact set, up to subsequences, it converges to some $(\widehat{w}_1, ..., \widehat{w}_n) \in \overline{\mathbb{B}}^n$. Then, thanks to the uniform convergence on compact sets of the function on the right side of equation (5.28), we get

$$\lim_{m\to\infty} \varphi_n\left(\frac{1}{m}\right) = d_n(\mathbb{B})^{-1} \prod_{j<k} \Big| \sum_{n\geq 1} (T_{k,j}(0))^{n-1} (\hat{w}_k^n - \hat{w}_j^n) a_n \Big|^{\frac{2}{n(n-1)}} .$$

Notice that $T_{k,j}(0) = 0$ for every $1 \leq j < k \leq n$, then

$$\lim_{m\to\infty} \varphi_n\left(\frac{1}{m}\right) = d_n(\mathbb{B})^{-1} \prod_{j<k} |(\hat{w}_k - \hat{w}_j) a_1|^{\frac{2}{n(n-1)}} .$$

By Definition 5.13, we obtain

$$\lim_{m\to\infty}\varphi_n\left(\frac{1}{m}\right)=d_n(\mathbb{B})^{-1}|a_1|\prod_{j<k}|(\hat{w}_k-\hat{w}_j)|^{\frac{2}{n(n-1)}}\leq|a_1|=|\partial_c f(0)|.$$

To prove the opposite inequality, notice that, for every choice of $\{\tilde{w}_1,...,\tilde{w}_n\}\subset\overline{\mathbb{B}}$,

$$\begin{aligned}&\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_k^n-w_j^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&\geq\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(\tilde{w}_k^n-\tilde{w}_j^n)a_n\Big|^{\frac{2}{n(n-1)}}.\end{aligned}$$

Hence

$$\begin{aligned}&\lim_{r\to0^+}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_k^n-w_j^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&\geq\lim_{r\to0^+}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(\tilde{w}_k^n-\tilde{w}_j^n)a_n\Big|^{\frac{2}{n(n-1)}}.\end{aligned}\tag{5.29}$$

Since (5.29) holds for every choice of $\{\tilde{w}_1,...,\tilde{w}_n\}\subset\overline{\mathbb{B}}$, we get

$$\begin{aligned}&\lim_{r\to0^+}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_k^n-w_j^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&\geq\max_{\tilde{w}_1,\dots,\tilde{w}_n\in\overline{\mathbb{B}}}\lim_{r\to0^+}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(\tilde{w}_k^n-\tilde{w}_j^n)a_n\Big|^{\frac{2}{n(n-1)}}.\end{aligned}$$

Therefore we conclude

$$\begin{aligned}&\lim_{r\to0^+}\varphi_n(r)=\lim_{r\to0^+}d_n(\mathbb{B})^{-1}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_k^n-w_j^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&\geq d_n(\mathbb{B})^{-1}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\lim_{r\to0^+}\max_{|q|=r}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_j^n-w_k^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&=d_n(\mathbb{B})^{-1}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\lim_{|q|\to0^+}\prod_{j<k}\Big|\sum_{n\geq1}(T_{k,j}(q))^{n-1}(w_j^n-w_k^n)a_n\Big|^{\frac{2}{n(n-1)}}\\&=d_n(\mathbb{B})^{-1}\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\prod_{j<k}|(w_j-w_k)a_1|^{\frac{2}{n(n-1)}}\\&=d_n(\mathbb{B})^{-1}|a_1|\max_{w_1,\dots,w_n\in\overline{\mathbb{B}}}\prod_{j<k}|w_j-w_k|^{\frac{2}{n(n-1)}}=d_n(\mathbb{B})^{-1}|a_1|d_n(\mathbb{B})=|a_1|=|\partial_c f(0)|.\end{aligned}$$

□

By means of Lemmas 5.17 and 5.18 it is direct to prove the following result.

**Theorem 5.19.** *Let $f$ be a regular function on $\mathbb{B}$ such that $\tilde{d}_n(f(\mathbb{B})) = d_n(\mathbb{B})$. Then*

$$\tilde{d}_n(f(r\mathbb{B})) \leq d_n(r\mathbb{B}) \quad \textit{for every} \quad r \in (0,1) \tag{5.30}$$

*and*

$$|\partial_c f(0)| \leq 1. \tag{5.31}$$

We believe that if equality holds in (5.30) for some $r \in (0,1)$ or in (5.31), then $f$ is affine, but we were not able to prove this statement. The techniques available to us do not help: on one side, it is easy to see that if $f$ is affine, $f(q) = a + qb$ with $a, b \in \mathbb{H}$, $|b| = 1$, then equality holds both in (5.30) and in (5.31); on the other side, we do not yet know, in general, if the converse holds. Indeed we tried to address the problem with different approaches that only revealed how delicate the question is.

# Chapter 6

# Quaternionic Hardy Spaces

A classical subject in complex function theory is the study of Hardy spaces of holomorphic functions in the complex open unit disc $\mathbb{D}$. This chapter is devoted to introduce the quaternionic counterparts of Hardy spaces and to show some of their basic properties. The study of these function spaces was carried out in the framework of a scientific collaboration with Professor Chiara de Fabritiis.

## 6.1 The spaces $H^p(\mathbb{B})$

Let $p \in (0, +\infty)$, $r \in [0, 1)$, and $I \in \mathbb{S}$. For any regular function $f$ in the unit ball $\mathbb{B}$, let $(f_I)_r$ be the function defined on the unit circle $\partial\mathbb{B}_I$ by

$$(f_I)_r(e^{I\theta}) = f(re^{I\theta})$$

and let $M_p(f_I, r)$ be the integral mean

$$M_p(f_I, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|(f_I)_r(e^{I\theta})\right|^p d\theta\right)^{\frac{1}{p}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{I\theta})\right|^p d\theta\right)^{\frac{1}{p}}. \qquad (6.1)$$

If $p = +\infty$, set $M_\infty(f, r)$ to be defined as

$$M_\infty(f, r) = \sup_{|q|<r} |f(q)|,$$

with the convention $M_\infty(f, 0) = |f(0)|$.

**Proposition 6.1.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function. Then for any $p \in (0, +\infty)$ and for any imaginary unit $I \in \mathbb{S}$, the function $r \mapsto M_p(f_I, r)$ is increasing on $[0, 1)$. For $p = +\infty$, the function $r \mapsto M_\infty(f, r)$ is increasing on $[0, 1)$ as well.*

*Proof.* The last part of the statement follows directly from the Maximum Modulus Principle 1.46. To prove the first part, fix $p \in (0, +\infty)$ and $I \in \mathbb{S}$. We will prove that the function

$$g : \mathbb{B}_I \to \mathbb{R}, \quad g : z \mapsto |f_I(z)|^p,$$

is a subharmonic function, and then the statement will follow from classical results, see for instance Theorem 17.5 in [46]. In order to obtain the subharmonicity of $g(z)$, we will first show that

$$h : \mathbb{B}_I \to \mathbb{R}, \quad h : z \mapsto \log\left(|f_I(z)|\right) \tag{6.2}$$

is a subharmonic function. Then, since $g(z) = e^{ph(z)}$ is the composition of an increasing convex function with a subharmonic function, we will conclude that also $g$ is subharmonic (see for instance Theorem 17.2 in [46]).

Suppose $f \not\equiv 0$ (otherwise the statement is trivially true). Then $h(z)$ is an upper semicontinuous function. Moreover in the set $\{z \in \mathbb{B}_I \mid f_I(z) \neq 0\}$ the function $h$ is subharmonic. In fact we have that

$$\begin{aligned}\Delta h(z) &= 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\log(|f_I(z)|) = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\frac{1}{2}\log\left(|f_I(z)|^2\right)\\ &= 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\log\left(|F(z)|^2 + |G(z)|^2\right)\end{aligned}$$

where $F, G$ are the splitting (holomorphic) functions of $f_I$ with respect to $J \in \mathbb{S}$, $J$ orthogonal to $I$. Hence (omitting the variable $z$)

$$\begin{aligned}\Delta h &= 2\frac{\partial}{\partial \bar{z}}\left(\frac{F'\overline{F} + G'\overline{G}}{|F|^2 + |G|^2}\right)\\ &= 2\frac{\left(|F'|^2 + |G'|^2\right)\left(|F|^2 + |G|^2\right) - \left(F'\overline{F} + G'\overline{G}\right)\left(F\overline{F'} + G\overline{G'}\right)}{\left(|F|^2 + |G|^2\right)^2}.\end{aligned} \tag{6.3}$$

If we denote by $\langle\, , \,\rangle$ the standard Hermitian product in $(L_I)^2 \simeq \mathbb{C}^2$, the numerator of the last term in equation (6.3) can be written in vector notation as

$$\begin{aligned}&|(F', G')|^2|(F, G)|^2 - \langle (F', G'), (F, G)\rangle\langle (F, G), (F', G')\rangle\\ &= |(F', G')|^2|(F, G)|^2 - |\langle (F', G'), (F, G)\rangle|^2.\end{aligned} \tag{6.4}$$

Schwarz inequality for Hermitian product yields that this quantity is always non-negative, and therefore we have that $h(z)$ is subharmonic where $f_I$ is non vanishing. It remains to show that $h$ is still subharmonic in a neighborhood of each zero of $f_I$. Recall that the zero set of a regular function intersected with a slice $L_I$, is a discrete subset of $L_I$ (see Theorem 1.28). Then for any zero $z_0$ of $f_I$, there exists a neighborhood $U_I \subset \mathbb{B}_I$ where $z_0$ is the only point where $f_I$ vanishes. Hence we have that, for all $r$ such that $z_0 + B_I(0, r) \subset U_I$, the submean property is trivially satisfied

$$-\infty = h(z_0) \leq \frac{1}{2\pi}\int_{\partial B_I(0,r)} h(z_0 + re^{I\theta})d\theta.$$

Since this condition implies the subharmonicity of $h$ near $z_0$, we can conclude the proof. □

**Remark 6.2.** The previous result is the analogue of the first statement of the Hardy convexity Theorem in the complex setting, see Theorem 1.5 in [17].

**Remark 6.3.** We point out that, despite what happens in the complex case, where the function $z \mapsto \log(|g(z)|)$ is actually harmonic for any non vanishing holomorphic function $g$, the function $h$ defined in equation (6.2) is only subharmonic in general. In fact $\Delta h = 0$ if and only if the last term in equation (6.4) equals zero, that is, if and only if the vector $(F(z), G(z)) \in (L_I)^2$, identified by the restriction of the regular function $f_I$, is parallel to the vector $(F'(z), G'(z))$ identified by the derivative of $f_I$. This happens for example if

$$\begin{cases} F'(z) = kF(z) \\ G'(z) = kG(z) \end{cases}$$

for some $k \in L_I$. In this case $F$ and $G$ are exponential (or constant) functions,

$$\begin{cases} F(z) = F(0)e^{kz} \\ G(z) = G(0)e^{kz} \end{cases}$$

Thanks to Proposition 6.1, we can give the following Definition.

**Definition 6.4.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function. If $p \in (0, +\infty)$, for any $I \in \mathbb{S}$, we set*

$$||f_I||_p = \lim_{r \to 1^-} M_p(f_I, r) = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(re^{I\theta}) \right|^p d\theta \right)^{\frac{1}{p}} \big( = \sup_{r \in (0,1)} M_p(f_I, r) \big)$$

*and*

$$||f||_p = \sup_{I \in \mathbb{S}} ||f_I||_p = \sup_{I \in \mathbb{S}} \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(re^{I\theta}) \right|^p d\theta \right)^{\frac{1}{p}}.$$

*If $p = +\infty$, we set*

$$||f||_\infty = \lim_{r \to 1^-} M_\infty(f, r) = \lim_{r \to 1^-} \sup_{|q| < r} |f(q)| = \sup_{q \in \mathbb{B}} |f(q)|.$$

**Remark 6.5.** Notice that $||f||_\infty$ is the uniform norm of $f$ on $\mathbb{B}$. Moreover, if we set for any $I \in \mathbb{S}$

$$||f_I||_\infty = \sup_{z \in \mathbb{B}_I} |f(z)|,$$

then we have

$$||f||_\infty = \sup_{I \in \mathbb{S}} ||f_I||_\infty.$$

For the sake of consistency, in this chapter we will adopt the notation $|| \cdot ||_\infty$ instead of the one introduced at the beginning of Chapter 2. The set where the uniform norm is taken will be $\mathbb{B}_I$ when considering the restriction $f_I$ (or its splitting components), and $\mathbb{B}$ when considering the function $f$.

**Definition 6.6.** *Let $p \in (0, +\infty]$. We define the quaternionic Hardy space $H^p(\mathbb{B})$ as*

$$H^p(\mathbb{B}) = \{f \colon \mathbb{B} \to \mathbb{H} \mid f \textit{ is regular and } \ ||f||_p < +\infty\}.$$

**Remark 6.7.** As in the complex case, for any $p \in (0, +\infty]$, the space $H^p(\mathbb{B})$ is a real vector space. Furthermore if (and only if) $p \geq 1$ the function $||\cdot||_p$ satisfies the triangle inequality, and hence it is a norm on $H^p(\mathbb{B})$. Since this norm endows $H^p(\mathbb{B})$ with the structure of complete metric space, we have that $H^p(\mathbb{B})$ is a Banach space for all $p \geq 1$. Moreover, the same relations of inclusions that hold for complex $H^p$ spaces, hold in the quaternionic setting. Namely, for any $p, q$ such that $0 < p < q \leq +\infty$, we have

$$H^q(\mathbb{B}) \subset H^p(\mathbb{B}).$$

### 6.1.1 A different choice of integral mean

Looking at the definition of complex Hardy spaces, one could wonder why, instead of the integral mean $M_p(f_I, r)$ defined in formula (6.1), taken on a circle, we did not choose an integral mean taken on a 3-dimensional sphere. Namely we could have defined

$$N_p(f, r) = \left(\frac{1}{2\pi^2 r^3} \int_{r\mathbb{S}^3} |f(q)|^p d\sigma_3(r\mathbb{S}^3)\right)^{\frac{1}{p}}, \tag{6.5}$$

where $\sigma_3(r\mathbb{S}^3)$ is the usual hypersurface measure of the 3-dimensional sphere $r\mathbb{S}^3$ and $2\pi^2 r^3$ is the 3-dimensional volume of the 3-dimensional sphere $r\mathbb{S}^3$. With this choice, we could have then taken its supremum

$$N_p(f) = \sup_{0<r<1} N_p(f, r).$$

First of all, it turns out that the class of regular functions $f$ such that $N_p(f)$ is finite does not coincide with the space $H^p(\mathbb{B})$ defined earlier. Indeed the following relation holds true.

**Proposition 6.8.** *The quaternionic Hardy space $H^p(\mathbb{B})$ is strictly contained in the class of regular functions $f$ such that $N_p(f)$ is finite.*

*Proof.* Notice first that we can describe an element $q \in r\mathbb{S}^3$ using the following change of variable

$$\begin{aligned} q &= g_r(\theta_1, \theta_2, \theta_3) \\ &= r\cos\theta_1 + r\sin\theta_1\cos\theta_2 i + r\sin\theta_1\sin\theta_2\cos\theta_3 j + r\sin\theta_1\sin\theta_2\sin\theta_3 k \end{aligned}$$

where $\theta_1 \in (0, 2\pi)$, and $\theta_2, \theta_3 \in (0, \pi)$. The hypersurface element of the 3-sphere in these coordinates is

$$r^3 \sin^2\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\theta_3.$$

Hence, for a regular function $f : \mathbb{B} \to \mathbb{H}$, we get

$$\begin{aligned} N_p(f)^p &= \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{r\mathbb{S}^3} |f(q)|^p d\sigma_3(r\mathbb{S}^3) \\ &= \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} |f(g_r(\theta_1,\theta_2,\theta_3))|^p r^3 \sin^2\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\theta_3. \end{aligned} \tag{6.6}$$

Now notice that, if

$$S_r = \{r\cos\theta_2 i + r\sin\theta_2\cos\theta_3 j + r\sin\theta_2\sin\theta_3 k \,|\, \theta_2,\theta_3 \in (0,\pi)\}$$

is a hemisphere of purely imaginary quaternions with radius $r$, its surface element in the coordinates $(\theta_2,\theta_3)$ is

$$d\sigma_2(S_r) = r^2 \sin\theta_2 d\theta_2 d\theta_3.$$

Hence we can write the last quantity in equation (6.6) as

$$\sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r}\int_0^{2\pi} |f(g_r(\theta_1,\theta_2,\theta_3)|^p r \sin^2\theta_1 d\theta_1 d\sigma_2(S_r).$$

Observe that if we set $I = \cos\theta_2 i + \sin\theta_2\cos\theta_3 j + \sin\theta_2\sin\theta_3 k \in \mathbb{S}$, we can write

$$f(g_r(\theta_1,\theta_2,\theta_3)) = f(r\cos\theta_1 + r\sin\theta_1 I) = f(re^{I\theta_1})$$

that implies

$$N_p(f)^p = \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} \left(\int_0^{2\pi} \left|f(re^{I\theta_1})\right|^p r\sin^2\theta_1 d\theta_1\right) d\sigma_2(S_r).$$

Clearly

$$\int_0^{2\pi} \left|f(re^{I\theta_1})\right|^p r\sin^2\theta_1 d\theta_1 \le \int_0^{2\pi} \left|f(re^{I\theta_1})\right|^p d\theta_1$$

for all $r \in (0,1)$ and for all $I \in \mathbb{S}$. Therefore if $f \in H^p(\mathbb{B})$, according to Definition 6.6, then also $N_p(f)$ is finite.
To prove that the two classes of functions do not coincide, we will make use of an example. Consider the function

$$f(q) = (1-q)^{-*} = (1-q)^{-1}.$$

This function is regular on $\mathbb{B}$ and it has a singularity on the boundary of the unit ball, precisely in $q = 1$. Let us show that $N_2(f)$ is finite while $||f||_2$ is not.

$$N_2(f)^2 = \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} \left(\int_0^{2\pi} \left|\frac{1}{1-re^{I\theta_1}}\right|^2 r\sin^2\theta_1 d\theta_1\right) d\sigma_2(S_r).$$

First, notice that there are no problems of convergence inside the unit disc. In fact, fix $r_0 \in (0,1)$ and set

$$M_0 = \max_{\substack{0\le r\le r_0\\ 0\le\theta_1\le 2\pi}} \left|\frac{1}{1-re^{I\theta_1}}\right| < +\infty.$$

Then we get

$$\begin{aligned}&\sup_{0\le r\le r_0} \frac{1}{2\pi^2 r^3}\int_{S_r}\left(\int_0^{2\pi}\left|\frac{1}{1-re^{I\theta_1}}\right|^2 r\sin^2\theta_1 d\theta_1\right)d\sigma_2(S_r)\\ &\le \sup_{0\le r\le r_0}\frac{1}{2\pi^2r^3}\int_{S_r}\int_0^{2\pi} M_0^2 r\sin^2\theta_1 d\theta_1 d\sigma_2(S_r) = \sup_{0\le r\le r_0}\frac{M_0^2}{2\pi^2r^3}\int_{r\mathbb{S}^3} d\sigma_3(r\mathbb{S}^3) = M_0^2.\end{aligned}$$

To study the behavior near the boundary, let us concentrate on the interior integral.

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{1}{1-re^{I\theta_1}}\right|^2 r\sin^2\theta_1 d\theta_1\\ &=\int_0^{2\pi}\frac{r\sin^2\theta_1}{(1-r\cos\theta_1-r\sin\theta_1 I)(1-r\cos\theta_1+r\sin\theta_1 I)}d\theta_1\\ &=\int_0^{2\pi}\frac{r\sin^2\theta_1}{1-2r\cos\theta_1+r^2}d\theta_1.\end{aligned}$$

Since $r^2+1\ge 2r$, we get that

$$\begin{aligned}\int_0^{2\pi}\frac{r\sin^2\theta_1}{1-2r\cos\theta_1+r^2}d\theta_1 &\le \int_0^{2\pi}\frac{r\sin^2\theta_1}{2r-2r\cos\theta_1}d\theta_1 = \int_0^{2\pi}\frac{\sin^2\theta_1}{2(1-\cos\theta_1)}d\theta_1\\ =\frac{1}{2}\int_0^{2\pi} 1+\cos\theta_1 d\theta_1 = \pi.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{r_0\le r<1}\frac{1}{2\pi^2r^3}\int_{S_r}\left(\int_0^{2\pi}\left|\frac{1}{1-re^{I\theta_1}}\right|^2 r\sin^2\theta_1 d\theta_1\right)d\sigma_2(S_r)\le \sup_{r_0\le r<1}\frac{1}{2\pi^2r^3}\int_{S_r}\pi d\sigma_2(S_r)\\ &=\sup_{r_0\le r<1}\frac{1}{2\pi r^3}\sigma_2(S_r) = \sup_{r_0\le r<1}\frac{2\pi r^2}{2\pi r^3} = \frac{1}{r_0} < +\infty,\end{aligned}$$

and hence $N_2(f)$ is finite. We point out that the proof could be easily adapted to show that $N_p(f)$ is finite for any $p\ge 2$. On the other hand

$$\begin{aligned}||f||_2^2 &= \sup_{I\in\mathbb{S}}\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{1-re^{I\theta}}\right|^2 d\theta = \sup_{I\in\mathbb{S}}\lim_{r\to1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-2r\cos\theta+r^2}d\theta\\ &=\lim_{r\to1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-2r\cos\theta+r^2}d\theta.\end{aligned}$$

Notice that the integrand function can be written as

$$\frac{1}{1-r^2}P_r(\theta)$$

where $P_r(\theta)$ denotes the classical Poisson kernel. Recalling that (see Chapter 3 of [39])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1 \quad \text{for any } 0 \leq r < 1,$$

we get that

$$\lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-2r\cos\theta + r^2}d\theta = \lim_{r\to 1^-}\frac{1}{1-r^2} = +\infty$$

and hence that $f$ does not belong to $H^2(\mathbb{B})$. Thanks to the relations of inclusion of $H^p$ spaces (see Remark 6.7), $f$ does not belong to $H^p(\mathbb{B})$, for any $p \geq 2$ either. □

The previous example suggests us that a definition of a quaternionic $H^p$ space that relies upon the $N_p$ norm is not convenient. In fact, the function $(1-q)^{-1}$ has real coefficients, and therefore, its restriction to $\mathbb{B}_I = \mathbb{B} \cap L_I$ is the holomorphic function

$$f_I(z) = (1-z)^{-1}.$$

It is well known that $f_I(z)$ does not belong to $H^p(\mathbb{B}_I)$, for all $p \geq 2$, hence it would be weird if its regular extension were in the quaternionic $H^p$ space. Furthermore, the biggest advantage of choosing to define the quaternionic $H^p$ spaces in terms of integral means taken on circles $\partial\mathbb{B}_I$, is that we can work slicewise. We will see soon that this allows us to use classical results in the theory of complex $H^p$ spaces, that apply to the splitting components of any regular function $f$. Let us begin by studying how $||f||_p$ is related with its value on a slice.

**Proposition 6.9.** *A regular function $f$ is in $H^p(\mathbb{B})$ for some $p \in (0, +\infty]$, if and only if there exists $I \in \mathbb{S}$ such that $||f_I||_p$ is bounded.*

*Proof.* Let $p \in (0, +\infty)$ and suppose first that $||f||_p < +\infty$. Then, for any $I \in \mathbb{S}$,

$$||f_I||_p \leq \sup_{I\in\mathbb{S}} ||f_I||_p = ||f||_p < +\infty.$$

To see the other implication, let $J \in \mathbb{S}$ be such that $||f_J||_p$ is finite, and use the Representation Formula 1.10 to write

$$f(re^{I\theta}) = \frac{1}{2}\big(f(re^{J\theta}) + f(re^{-J\theta})\big) + \frac{IJ}{2}\big(f(re^{-J\theta}) - f(re^{J\theta})\big)$$

for any $I \in \mathbb{S}, r \in [0,1)$. Hence

$$\begin{aligned}
||f||_p^p &= \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{I\theta})\right|^p d\theta \\
&= \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{2}\big(f(re^{J\theta})+f(re^{-J\theta})\big)+\frac{IJ}{2}\big(f(re^{-J\theta})-f(re^{J\theta})\big)\right|^p d\theta \\
&\leq \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big(\left|f(re^{J\theta})\right|+\left|f(re^{-J\theta})\right|+\left|f(re^{-J\theta})\right|+\left|f(re^{J\theta})\right|\Big)^p d\theta \\
&= \lim_{r\to 1^-} \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big(2\left|f(re^{J\theta})\right|+2\left|f(re^{-J\theta})\right|\Big)^p d\theta
\end{aligned} \tag{6.7}$$

where last integral does not depend on $I \in \mathbb{S}$. If $p \geq 1$, taking into account the convexity of the map $x \mapsto x^p$, we get that the last term in inequality (6.7) is bounded by

$$\begin{aligned}
&\lim_{r\to 1^-} \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big(2\left|f(re^{J\theta})\right|+2\left|f(re^{-J\theta})\right|\Big)^p d\theta \\
&\leq \lim_{r\to 1^-} \frac{2^p 2^{p-1}}{4\pi}\int_{-\pi}^{\pi} \Big(\left|f(re^{J\theta})\right|^p+\left|f(re^{-J\theta})\right|^p\Big) d\theta \\
&= \lim_{r\to 1^-} \frac{2^{2p}}{4\pi}\int_{-\pi}^{\pi} \left|f(re^{J\theta})\right|^p d\theta = 2^{2p-1}||f_J||_p^p < +\infty.
\end{aligned}$$

On the other hand, if $0 < p < 1$, taking into account the subadditivity on the positive real axis of the map $x \mapsto x^p$ (it is concave and maps 0 to 0), we can bound the last quantity in inequality (6.7) as

$$\begin{aligned}
&\lim_{r\to 1^-} \frac{1}{4\pi}\int_{-\pi}^{\pi} \Big(2\left|f(re^{J\theta})\right|+2\left|f(re^{-J\theta})\right|\Big)^p d\theta \\
&\leq \lim_{r\to 1^-} \frac{2^p}{4\pi}\int_{-\pi}^{\pi} \Big(\left|f(re^{J\theta})\right|^p+\left|f(re^{-J\theta})\right|^p\Big) d\theta \\
&= \lim_{r\to 1^-} \frac{2^p}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{J\theta})\right|^p d\theta \\
&= 2^p||f_J||_p^p < +\infty.
\end{aligned}$$

Let $p = +\infty$. On one side, if $f$ is bounded, then for any $I \in \mathbb{S}$

$$||f_I||_\infty = \sup_{z\in\mathbb{B}_I} |f_I(z)| \leq \sup_{q\in\mathbb{B}} |f(q)| = ||f||_\infty < +\infty.$$

On the other side, thanks to Proposition 1.12, if $J \in \mathbb{S}$ is such that $||f_J||_\infty < +\infty$ we have that

$$||f||_\infty \leq 2||f_J||_\infty < +\infty.$$

□

**Remark 6.10.** In particular, we get that if $||f_J||_p < +\infty$ for some $J \in \mathbb{S}$, we have the following inequalities

$$\begin{aligned} ||f_J||_p \leq ||f||_p \leq 2||f_J||_p, &\quad \text{if } p \in (0,1), \\ ||f_J||_p \leq ||f||_p \leq 2^{\frac{2p-1}{p}}||f_J||_p, &\quad \text{if } p \in [1,+\infty), \end{aligned}$$

and, if $p = +\infty$,

$$||f_J||_\infty \leq ||f||_\infty \leq 2||f_J||_\infty.$$

The key fact that allows us to apply classical results to the splitting components of a function in $H^p(\mathbb{B})$ is the following.

**Proposition 6.11.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0,+\infty]$. Then for any $I \in \mathbb{S}$, if the splitting of $f$ on $L_I$ with respect to $J \in \mathbb{S}$, $J \perp I$, is $f_I(z) = F(z) + G(z)J$, then the holomorphic functions $F$ and $G$ are both in $H^p(\mathbb{B}_I)$.*

*Proof.* Let $I \in \mathbb{S}$ and consider, for any $z \in \mathbb{B}_I$, the splitting $f_I(z) = F(z) + G(z)J$. Then, for any $z \in \mathbb{B}_I$,

$$|f_I(z)| = \sqrt{|F(z)|^2 + |G(z)|^2} \geq |F(z)| \tag{6.8}$$

and

$$|f_I(z)| = \sqrt{|F(z)|^2 + |G(z)|^2} \geq |G(z)|. \tag{6.9}$$

Hence, for $p \in (0,+\infty)$,

$$\begin{aligned} +\infty > ||f||_p^p = \sup_{J\in\mathbb{S}} ||f_J||_p^p \geq ||f_I||_p^p = \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f_I(re^{I\theta})\right|^p d\theta \\ \geq \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{I\theta})\right|^p d\theta = ||F||_p^p \end{aligned}$$

and analogously

$$+\infty > ||f||_p^p \geq ||f_I||_p^p \geq ||G||_p^p.$$

For $p = +\infty$, equations (6.8) and (6.9) directly imply that

$$||F||_\infty \leq ||f_I||_\infty \leq ||f||_\infty < +\infty,$$

and

$$||G||_\infty \leq ||f_I||_\infty \leq ||f||_\infty < +\infty.$$

□

The natural guess that if a function is in $H^p(\mathbb{B})$ then its regular conjugate is in $H^p(\mathbb{B})$ as well, is in fact true.

**Proposition 6.12.** *Let $p \in (0,+\infty]$ and let $f \in H^p(\mathbb{B})$. Then also the regular conjugate $f^c$ belongs to $H^p(\mathbb{B})$.*

*Proof.* If $p = +\infty$ the proof follows directly by Corollary 2.4. Consider then $p \in (0, +\infty)$. For any $I \in \mathbb{S}$, if $f$ splits on $\mathbb{B}_I$ as

$$f(z) = F(z) + G(z)J,$$

where $J \in \mathbb{S}$ is orthogonal to $I$ and $F, G$ are holomorphic functions, then, recalling Definition 1.21, we get that on the same slice $f^c$ can be written as

$$f_I^c(z) = \overline{F(\overline{z})} - G(z)J.$$

Therefore, for any $r \in [0, 1)$,

$$\begin{aligned} M_p(f_I^c, r)^p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f_I^c(re^{I\theta})\right|^p d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\overline{F(re^{-I\theta})} - G(re^{I\theta})J\right|^p d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\left|\overline{F(re^{-I\theta})}\right|^2 + \left|G(re^{I\theta})\right|^2\right)^{\frac{p}{2}} d\theta. \end{aligned}$$

If $0 < p < 2$, thanks to the subadditivity on the positive real axis of the map $x \mapsto x^{p/2}$, we get

$$\begin{aligned} M_p(f_I^c, r)^p &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\left(\left|\overline{F(re^{-I\theta})}\right|^2\right)^{\frac{p}{2}} + \left(\left|G(re^{I\theta})\right|^2\right)^{\frac{p}{2}}\right) d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\left|\overline{F(re^{-I\theta})}\right|^p + \left|G(re^{I\theta})\right|^p\right) d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\overline{F(re^{-I\theta})}\right|^p d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G(re^{I\theta})\right|^p d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{I\theta})\right|^p d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G(re^{I\theta})\right|^p d\theta \\ &= M_p(F, r)^p + M_p(G, r)^p. \end{aligned}$$

Proposition 6.11 yields that both $F$ and $G$ belong to $H^p(\mathbb{B}_I)$, hence

$$\begin{aligned} ||f_I^c||_p^p &= \lim_{r \to 1^-} M_p(f_I^c, r)^p \leq \lim_{r \to 1^-} \left(M_p(F, r)^p + M_p(G, r)^p\right) \\ &= ||F||_p^p + ||G||_p^p < +\infty \end{aligned}$$

and therefore we obtain that $f^c(q) \in H^p(\mathbb{B})$.
If $2 \leq p < +\infty$, thanks to the convexity of the map $x \mapsto x^{p/2}$, we can bound $M_p(f_I^c, r)$ as follows

$$\begin{aligned} M_p(f_I^c, r)^p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\left|\overline{F(re^{-I\theta})}\right|^2 + \left|G(re^{I\theta})\right|^2\right)^{\frac{p}{2}} d\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} 2^{\frac{p}{2}-1}\left(\left(\left|\overline{F(re^{-I\theta})}\right|^2\right)^{\frac{p}{2}} + \left(\left|G(re^{I\theta})\right|^2\right)^{\frac{p}{2}}\right) d\theta \\ &= 2^{\frac{p}{2}-1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\overline{F(re^{-I\theta})}\right|^p d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G(re^{I\theta})\right|^p d\theta\right) \\ &= 2^{\frac{p}{2}-1}\left(M_p(F, r)^p + M_p(G, r)^p\right). \end{aligned}$$

Hence, as before,

$$\begin{aligned}||f_I^c||_p^p &= \lim_{r\to 1^-} M_p(f_I^c, r)^p \leq 2^{\frac{p}{2}-1} \lim_{r\to 1^-} \left(M_p(F,r)^p + M_p(G,r)^p\right)\\ &= 2^{\frac{p}{2}-1}\left(||F||_p^p + ||G||_p^p\right) < +\infty.\end{aligned}$$

We can now conclude that, for any $p \in (0,+\infty]$, if $f \in H^p(\mathbb{B})$, then also $f^c$ does. □

For the symmetrization of a function in $H^p(\mathbb{B})$ the following result holds true.

**Proposition 6.13.** *For any $p \in (0,+\infty)$, if $f \in H^p(\mathbb{B})$, then the symmetrization $f^s \in H^{\frac{p}{2}}(\mathbb{B})$. Moreover if $f \in H^\infty(\mathbb{B})$ then also $f^s$ does.*

*Proof.* Let $f \in H^p(\mathbb{B})$ for some $0 < p < +\infty$. For any $r \in [0,1), I \in \mathbb{S}$ (such that $f(re^{I\theta}) \neq 0$) we have

$$\left|f^s(re^{I\theta})\right| = \left|f * f^c(re^{I\theta})\right| = \left|f(re^{I\theta})f^c(f(re^{I\theta})^{-1}re^{I\theta}f(re^{I\theta}))\right| = \left|f(re^{I\theta})f^c(re^{J\theta})\right|$$

for some $J \in \mathbb{S}$. Hence, recalling Proposition 2.3,

$$\left|f^s(re^{I\theta})\right| \leq \sup_{I\in\mathbb{S}}\left|f(re^{I\theta})\right| \sup_{I\in\mathbb{S}}\left|f^c(re^{I\theta})\right| = \left(\sup_{I\in\mathbb{S}}\left|f(re^{I\theta})\right|\right)^2 = \left|f(re^{K\theta})\right|^2 \tag{6.10}$$

for some $K \in \mathbb{S}$ (since $\mathbb{S}$ is compact). for such a $K$ inequality (6.10) implies

$$\begin{aligned}+\infty > ||f||_p^p \geq ||f_K||_p^p &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{K\theta})\right|^p d\theta\\ &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f^s(re^{I\theta})\right|^{\frac{p}{2}} d\theta = M_{\frac{p}{2}}(f_I^s, r)^{\frac{p}{2}}.\end{aligned}$$

Taking the limit for $r \to 1^-$, and the supremum in $I \in \mathbb{S}$, we get that

$$+\infty > ||f||_p^p \geq \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} M_{\frac{p}{2}}(f_I^s, r)^{\frac{p}{2}} = ||f^s||_{p/2}^{p/2},$$

and hence that $f^s \in H^{\frac{p}{2}}(\mathbb{B})$. If instead $f \in H^\infty(\mathbb{B})$, observe that equation (6.10) directly implies that, for any $q \in \mathbb{B}$,

$$|f^s(q)| \leq \sup_{q\in\mathbb{B}} |f(q)|^2 = ||f||_\infty^2.$$

Hence

$$||f^s||_\infty = \sup_{q\in\mathbb{B}} |f^s(q)| \leq ||f||_\infty^2 < +\infty.$$

□

### 6.1.2 Boundary values

A very important result, that in the classic case is quite laborious to reach, is that functions in $H^p(\mathbb{B})$ have radial limit along almost any ray.

**Proposition 6.14.** *Let $f : \mathbb{B} \to \mathbb{H}$ be a regular function such that $||f||_p$ is finite for some $p \in (0, +\infty]$. Then for any $I \in \mathbb{S}$, the limit*

$$\lim_{r \to 1^-} f(re^{I\theta}) = \tilde{f}(e^{I\theta})$$

*exists for almost every $\theta \in [-\pi, \pi)$.*

*Proof.* Thanks to Proposition 6.11, if $||f||_p$ is finite, then for any $I \in \mathbb{S}$, if $f$ splits on $L_I$ as $f_I(z) = F(z) + G(z)J$, then the holomorphic functions $F$ and $G$ are both in $H^p(\mathbb{B}_I)$. Classical results in the theory of $H^p$ spaces (see e.g. [39]) yield that the radial limits

$$\lim_{r \to 1^-} F(re^{I\theta}) = \widetilde{F}(e^{I\theta}) \quad \text{and} \quad \lim_{r \to 1^-} G(re^{I\phi}) = \widetilde{G}(e^{I\phi})$$

exist respectively for almost every $\theta$ and for almost every $\phi$. Clearly, the set

$$\{\theta \in [-\pi, \pi) \mid \widetilde{F}(e^{I\theta}) \text{ is not defined}\} \cup \{\phi \in [-\pi, \pi) \mid \widetilde{G}(e^{I\phi}) \text{ is not defined}\}$$

has measure zero, hence we can conclude that the radial limit

$$\lim_{r \to 1^-} f(re^{I\theta}) = \lim_{r \to 1^-} (F(re^{I\theta}) + G(re^{I\theta})J) = \widetilde{F}(re^{I\theta}) + \widetilde{G}(re^{I\theta})J = \tilde{f}(e^{I\theta})$$

exists for almost every $\theta \in [-\pi, \pi)$. □

**Remark 6.15.** Notice that in the quaternionic setting, we have more than radial limit along almost any direction, because on *each* slice, the radial limit exists along almost any ray.

**Remark 6.16.** From now on, we will denote by $\tilde{f}$ the radial limit of a function $f$, and, when it exists almost everywhere, we will consider it as an actual function defined on $\partial\mathbb{B}$. We point out that radial limits of regular functions are measurable functions.

As in the complex case, radial limits of functions in $H^p(\mathbb{B})$ cannot vanish on a subset of positive measure of the boundary of the ball. Indeed in the quaternionic case the statement is slightly stronger: on each slice the radial limit is non-zero along almost every ray.

**Proposition 6.17.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$, $f \not\equiv 0$. Then, for any $I \in \mathbb{S}$, for almost every $\theta \in [-\pi, \pi)$,*

$$\lim_{r \to 1^-} f(re^{I\theta}) = \tilde{f}(e^{I\theta}) \neq 0.$$

*Proof.* Let $I \in \mathbb{S}$ and let $J \in \mathbb{S}$, $J$ orthogonal to $I$. If $f$ splits on $\mathbb{B}_I$ as

$$f_I(z) = F(z) + G(z)J,$$

then the splitting components $F$ and $G$ are in $H^p(\mathbb{B}_I)$ and, thanks to the Identity Principle 1.8, at least one of them is not identically vanishing. Suppose that $F \not\equiv 0$ on $\mathbb{B}_I$. Theorem 17.18 in [46] yields that

$$\lim_{r \to 1^-} F(re^{I\theta}) = \widetilde{F}(e^{I\theta}) \neq 0 \quad \text{for almost every } \theta \in [-\pi, \pi).$$

Thanks to the orthogonality of $I$ and $J$, we easily conclude that for almost every $\theta \in [-\pi, \pi)$,

$$\tilde{f}(e^{I\theta}) = \widetilde{F}(e^{I\theta}) + \widetilde{G}(e^{I\theta})J \neq 0.$$

□

This easy consequence of the previous result will be used in the sequel.

**Remark 6.18.** If $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$, $f \not\equiv 0$, we have that for any $I \in \mathbb{S}$ and for almost every $\theta \in [-\pi, \pi)$, there exists $r_0 > 0$ such that $f(re^{I\theta}) \neq 0$ for all $r \in [r_0, 1)$.

In order to define the $*$-product and the $*$-inverse for radial limits, we begin by proving the following statement.

**Proposition 6.19.** *Let $f \in H^p(\mathbb{B})$, $f \not\equiv 0$, and $g \in H^q(\mathbb{B})$ for some $p, q \in (0, +\infty]$, and let $\tilde{f}$ and $\tilde{g}$ be their (almost everywhere) radial limits. For any $I \in \mathbb{S}$, for almost every $\theta \in [-\pi, \pi)$,*

$$\lim_{r \to 1^-} f * g(re^{I\theta}) = \tilde{f}(e^{I\theta})\tilde{g}(\tilde{f}(e^{I\theta})^{-1} e^{I\theta} \tilde{f}(e^{I\theta}))$$

*and*

$$\lim_{r \to 1^-} f^{-*} * g(re^{I\theta}) = \tilde{f}(\tilde{f}^c(e^{I\theta})^{-1} e^{I\theta} \tilde{f}^c(e^{I\theta}))^{-1} \tilde{g}(\tilde{f}^c(e^{I\theta})^{-1} e^{I\theta} \tilde{f}^c(e^{I\theta})).$$

*Proof.* Proposition 6.17 yields that if $f$ is not vanishing identically, then for any $I \in \mathbb{S}$, $\tilde{f}(e^{I\theta}) \neq 0$ for almost every $\theta \in [-\pi, \pi)$, and the same holds for $f^c$. Set

$$T(q) = f(q)^{-1} q f(q).$$

Thanks to Remark 6.18, $T(re^{I\theta}) = f(re^{I\theta})^{-1} re^{I\theta} f(re^{I\theta})$ is well defined for any $I \in \mathbb{S}$, for almost every $\theta \in [-\pi, \pi)$, and for $1 - r$ sufficiently small. Moreover

$$\lim_{r \to 1^-} T(re^{I\theta}) = \tilde{f}(e^{I\theta})^{-1} e^{I\theta} \tilde{f}(e^{I\theta}) = \widetilde{T}(e^{I\theta}).$$

Given any $I \in \mathbb{S}$, set $J(r, \theta) = f(re^{I\theta})^{-1} I f(re^{I\theta}) \in \mathbb{S}$; then, for almost every $\theta$ the radial limit

$$\lim_{r \to 1^-} J(r, \theta) = \tilde{J}(\theta)$$

exists and belongs to $\mathbb{S}$. Hence we can write

$$\lim_{r\to 1^-} T(re^{I\theta}) = \lim_{r\to 1^-} \big(r\cos\theta + r\sin\theta(f(re^{I\theta})^{-1}If(re^{I\theta})\big) = \cos\theta + (\sin\theta)\tilde{J}(\theta).$$

Using the Representation Formula 1.10 twice, for almost every $\theta$ we can write

$$\begin{aligned}
\lim_{r\to 1^-} g(T(re^{I\theta})) &= \lim_{r\to 1^-} (g(r\cos\theta + r\sin\theta J(r,\theta))) \\
&= \lim_{r\to 1^-} \left(\frac{1}{2}\big(g(re^{I\theta}) + g(re^{-I\theta})\big) + \frac{J(r,\theta)I}{2}\big(g(re^{-I\theta}) - g(re^{I\theta})\big)\right) \\
&= \frac{1}{2}\big(\tilde{g}(e^{I\theta}) + \tilde{g}(e^{-I\theta})\big) + \frac{\tilde{J}(\theta)I}{2}\big(\tilde{g}(e^{-I\theta}) - \tilde{g}(e^{I\theta})\big) \\
&= \lim_{r\to 1^-} \left(\frac{1}{2}\big(g(re^{I\theta}) + g(re^{-I\theta})\big) + \frac{\tilde{J}(\theta)I}{2}\big(g(re^{-I\theta}) - g(re^{I\theta})\big)\right) \\
&= \lim_{r\to 1^-} g(re^{\tilde{J}(\theta)\theta}) = \tilde{g}(e^{\tilde{J}(\theta)\theta}) = \tilde{g}(\widetilde{T}(e^{I\theta}))
\end{aligned}$$

Hence, recalling Proposition 1.20, we have

$$\lim_{r\to 1^-} f * g(re^{I\theta}) = \lim_{r\to 1^-} f(re^{I\theta})g(T(re^{I\theta})) = \tilde{f}(e^{I\theta})\tilde{g}(\widetilde{T}(e^{I\theta})).$$

The same arguments apply also to the proof for the regular quotient. □

We are now ready to give the announced definitions.

**Definition 6.20.** *Let $f \in H^p(\mathbb{B})$ and $g \in H^q(\mathbb{B})$ for some $p, q \in (0, +\infty]$. For any $I \in \mathbb{S}$, for almost every $\theta \in [-\pi, \pi)$, let $\tilde{f}(e^{I\theta})$ and $\tilde{g}(e^{I\theta})$ be the radial limits of $f$ and $g$. We define the $*$-product of $\tilde{f}$ and $\tilde{g}$ as*

$$\tilde{f} * \tilde{g}(e^{I\theta}) = \lim_{r\to 1^-} f * g(re^{I\theta})$$

*for almost every $\theta$. If moreover $f \not\equiv 0$, we define the $*$-quotient of $\tilde{f}$ and $\tilde{g}$ as*

$$\tilde{f}^{-*} * \tilde{g}(e^{I\theta}) = \lim_{r\to 1^-} f^{-*} * g(re^{I\theta})$$

*for almost every $\theta$. In particular, if $g \equiv 1$, we obtain the definition of the $*$-inverse of $\tilde{f}$.*

Thanks to the existence of the radial limit it is possible to obtain the basic integral representations.

**Theorem 6.21.** *Let $f \in H^1(\mathbb{B})$. Then, for any $I \in \mathbb{S}$, $f_I$ is the Poisson integral and the Cauchy integral of its radial limit $\tilde{f}_I$, i.e.,*

$$f_I(re^{I\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\tilde{f}_I(e^{It})dt$$

*and*

$$f_I(z) = \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \frac{d\zeta}{\zeta - z}\tilde{f}_I(\zeta)$$

*for any* $r \in [0,1)$, $\theta \in [-\pi,\pi)$, *and* $z \in \mathbb{B}_I$.

*Proof.* Since $f \in H^1(\mathbb{B})$, Proposition 6.11 yields that, for any $I \in \mathbb{S}$, if $f$ splits on $\mathbb{B}_I$ with respect to $J$, as

$$f_I(z) = F(z) + G(z)J,$$

the splitting components $F$ and $G$ are in $H^1(\mathbb{B}_I)$. Clearly their radial limits are related (for almost every $\theta$) by the equation

$$\tilde{f}_I(e^{I\theta}) = \widetilde{F}(e^{I\theta}) + \widetilde{G}(e^{I\theta})J.$$

Thanks to classical results in this setting, (see e.g. [46], Corollary to Theorem 17.12) we have that both $F$ and $G$ are the Poisson and the Cauchy integrals of their radial limits $\widetilde{F}$ and $\widetilde{G}$. Hence, for the Cauchy integral,

$$\begin{aligned} f_I(z) = F(z) + G(z)J &= \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \frac{\widetilde{F}(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \frac{\widetilde{G}(\zeta)}{\zeta - z}d\zeta J \\ &= \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \frac{d\zeta}{\zeta - z}\big(\widetilde{F}(\zeta) + \widetilde{G}(\zeta)J\big) = \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \frac{d\zeta}{\zeta - z}\tilde{f}_I(\zeta). \end{aligned}$$

Analogously for the Poisson integral, if $z = re^{I\theta}$ for some $r \in (0,1), \theta \in [-\pi,\pi)$, and if $P_r(\theta)$ is the classical Poisson kernel (see e.g. [39], Chapter 3),

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2},$$

then

$$\begin{aligned} f_I(re^{I\theta}) &= F(re^{I\theta}) + G(re^{I\theta})J \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\widetilde{F}(e^{It})dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\widetilde{G}(e^{It})dtJ \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\big(\widetilde{F}(e^{It}) + \widetilde{G}(e^{It})J\big)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\tilde{f}_I(e^{It})dt. \end{aligned}$$

□

Thanks to the relations of inclusion of $H^p$ spaces (see Remark 6.7) we obtain that the same result holds for all $p \in [1,+\infty]$.

**Corollary 6.22.** *For any* $p \in [1,+\infty]$, *if* $f \in H^p(\mathbb{B})$, *then for any* $I \in \mathbb{S}$, $f_I$ *is both the Cauchy integral and the Poisson integral of* $\tilde{f}_I$.

Our next goal is to show that, for any $p \in (0, +\infty]$, the radial limits $\tilde{f}_I$ of the restrictions of a function $f$ in $H^p(\mathbb{B})$, are $L^p$ functions on the circle $\partial\mathbb{B}_I$.

**Definition 6.23.** *Let $g$ be a (quaternion valued) function defined (almost everywhere) on $\partial\mathbb{B}_I$, such that $|g|$ is measurable on $\partial\mathbb{B}_I$. If $p \in (0, +\infty)$ we set $||g||_{L^p}$ to be the integral mean*

$$||g||_{L^p} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g(e^{I\theta})\right|^p d\theta\right)^{\frac{1}{p}},$$

*If $p = +\infty$, we set*

$$||g||_{L^\infty} = \operatorname{ess}\sup_{\theta\in(-\pi,\pi)} \left|g(e^{I\theta})\right|.$$

*For any $p \in (0, +\infty]$, we denote by $L^p(\partial\mathbb{B}_I)$ the standard $L^p$ space,*

$$L^p(\partial\mathbb{B}_I) = \{g : \partial\mathbb{B}_I \to \mathbb{H} \ \mid \ |g| \text{ is measurable and } \ ||g||_{L^p} < +\infty\}.$$

Recall that, if $f$ is a regular function on $\mathbb{B}$, then for any $I \in \mathbb{S}$ and for any $r \in [0, 1)$ we denote by $(f_I)_r$ the function defined on $\partial\mathbb{B}_I$ as

$$(f_I)_r(e^{I\theta}) = f(re^{I\theta}),$$

and notice that, for any $p \in (0, +\infty)$,

$$||(f_I)_r||_{L^p} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{I\theta})\right|^p d\theta\right)^{\frac{1}{p}} = M_p(f_I, r).$$

**Proposition 6.24.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty)$. Then, for any $I \in \mathbb{S}$, the function $\tilde{f}_I - (f_I)_r$ belongs to $L_p(\partial\mathbb{B}_I)$ and*

$$\lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||_{L^p} = 0.$$

*Proof.* The analogous statement holds true in the classical setting. In fact, if $f_I$ splits with respect to $J \in \mathbb{S}$, $J \perp I$, as

$$f_I(z) = F_I(z) + G_I(z)J,$$

then Theorem 2.6 in [17] yields that for both $F_I$ and $G_I$ (holomorphic functions in $H^p(\mathbb{B}_I)$)

$$\lim_{r\to 1^-} ||\widetilde{F}_I - (F_I)_r||_{L^p} = 0$$

and

$$\lim_{r\to 1^-} ||\widetilde{G}_I - (G_I)_r||_{L^p} = 0.$$

Hence, if $1 \leq p < +\infty$,

$$\begin{aligned}
\lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||_{L^p} &= \lim_{r\to 1^-} ||\widetilde{F}_I + \widetilde{G}_I J - (F_I)_r - (G_I)_r J||_{L^p} \\
&\leq \lim_{r\to 1^-} \left(||\widetilde{F}_I - (F_I)_r||_{L^p} + ||\widetilde{G}_I J - (G_I)_r J||_{L^p}\right) \\
&= \lim_{r\to 1^-} ||\widetilde{F}_I - (F_I)_r||_{L^p} + \lim_{r\to 1^-} ||\widetilde{G}_I - (G_I)_r||_{L^p} = 0.
\end{aligned}$$

If instead $0 < p < 1$, consider

$$\begin{aligned}
\lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||^p_{L^p} &= \lim_{r\to 1^-} ||\widetilde{F}_I + \widetilde{G}_I J - (F_I)_r - (G_I)_r J||^p_{L^p} \\
&= \lim_{r\to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\widetilde{F}_I - (F_I)_r + (\widetilde{G}_I - (G_I)_r)J\right|^p d\theta \\
&= \lim_{r\to 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\left|\widetilde{F}_I - (F_I)_r\right|^2 + \left|\widetilde{G}_I - (G_I)_r)\right|^2\right)^{\frac{p}{2}} d\theta
\end{aligned}$$

where last equality is due to the orthogonality of $I$ and $J$. Then, recalling that for any $0 < p < 2$ the map $x \mapsto x^{p/2}$ is subadditive on the positive real axis, we have

$$\begin{aligned}
\lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||^p_{L^p} &\leq \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\left|\widetilde{F}_I - (F_I)_r\right|^2\right)^{\frac{p}{2}} + \left(\left|\widetilde{G}_I - (G_I)_r)\right|^2\right)^{\frac{p}{2}} d\theta\right) \\
&= \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\widetilde{F}_I - (F_I)_r\right|^p d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\widetilde{G}_I - (G_I)_r)\right|^p d\theta\right) \\
&= \lim_{r\to 1^-} \left(||\widetilde{F}_I - (F_I)_r||^p_{L^p} + ||\widetilde{G}_I - (G_I)_r||^p_{L^p}\right) \\
&= \lim_{r\to 1^-} ||\widetilde{F}_I - (F_I)_r||^p_{L^p} + \lim_{r\to 1^-} ||\widetilde{G}_I - (G_I)_r||^p_{L^p} = 0.
\end{aligned}$$

□

Now we are able to prove the desired result.

**Proposition 6.25.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$. Then, for any $I \in \mathbb{S}$, the function $\tilde{f}_I : \partial\mathbb{B}_I \to \mathbb{H}$ defined, for almost every $\theta \in [-\pi, \pi)$, by*

$$\tilde{f}(e^{I\theta}) = \lim_{r\to 1^-} f(re^{I\theta}),$$

*does belong to $L^p(\partial\mathbb{B}_I)$. Moreover*

$$||\tilde{f}_I||_{L^p} = ||f_I||_p.$$

*Proof.* Let $I \in \mathbb{S}$ and let $\tilde{f}_I$ be the radial limit of $f_I$. Thanks to Proposition 6.24, for any $0 < p < +\infty$,

$$\lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||_{L^p} = 0.$$

If $1 \le p < +\infty$, for any $r \in [0,1)$ we have

$$\big| ||\tilde{f}_I||_{L^p} - ||(f_I)_r||_{L^p} \big| \le ||\tilde{f}_I - (f_I)_r||_{L^p}.$$

Then

$$\lim_{r\to 1^-} \big| ||\tilde{f}_I||_{L^p} - ||(f_I)_r||_{L^p} \big| \le \lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||_{L^p} = 0.$$

Therefore

$$||f_I||_p = \lim_{r\to 1^-} M_p(f_I, r) = \lim_{r\to 1^-} ||(f_I)_r||_{L^p} = ||\tilde{f}_I||_{L^p}.$$

If $0 < p < 1$, since for any $r \in [0,1)$

$$\big| ||\tilde{f}_I||^p_{L^p} - ||(f_I)_r||^p_{L^p} \big| \le ||\tilde{f}_I - (f_I)_r||^p_{L^p},$$

then

$$\lim_{r\to 1^-} \big| ||\tilde{f}_I||^p_{L^p} - ||(f_I)_r||^p_{L^p} \big| \le \lim_{r\to 1^-} ||\tilde{f}_I - (f_I)_r||^p_{L^p} = 0,$$

and hence

$$||f_I||_p = \lim_{r\to 1^-} M_p(f_I, r) = \lim_{r\to 1^-} ||(f_I)_r||_{L^p} = ||\tilde{f}_I||_{L^p}.$$

For the case $p = +\infty$, on one side, for almost every $\theta$,

$$\big|\tilde{f}_I(e^{I\theta})\big| = \big| \lim_{r\to 1^-} f_I(re^{I\theta})\big| = \lim_{r\to 1^-} \big|f_I(re^{I\theta})\big| \le \sup_{0\le r<1} \big|f_I(re^{I\theta})\big|.$$

Hence

$$||\tilde{f}_I||_{L^\infty} = \operatorname{ess} \sup_{\theta\in(-\pi,\pi)} \big|\tilde{f}_I(e^{I\theta})\big| \le \operatorname{ess} \sup_{\theta\in(-\pi,\pi)} \sup_{0\le r<1} \big|f_I(re^{I\theta})\big| = \sup_{z\in\mathbb{B}_I} |f(z)| = ||f_I||_\infty.$$

On the other side, since $f_I$ is the Poisson integral of $\tilde{f}_I$ (see Theorem 6.21), for any $r \in [0,1)$ and for any $\theta \in [-\pi,\pi)$, we have

$$\big|f_I(re^{I\theta})\big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|P_r(\theta - t)\tilde{f}_I(e^{It})\big| dt.$$

Let us recall the properties of the classical Poisson kernel (see for instance Chapter 3 of [39]), namely

$$P_r(\theta) \ge 0 \quad \text{for any } \theta \in [-\pi,\pi) \text{ and for any } 0 \le r < 1$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta) d\theta = 1 \quad \text{for any } 0 \le r < 1.$$

Thanks to these properties, we get that for any $r \in [0,1)$ and for any $\theta \in [-\pi,\pi)$,

$$\big|f_I(re^{I\theta})\big| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\big|\tilde{f}_I(e^{It})\big| dt \le ||\tilde{f}_I||_{L^\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) dt = ||\tilde{f}_I||_{L^\infty}.$$

Therefore we obtain also the opposite inequality

$$||f_I||_\infty = \sup_{\substack{0\le r<1\\ \theta\in(-\pi,\pi)}} \big|f_I(re^{I\theta})\big| \le ||\tilde{f}_I||_{L^\infty},$$

hence concluding the proof. □

## 6.2 An integral representation

Let us present another integral representation that allows us to recover the value of a function, regular up to the closure of the unit ball $\overline{\mathbb{B}} \subset \mathbb{H}$, from its value on the boundary of a disc $\mathbb{B}_I$.
Let $f$ be a function regular on $\overline{\mathbb{B}}$ and let $I$ be an element of $\mathbb{S}$. The Splitting Lemma 1.7 implies that for any $J \in \mathbb{S}$, $J$ orthogonal to $I$, there exist two holomorphic functions $F_I, G_I : \overline{\mathbb{B}}_I \to L_I$, such that, for any $z \in \overline{\mathbb{B}}_I$, $f_I(z) = F_I(z) + G_I(z)J$. Classical results in the complex setting, see [39], yield that the splitting components $F_I$ and $G_I$ can be represented by means of the holomorphic kernel $H_\theta$ defined, for all $z \in \mathbb{B}_I$, by

$$H_\theta(z) = \frac{e^{I\theta} + z}{e^{I\theta} - z},$$

where $\theta \in [-\pi, \pi)$. We recall that the kernel $H_\theta(z)$, if $z = re^{It}$, is such that

$$\mathrm{Re}\left(H_\theta(re^{It})\right) = \mathrm{Re}\left(\frac{1 + re^{I(t-\theta)}}{1 - re^{I(t-\theta)}}\right) = \frac{1 - r^2}{1 + r^2 - 2r\cos(t-\theta)}$$

namely such that

$$\mathrm{Re}\left(H_\theta(re^{It})\right) = P_r(t - \theta)$$

where $P_r(t)$ is the classical Poisson kernel (see [39], Chapter 3).

The integral representation that we obtain for $F_I$ and $G_I$ is the following.

$$\begin{aligned} F_I(z) &= \mathrm{Re}\, F_I(0) - \overline{F_I(0)} + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_\theta(z)\,\mathrm{Re}\, F_I(e^{I\theta})d\theta \\ &= \mathrm{Re}\, F_I(0) - \overline{F_I(0)} + \frac{1}{2\pi I}\int_{-\pi}^{\pi} \frac{e^{I\theta} + z}{e^{I\theta} - z}\,\mathrm{Re}\, F_I(e^{I\theta})d\theta \end{aligned}$$

and

$$\begin{aligned} G_I(z) &= \mathrm{Re}\, G_I(0) - \overline{G_I(0)} + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_\theta(z)\,\mathrm{Re}\, G_I(e^{I\theta})d\theta \\ &= \mathrm{Re}\, G_I(0) - \overline{G_I(0)} + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta} + z}{e^{I\theta} - z}\,\mathrm{Re}\, G_I(e^{I\theta})d\theta. \end{aligned}$$

If we set

$$C_0 = \mathrm{Re}\, F_I(0) + \mathrm{Re}\, G_I(0)J - \left(\overline{F_I(0)} + \overline{G_I(0)}J\right),$$

we can represent $f_I$ in $\mathbb{B}_I$ in the following manner

$$\begin{aligned} f_I(z) &= F_I(z) + G_I(z)J \\ &= C_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta} + z}{e^{I\theta} - z}\,\mathrm{Re}\, F_I(e^{I\theta})d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta} + z}{e^{I\theta} - z}\,\mathrm{Re}\, G_I(e^{I\theta})Jd\theta \\ &= C_0 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta} + z}{e^{I\theta} - z}\left(\mathrm{Re}\, F_I(e^{I\theta}) + \mathrm{Re}\, G_I(e^{I\theta})J\right)d\theta. \end{aligned} \tag{6.11}$$

In order to understand what actually are the constant $C_0$ and the function $z \mapsto \operatorname{Re} F_I(z) + \operatorname{Re} G_I(z)J$ appearing in formula (6.11), consider the splitting of $f$ on $L_J$ with respect to $IJ = K$,

$$f_J(w) = F_J(w) + G_J(w)K,$$

where $w \in \overline{\mathbb{B}}_J$. On the real axis we can express the value of $F_J$ (and of $G_J$) in terms of the splitting functions $F_I, G_I$. In fact since $\overline{\mathbb{B}} \cap \mathbb{R} = \overline{\mathbb{B}}_I \cap \overline{\mathbb{B}}_J$, for any $x \in \overline{\mathbb{B}} \cap \mathbb{R}$ we have

$$f_J(x) = f_I(x) = \operatorname{Re} F_I(x) + \operatorname{Im} F_I(x)I + \operatorname{Re} G_I(x)J + \operatorname{Im} G_I(x)IJ,$$

and hence

$$F_J(x) = \operatorname{Re} F_I(x) + \operatorname{Re} G_I(x)J. \tag{6.12}$$

Therefore we get

$$C_0 = \operatorname{Re} F_I(0) + \operatorname{Re} G_I(0)J - (\overline{F_I(0)} + \overline{G_I(0)}J) = F_J(0) - (\overline{F_I(0)} + \overline{G_I(0)}J).$$

Moreover, equality (6.12) and the Identity Principle 1.8 imply that, if $\operatorname{ext}(F_J)(q)$ denotes as usual the unique regular extension of $F_J$ to $\overline{\mathbb{B}}$,

$$\operatorname{Re} F_I(z) + \operatorname{Re} G_I(z)J = \operatorname{ext}(F_J)(z) \quad \text{for any } z \in \overline{\mathbb{B}}_I.$$

Hence the integral representation of $f_I$ on $\mathbb{B}_I$ is

$$f_I(z) = F_J(0) - (\overline{F_I(0)} + \overline{G_I(0)}J) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta}+z}{e^{I\theta}-z} \operatorname{ext}(F_J)(e^{I\theta})d\theta. \tag{6.13}$$

Formula (6.13) allows us also to represent the function $f_I$ evaluated at $\overline{z} \in \mathbb{B}_I$,

$$f_I(\overline{z}) = F_J(0) - (\overline{F_I(0)} + \overline{G_I(0)}J) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta}+\overline{z}}{e^{I\theta}-\overline{z}} \operatorname{ext}(F_J)(e^{I\theta})d\theta.$$

Knowing the values of $f$ at $z = x + yI$ and at $\overline{z} = x - yI$, using the Representation Formula 1.10, we can retrieve the value of $f$ at $q = x + yL$ for any $L \in \mathbb{S}$,

$$f(q) = \frac{1}{2}\left(f(z) + f(\overline{z})\right) + \frac{LI}{2}\left(f(\overline{z}) - f(z)\right).$$

Thus we can recover the wanted integral representation of $f(q)$, for all $q \in \mathbb{B}$, as

$$\begin{aligned} f(q) &= F_J(0) - (\overline{F_I(0)} + \overline{G_I(0)}J) \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}\left(\frac{e^{I\theta}+z}{e^{I\theta}-z} + \frac{e^{I\theta}+\overline{z}}{e^{I\theta}-\overline{z}}\right) + \frac{LI}{2}\left(\frac{e^{I\theta}+\overline{z}}{e^{I\theta}-\overline{z}} - \frac{e^{I\theta}+z}{e^{I\theta}-z}\right)\right] \operatorname{ext}(F_J)(e^{I\theta})d\theta. \end{aligned} \tag{6.14}$$

Consider now the unique regular extension of the holomorphic kernel $H_\theta$ to the entire ball $\mathbb{B}$, namely the regular kernel

$$\operatorname{ext}(H_\theta)(q) = (e^{I\theta} - q)^{-*} * (e^{I\theta} + q).$$

Notice that the Representation Formula 1.10 applied to the function $\text{ext}(H_\theta)(q)$, yields that

$$\frac{1}{2}\left(\frac{e^{I\theta}+z}{e^{I\theta}-z}+\frac{e^{I\theta}+\overline{z}}{e^{I\theta}-\overline{z}}\right)+\frac{LI}{2}\left(\frac{e^{I\theta}+\overline{z}}{e^{I\theta}-\overline{z}}-\frac{e^{I\theta}+z}{e^{I\theta}-z}\right)=(e^{I\theta}-q)^{-*}*(e^{I\theta}+q),$$

hence, from equation (6.14), we can conclude

$$f(q)=F_J(0)-(\overline{F_I(0)}+\overline{G_I(0)}J)+\frac{1}{2\pi}\int_{-\pi}^{\pi}\left((e^{I\theta}-q)^{-*}*(e^{I\theta}+q)\right)\text{ext}(F_J)(e^{I\theta})d\theta.$$

We point out that we can write the previous result also with an intrinsic notation. For, set $\zeta = e^{I\theta}$, $\zeta \in \partial\mathbb{B}_I$, and consider the corresponding holomorphic kernel $H_\zeta$ defined, for $z \in \mathbb{B}_I$, by

$$H_\zeta(z)=\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}=\frac{\zeta+z}{\zeta-z}.$$

In this new variable the differential changes as follows

$$d\zeta = Ie^{I\theta}d\theta,$$

Hence we can write

$$\begin{aligned}F_I(z)&=\operatorname{Re}F_I(0)-\overline{F_I(0)}+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}H_\zeta(z)\frac{d\zeta}{\zeta}\operatorname{Re}F_I(\zeta)\\&=\operatorname{Re}F_I(0)-\overline{F_I(0)}+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\operatorname{Re}F_I(\zeta)\end{aligned}$$

and

$$\begin{aligned}G_I(z)&=\operatorname{Re}G_I(0)-\overline{G_I(0)}+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}H_\zeta(z)\frac{d\zeta}{\zeta}\operatorname{Re}G_I(\zeta)\\&=\operatorname{Re}G_I(0)-\overline{G_I(0)}+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\operatorname{Re}G_I(\zeta),\end{aligned}$$

obtaining

$$\begin{aligned}f_I(z)&=C_0+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\operatorname{Re}F_I(\zeta)+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\operatorname{Re}G_I(\zeta)J\\&=C_0+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}(\operatorname{Re}F_I(\zeta)+\operatorname{Re}G_I(\zeta)J)\end{aligned}$$

and

$$f_I(\overline{z})=C_0+\frac{1}{2\pi I}\int_{\partial\mathbb{B}_I}\frac{\zeta+\overline{z}}{\zeta-\overline{z}}\frac{d\zeta}{\zeta}(\operatorname{Re}F_I(\zeta)+\operatorname{Re}G_I(\zeta)J).$$

Proceeding as in the parametric version, applying the Representation Formula 1.10, we can write

$$f(q) = F_J(0) - (\overline{F_I(0)} + \overline{G_I(0)}J) + \frac{1}{2\pi I}\int_{\partial\mathbb{B}_I} \big((\zeta - q)^{-*} * (\zeta + q)\big)\frac{d\zeta}{\zeta}\,\mathrm{ext}(F_J)(\zeta),$$

where

$$(\zeta - q)^{-*} * (\zeta + q)$$

is the unique regular extension to the entire ball $\mathbb{B}$ of the holomorphic kernel $H_\zeta$,

$$\mathrm{ext}(H_\zeta)(q) = (\zeta - q)^{-*} * (\zeta + q).$$

## 6.3 Factorization theorems

In the classical setting it is possible to decompose a holomorphic function in $H^p(\mathbb{D})$ into its *inner* and *outer* factors, see Chapter 5 of [39]. The quaternionic counterparts are defined as follows.

**Definition 6.26.** *A regular function $E \in H^1(\mathbb{B})$ is an* outer function *if for any $f \in H^1(\mathbb{B})$ such that $|\widetilde{E}(q)| = |\tilde{f}(q)|$ for almost any $q \in \partial\mathbb{B}$, we have*

$$|E(q)| \geq |f(q)| \quad \textit{for any } q \in \mathbb{B}.$$

An outer function can be then interpreted as a solution of an extremal problem. In the complex setting, the definition of outer function can be given equivalently in terms of a never-vanishing holomorphic function, expressed by means of the integral representation introduced in Section 6.2 (see [39], Chapter 5). This correspondence fails to be true for regular functions, since in general we can not reproduce the same construction.

**Definition 6.27.** *A regular function $\mathcal{I} \in H^\infty(\mathbb{B})$ is an* inner function *if $|\mathcal{I}(q)| \leq 1$ for any $q \in \mathbb{B}$ and $|\widetilde{\mathcal{I}}(q)| = 1$ for almost any $q \in \partial\mathbb{B}$.*

We recall that, in the complex setting, each inner function can be factored into a product of two distinct types of inner functions, namely a *Blaschke product* and a *singular function*. Let us state here how we define the quaternionic analogues of singular functions, and we will study in the next section the analogues of Blaschke products.

**Definition 6.28.** *An inner function $f \in H^\infty(\mathbb{B})$ is a* singular function *if $f$ is non-vanishing on $\mathbb{B}$.*

### 6.3.1 Extraction of the Blaschke product

In the complex setting there are two possible approaches to the factorization of an $H^p$ function (compare [17, 39]). The first one is to begin by the extraction of the outer factor, thus obtaining the inner one. At this point, extracting the zeros one separates the Blaschke product and the singular part. The other possibility is to start with

the extraction of the zeros, thus identifying the Blaschke product, and then separate the outer factor from the singular one. In the quaternionic setting, since we can not reproduce the construction of the outer factor, let us begin with the extraction of the zeros of a function $f$ in $H^p(\mathbb{B})$ for $p \in (0, +\infty]$.
Thanks to the characterization of the zero set of regular functions (see Theorem 1.28), recalling Definition 1.32, we can build a sequence representing the zeros of $f$.

**Definition 6.29.** *Let $f$ be a regular function.The* zero sequence *of $f$ is a sequence $\{a_n\}_{n\in\mathbb{N}}$, contained in the zero set of $f$, composed as follows: the isolated zeros are listed according to their isolated multiplicity; the spherical zeros are represented by any element that generates the 2-sphere of zeros together with its conjugate, listed according to their spherical multiplicity. Namely, if $a_n$ generates a spherical zero (not containing $a_{n-1}$) with spherical multiplicity $2m$, then $a_{n+2k} = a_n$ and $a_{n+2k+1} = \overline{a}_n$ for all $k = 0, \dots, m-1$.*

In analogy with the complex case, we can give the following two definitions (previously introduced in [4]).

**Definition 6.30.** *Let $a$ be a point in $\mathbb{B}$. If $a \neq 0$, the* Blaschke factor *associated with $a$ is the regular Moebius transformation defined as*

$$M_a(q) = (1 - q\overline{a})^{-*} * (a - q)\frac{\overline{a}}{|a|}.$$

*If instead $a = 0$, we set $M_a(q) = q$.*

**Definition 6.31.** *If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of points in $\mathbb{B}$ such that the infinite regular product*

$$B(q) = \prod_{n\geq 0}^{*} M_{a_n}(q)$$

*converges uniformly on compact sets of $\mathbb{B}$, then $B$ is called* Blaschke product*, and it defines a regular function on $\mathbb{B}$ (see Theorem 1.35).*

Blaschke products are examples of inner functions.

**Proposition 6.32.** *Let*

$$B(q) = \prod_{n\geq 0}^{*} M_{a_n}(q)$$

*be a Blaschke product. Then $|B(q)| \leq 1$ for any $q \in \mathbb{B}$ and $|\widetilde{B}(q)| = 1$ for almost any $q \in \partial\mathbb{B}$.*

*Proof.* Consider the finite (regular) products $B_k$,

$$B_k(q) = \prod_{n=0}^{k}{}^{*} M_{a_n}(q) = \prod_{n=0}^{k} M_{a_n}(T_n(q)),$$

where (for any $n = 0, \dots, k$) $T_n$ is defined iteratively (outside the zero set of $B_{n-1}$), in view of Proposition 1.20, by

$$T_n(q) = \left(\prod_{j=0}^{n-1}{}^{*} M_{a_j}(q)\right)^{-1} q \prod_{j=0}^{n-1}{}^{*} M_{a_j}(q) = \left(\prod_{j=0}^{n-1} M_{a_j}(T_j(q))\right)^{-1} q \prod_{j=0}^{n-1} M_{a_j}(T_j(q)).$$

Since $|T_n(q)| = |q| < 1$ for any $q \in \mathbb{B}$ and since each factor $M_{a_n}$ is bounded in modulus by 1 on $\mathbb{B}$ (see Proposition 1.55), we get that we can bound each finite product,

$$|B_k(q)| = \prod_{n=0}^{k} |M_{a_n}(T_n(q))| < 1.$$

Thanks to the uniform convergence on compact sets of the finite products to $B$ we get that

$$|B(q)| \leq 1 \quad \text{for any } q \in \mathbb{B}.$$

Hence $B \in H^\infty(\mathbb{B})$ and therefore for any $I \in \mathbb{S}$, for almost any $\theta \in [-\pi, \pi)$ there exists the radial limit

$$\lim_{r \to 1^-} B(re^{I\theta}) = \widetilde{B}(e^{I\theta}).$$

The same clearly holds true for any finite product $B_k \in H^\infty(\mathbb{B})$. Consider then, the regular functions

$$B_k^{-*} * B(q),$$

for any $k \in \mathbb{N}$. Each of them is a Blaschke product (hence a bounded regular function) and the limit

$$\lim_{k \to \infty} B_k^{-*} * B(q) \equiv 1$$

uniformly on compact sets of $\mathbb{B}$. Let $k \in \mathbb{N}$, and set

$$\tau_k(q) = \big(B_k^c(q)\big)^{-1} q B_k^c(q).$$

Observe that the finite regular product $B_k$ (as well as its regular conjugate $B_k^c$) is regular up to the closure of $\mathbb{B}$. Moreover, both $B_k$ and $B_k^c$ only have finitely many zeros in the interior of $\mathbb{B}$, and hence (see Proposition 1.38) $\tau_k$ is a diffeomorphism of a neighborhood of $\partial\mathbb{B}$ onto itself, it maps the boundary of $\mathbb{B}$ onto itself and it has inverse

$$\tau_k^{-1}(q) = (B_k(q))^{-1} q B_k(q).$$

Let $I \in \mathbb{S}$. Proposition 1.38 yields that we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \big|B_k^{-*} * B(\tau_k^{-1}(re^{I\theta}))\big| d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big|B_k(re^{I\theta})\big|^{-1} \big|B(re^{I\theta})\big| d\theta$$
$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(\max_{\theta \in [-\pi,\pi]} \big|B_k(re^{I\theta})\big|^{-1}\big) \big|B(re^{I\theta})\big| d\theta.$$

Since $B_k$ is a finite Blaschke product, it maps $\partial\mathbb{B}$ to itself and

$$\lim_{r\to 1^-} \left|B_k(re^{I\theta})\right| = \left|\widetilde{B}_k(e^{I\theta})\right| = 1,$$

for any $\theta$. Hence, for any $\varepsilon > 0$ there exists $r(\varepsilon)$ such that for any $r(\varepsilon) \le r < 1$

$$\max_{\theta\in[-\pi,\pi]} \left|B_k(re^{I\theta})\right|^{-1} \le 1+\varepsilon.$$

Therefore, for any $\varepsilon > 0$ there exists $r$ sufficiently near to 1 such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|B_k^{-*} * B(\tau_k^{-1}(re^{I\theta}))\right| d\theta \le \frac{1+\varepsilon}{2\pi}\int_{-\pi}^{\pi} \left|B(re^{I\theta})\right| d\theta.$$

Thanks to Proposition 6.1 and since $||\widetilde{B}||_{L^\infty} = ||B||_\infty \le 1$, we have then that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|B_k^{-*} * B(\tau_k^{-1}(re^{I\theta}))\right| d\theta \le \frac{1+\varepsilon}{2\pi}\int_{-\pi}^{\pi} \left|\tilde{B}(e^{I\theta})\right| d\theta \le 1+\varepsilon. \tag{6.15}$$

Set

$$J_k(r,\theta) = \left(B_k(re^{I\theta})\right)^{-1} I B_k(re^{I\theta}) \in \mathbb{S},$$

so that

$$\tau_k^{-1}(re^{I\theta}) = r\cos\theta + (r\sin\theta)J_k(r,\theta).$$

Since $\tau_k^{-1}$ is a diffeomorphism of $\partial\mathbb{B}$ onto itself, we get that for every $\theta$ there exists the limit

$$\lim_{r\to 1^-} J_k(r,\theta) = \tilde{J}_k(\theta).$$

Using the Representation Formula 1.10 we can write

$$\begin{aligned}
&\left|B_k^{-*} * B(\tau_k^{-1}(re^{I\theta}))\right| \\
&= \left|\frac{1}{2}\left(B_k^{-*}*B(re^{I\theta})+B_k^{-*}*B(re^{-I\theta})\right) + \frac{J_k(r,\theta)I}{2}\left(B_k^{-*}*B(re^{-I\theta})-B_k^{-*}*B(re^{I\theta})\right)\right|
\end{aligned}$$

Hence, recalling Proposition 6.19, applying twice the Representation Formula 1.10, we get, for almost every $\theta$,

$$\begin{aligned}
&\lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|B_k^{-*} * B(\tau_k^{-1}(re^{I\theta}))\right| \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2}\left(\widetilde{B}_k^{-*}*\widetilde{B}(e^{I\theta})+\widetilde{B}_k^{-*}*\widetilde{B}(e^{-I\theta})\right) + \frac{\tilde{J}_k(\theta)I}{2}\left(\widetilde{B}_k^{-*}*\widetilde{B}(e^{-I\theta})-\widetilde{B}_k^{-*}*\widetilde{B}(e^{I\theta})\right)\right| d\theta \\
&= \lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2}\left(B_k^{-*} * B(re^{I\theta}) + B_k^{-*} * B(re^{-I\theta})\right)\right. \\
&\qquad\qquad\left. + \frac{\tilde{J}_k(\theta)I}{2}\left(B_k^{-*} * B(re^{-I\theta}) - B_k^{-*} * B(re^{I\theta})\right)\right| d\theta \\
&= \lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|B_k^{-*}*B(re^{\tilde{J}_k(\theta)\theta})\right| d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widetilde{B}_k^{-*} * \widetilde{B}(\tilde{\tau}_k^{-1}(e^{I\theta}))\right|
\end{aligned}$$

where $\tilde{\tau}_k^{-1}$ is the radial limit of $\tau_k^{-1}$. Recalling inequality (6.15), we get then that for any $\varepsilon > 0$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widetilde{B}_k^{-*}*\widetilde{B}(\tilde{\tau}_k^{-1}(e^{I\theta}))\right| = \lim_{r\to 1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|B_k^{-*}*B(\tau_k^{-1}(re^{I\theta}))\right|$$
$$\le \frac{1+\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|\tilde{B}(e^{I\theta})\right|d\theta \le 1+\varepsilon.$$

Now we want to take the limit for $k\to+\infty$ of the previous inequality. For almost every $\theta$, the limit

$$\lim_{k\to+\infty}\tilde{\tau}_k^{-1}(e^{I\theta}) = \lim_{k\to+\infty}\left(\widetilde{B}_k(e^{I\theta})\right)^{-1}e^{I\theta}\widetilde{B}_k(e^{I\theta}) = \left(\widetilde{B}(e^{I\theta})\right)^{-1}e^{I\theta}\widetilde{B}(e^{I\theta}) = \tilde{\tau}^{-1}(e^{I\theta})$$

does exist and it coincides with

$$\lim_{k\to+\infty}\tilde{\tau}_k^{-1}(e^{I\theta}) = \lim_{k\to+\infty}(\cos\theta + (\sin\theta)\widetilde{J}_k(\theta)) = \cos\theta + (\sin\theta)\widetilde{J}(\theta),$$

which implies that $\widetilde{J}_k(\theta)$ converges for almost every $\theta$. Using again the Representation Formula 1.10, we have then that

$$\lim_{k\to+\infty}\left|\widetilde{B}_k^{-*}*\widetilde{B}(\tilde{\tau}_k^{-1}(e^{I\theta}))\right|$$
$$= \lim_{k\to+\infty}\left|\frac{1}{2}\left(\widetilde{B}_k^{-*}*\widetilde{B}(e^{I\theta})+\widetilde{B}_k^{-*}*\widetilde{B}(e^{-I\theta})\right) + \frac{\tilde{J}_k(\theta)I}{2}\left(\widetilde{B}_k^{-*}*\widetilde{B}(e^{-I\theta})-\widetilde{B}_k^{-*}*\widetilde{B}(e^{I\theta})\right)\right|.$$

Recalling that $B_k^{-*}*B$ converges uniformly on compact sets to the function constantly equal to 1, and that $\tilde{J}_k(\theta)$ converges, we obtain

$$\lim_{k\to+\infty}\left|\widetilde{B}_k^{-*}*\widetilde{B}(\tilde{\tau}_k^{-1}(e^{I\theta}))\right| \equiv 1.$$

Therefore we get that, for any $\varepsilon > 0$,

$$1 = \lim_{k\to+\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\widetilde{B}_k^{-*}*\widetilde{B}(\tilde{\tau}_k^{-1}(e^{I\theta}))\right|d\theta \le \frac{1+\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|\tilde{B}(e^{I\theta})\right|d\theta \le 1+\varepsilon,$$

that finally implies

$$\left|\tilde{B}(e^{I\theta})\right| = 1$$

for almost every $\theta$.

□

In order to show that the zero sequence of a function $f$ in $H^p(\mathbb{B})$ is such that the Blaschke product associated with it is convergent, we will use classical results in the theory of complex $H^p$ spaces, that apply to the symmetrization of $f$.

**Remark 6.33.** Let $I$ be an imaginary unit. Recall that the symmetrization $f^s$ of a regular function $f$ behaves exactly as a holomorphic function on the slice $L_I$. Hence, if $f^s$ is in $H^p(\mathbb{B})$ (and therefore in $H^p(\mathbb{B}_I)$) for some $p \in (0, +\infty]$, and $f^s \not\equiv 0$, classical results (see e.g. Theorem 15.23 in [46]) yield that, if $\{a_n^I\}_{n \geq 0}$ is the sequence of zeros of $f^s$ in $\mathbb{B}_I$, listed according to their multiplicity, then the Blaschke condition

$$\sum_{n \geq 0} (1 - |a_n^I|) < +\infty$$

is fulfilled.

Consequently,

**Proposition 6.34.** *Let $f \in H^p(\mathbb{B})$, $f \not\equiv 0$ and let $\{b_n\}_{n \in \mathbb{N}}$ be its zero sequence. Then $\{b_n\}_{n \in \mathbb{N}}$ satisfies the Blaschke condition*

$$\sum_{n \geq 0} (1 - |b_n|) < +\infty.$$

*Proof.* Let us consider the symmetrization of $f$, $f^s$. Thanks to Remark 6.33, for any $I \in \mathbb{S}$, if $\{a_n^I\}_{n \in \mathbb{N}}$ is the sequence of zeros of $f^s$ on $\mathbb{B}_I$, then it satisfies the Blaschke contidion. If $b_n = b_{1,n} + b_{2,n} I_n$, let us set $b_n^I = b_{1,n} + b_{2,n} I$ for all $n \in \mathbb{N}$. Then $|b_n| = |b_n^I|$ for all $n \in \mathbb{N}$, and $\{b_n^I\}_{n \in \mathbb{N}} \subseteq \{a_n^I\}_{n \in \mathbb{N}}$. Therefore (recalling that $|a_n^I| < 1$)

$$\sum_{n \geq 0} (1 - |b_n|) = \sum_{n \geq 0} \left(1 - |b_n^I|\right) \leq \sum_{n \geq 0} \left(1 - |a_n^I|\right) < +\infty.$$

□

The previous result implies that the Blaschke product built from the zeros of a regular function $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$ does converge uniformly on compact sets (compare with [4]).

**Proposition 6.35.** *Let $f$ be in $H^p(\mathbb{B})$ for some $p \in (0, +\infty]$ and let $\{a_n\}_{n \geq 0}$ be its zero sequence. If $M_{a_n}(q)$ denotes the Blaschke factor associated with $a_n$,*

$$M_{a_n}(q) = (1 - q\bar{a}_n)^{-*} * (a_n - q) \frac{\bar{a}_n}{|a_n|},$$

*then the Blaschke product*

$$B(q) = \prod_{n \geq 0}^{*} M_{a_n}(q)$$

*converges uniformly on compact sets of $\mathbb{B}$. Moreover, the function $B$ is regular on $\mathbb{B}$.*

*Proof.* Thanks to Theorem 1.34, we know that the infinite regular product

$$\prod_{n\geq 0}^{*} M_{a_n}(q) \tag{6.16}$$

converges uniformly on compact sets if and only if the infinite product

$$\prod_{n\geq 0} M_{a_n}(q)$$

does. Furthermore Theorem 1.33 gives us a sufficient condition for the uniform convergence on compact sets of

$$\prod_{n\geq 0} M_{a_n}(q),$$

namely the uniform convergence on compact sets of the series

$$\sum_{n\geq 0} |1 - M_{a_n}(q)|.$$

To estimate the latter, for any $n \in \mathbb{N}$ let us set $T_n(q) = (1 - qa_n)^{-1}q(1 - qa_n)$ (in view of Proposition 1.38), and consider

$$\begin{aligned}
|1 - M_{a_n}(q)| &= \left|1 - (1 - q\overline{a}_n)^{-*} * (a_n - q)\frac{\overline{a}_n}{|a_n|}\right| \\
&= \left|1 - (1 - T_n(q)\overline{a}_n)^{-1}(a_n - T_n(q))\frac{\overline{a}_n}{|a_n|}\right| \\
&= \left|(1 - T_n(q)\overline{a}_n)^{-1}\left((1 - T_n(q)\overline{a}_n) - (a_n - T_n(q))\frac{\overline{a}_n}{|a_n|}\right)\right| \\
&= \left|(1 - T_n(q)\overline{a}_n)^{-1}\left(1 - T_n(q)\overline{a}_n + T_n(q)\frac{\overline{a}_n}{|a_n|} - |a_n|\right)\right| \\
&= |(1 - T_n(q)\overline{a}_n)|^{-1}\left|(1 - |a_n|)\left(1 + T_n(q)\frac{\overline{a}_n}{|a_n|}\right)\right| \\
&= |(1 - T_n(q)\overline{a}_n)|^{-1}(1 - |a_n|)\left|1 + T_n(q)\frac{\overline{a}_n}{|a_n|}\right|.
\end{aligned}$$

Recalling that $|T_n(q)| = |q| < 1$ and $|a_n| < 1$, we obtain

$$|1 - M_{a_n}(q)| \leq (1 - |q|)^{-1}(1 - |a_n|)2.$$

Recalling that the zero sequence of a function in $H^p(\mathbb{B})$ satisfies the Blaschke condition,

$$\sum_{n\geq 0}(1 - |a_n|) < +\infty,$$

we get that

$$\sum_{n\geq 0} |1 - M_{a_n}(q)| < +\infty$$

and hence the convergence of the infinite regular product (6.16). The regularity of $B(q)$ is guaranteed by Theorem 1.35 $\square$

We point out that the convergence of $B(q)$ depends only on the moduli $|a_n|$, $n \in \mathbb{N}$. This means that we can build a Blaschke product $\widehat{B}(q)$ having the same zeros (with the same multiplicities) of a given regular function $f \in H^p(\mathbb{B})$ . In fact, in order to make $\widehat{B}(q)$ vanish exactly at the zero set of $f$, we have to consider the product of Blaschke factors associated with suitable conjugates of the points $a_n$, lying on the same 2-spheres $x_n + y_n\mathbb{S}$ generated by $a_n$, taking into account Proposition 1.20.

**Corollary 6.36.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be the zero sequence of a regular function $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$. Then there exists a Blaschke product $\widehat{B}(q)$ having the same zero sequence.*

*Proof.* We have to build a sequence $\{\widehat{a}_n\}_{n\in\mathbb{N}}$, where each $\widehat{a}_n$ is a conjugate of $a_n$, such that the Blaschke product associated with it

$$\widehat{B}(q) = \prod_{n\geq 0}^{*} M_{\widehat{a}_n}(q)$$

has $\{a_n\}_{n\in\mathbb{N}}$ as it zero sequence. The convergence of the Blaschke product is guaranteed by Proposition 6.35.
Since regular multiplication does not conjugate the zeros of the first function in the $*$-product, the first term of the sequence will be equal to $a_0$,

$$\widehat{a}_0 = a_0.$$

For the second, we need to find $\widehat{a}_1$ such that

$$M_{\widehat{a}_0}(q) * M_{\widehat{a}_1}(q) = M_{a_0}(q) * (1 - q\overline{\widehat{a}}_1)^{-*} * (\widehat{a}_1 - q)\frac{\overline{\widehat{a}}_1}{|\widehat{a}_1|}$$

vanishes at $q = a_1$. Notice that, for any $k \in \mathbb{N}$,

$$(1 - q\overline{\widehat{a}}_k)^{-*} * (\widehat{a}_k - q) = (\widehat{a}_k - q) * (1 - q\overline{\widehat{a}}_k)^{-*},$$

because

$$(\widehat{a}_k - q) * (1 - q\overline{\widehat{a}}_k) = (1 - q\overline{\widehat{a}}_k) * (\widehat{a}_k - q).$$

Hence

$$M_{\widehat{a}_0}(q) * M_{\widehat{a}_1}(q) = M_{a_0}(q) * (\widehat{a}_1 - q) * (1 - q\overline{\widehat{a}}_1)^{-*}\frac{\overline{\widehat{a}}_1}{|\widehat{a}_1|},$$

and, thanks to Proposition 1.20, we can write

$$M_{\widehat{a}_0}(q) * M_{\widehat{a}_1}(q) = (M_{a_0}(q)(\widehat{a}_1 - T_1(q))) * \left((1 - q\overline{\widehat{a}}_1)^{-*}\frac{\overline{\widehat{a}}_1}{|\widehat{a}_1|}\right),$$

where

$$T_1(q) = (M_{a_0}(q))^{-1}\, qM_{a_0}(q).$$

Therefore, if we want that this product vanishes at $a_1$, we need to set

$$\widehat{a}_1 = T_1(a_1).$$

We can iterate this process, setting, for any $n \geq 1$,

$$T_n(q) = \left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right)^{-1} q\left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right),$$

so that

$$\begin{aligned}\left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right) * M_{\widehat{a}_n}(q) &= \left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right) * (\widehat{a}_n - q) * (1 - q\overline{\widehat{a}}_n)^{-*}\frac{\overline{\widehat{a}}_n}{|\widehat{a}_n|} \\ &= \left(\left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right)(\widehat{a}_n - T_n(q))\right) * \left((1 - q\overline{\widehat{a}}_n)^{-*}\frac{\overline{\widehat{a}}_n}{|\widehat{a}_n|}\right)\end{aligned}$$

Hence, if we want that

$$\left(\prod_{k=0}^{n-1}{}^{*} M_{\widehat{a}_k}(q)\right) * M_{\widehat{a}_n}(q)$$

vanishes at $q = a_n$, we have to set

$$\widehat{a}_n = T_n(a_n).$$

□

**Remark 6.37.** Since we transform the zeros of $f$ by conjugation, all real and spherical zeros of $f$ are not modified by this process.

We are now ready to prove our first result in the direction of a factorization of functions in $H^p(\mathbb{B})$.

**Theorem 6.38.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$. Then we can factor $f$ as*

$$f(q) = h * g(q)$$

*where $h$ and $g$ are regular functions on $\mathbb{B}$ such that $h(q) \neq 0$ for any $q \in \mathbb{B}$ and $g$ is an inner function (more precisely a Blaschke product).*

*Proof.* Let $\{a_n\}_{n\in\mathbb{N}}$ be the zero sequence of $f$, and let us subdivide it as

$$\{a_n\}_{n\in\mathbb{N}} = \{\alpha_n\}_{n\in\mathbb{N}} \cup \{\beta_n\}_{n\in\mathbb{N}}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is the sequence of spherical and real zeros, while $\{\beta_n\}_{n\in\mathbb{N}}$ is the sequence of isolated (non-real) ones. Let $M_{a_n}$ denote the Blaschke factor associated with $a_n$

$$M_{a_n}(q) = (1 - q\overline{a}_n)^{-*} * (a_n - q)\frac{\overline{a}_n}{|a_n|}.$$

Since $\{\alpha_n\}_{n\in\mathbb{N}}$ is contained in the zero sequence of the function $f$, clearly

$$\sum_{n\geq 0}(1 - |\alpha_n|) \leq \sum_{n\geq 0}(1 - |a_n|) < +\infty.$$

Hence the Blaschke product associated with $\{\alpha_n\}_{n\in\mathbb{N}}$

$$B_\alpha(q) = \prod_{n\geq 0}^{*} M_{\alpha_n}(q)$$

converges uniformly on compact sets thus defining a regular function, vanishing exactly at the spherical and real zeros of $f$. The function $B_\alpha(q)$ is slice preserving, in fact it contains Blaschke factors associated with real points or factors of the type

$$M_{\alpha_n} * M_{\overline{\alpha}_n} = M^s_{\alpha_n}.$$

In both cases they are factors with real coefficients.
Let $f_\beta(q)$ be the function defined as

$$f_\beta(q) = B_\alpha^{-*} * f(q) = B_\alpha(q)^{-1} f(q),$$

so that we can write

$$f(q) = B_\alpha(q) f_\beta(q).$$

Then $f_\beta$ is regular on $\mathbb{B}$ and its zero sequence coincides with $\{\beta_n\}_{n\in\mathbb{N}}$. The idea is to make spherical all the zeros of $f_\beta$. In order to do it, we want to find a Blaschke product $B_{\overline{\beta}}$ such that

$$f_\beta * B_{\overline{\beta}}(q)$$

vanishes at all spheres generated by $\{\beta_n\}_{n\in\mathbb{N}}$, namely so that its zero sequence is

$$\{\beta_n, \overline{\beta}_n\}_{n\in\mathbb{N}}.$$

We can build $B_{\overline{\beta}}$ with a similar process of the one used in the proof of Corollary 6.36. If

$$B_{\overline{\beta}}(q) = \prod_{n\geq 0}^{*} M_{\widehat{\beta}_n}(q),$$

we can define the sequence $\{\widehat{\beta}_n\}_{n\in\mathbb{N}}$ iteratively as follows. The first one, $\widehat{\beta}_0$, is such that

$$f_\beta * M_{\widehat{\beta}_0}(q) = f_\beta * \big(\widehat{\beta}_0 - q\big) * \big(1 - q\overline{\widehat{\beta}}_0\big)^{-*}\frac{\overline{\widehat{\beta}}_0}{|\widehat{\beta}_0|}$$

vanishes in $q = \beta_0$ and in $q = \overline{\beta}_0$. Since $f_\beta$ vanishes in $q = \beta_0$ and $f$ does not vanish in $q = \overline{\beta}_0$, if $T_0$ is defined as

$$T_0(q) = (f_\beta(q))^{-1}\, q f_\beta(q),$$

then we can write

$$f_\beta * M_{\widehat{\beta}_0}(q) = \big(f_\beta(q)\big(\widehat{\beta}_0 - T_0(q)\big)\big) * \left(\big(1 - q\overline{\widehat{\beta}}_0\big)^{-*}\frac{\overline{\widehat{\beta}}_0}{|\widehat{\beta}_0|}\right).$$

Hence, if we set

$$\widehat{\beta}_0 = T_0(\overline{\beta}_0),$$

we have that

$$f_\beta * M_{\widehat{\beta}_0}(q)$$

vanishes both in $q = \beta_0$ and in $q = \overline{\beta}_0$. As we have done to prove Corollary 6.36, we can iterate the process, setting, for any $n \geq 1$,

$$T_n(q) = \left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}(q)\right)^{-1} q\left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}(q)\right)$$

so that

$$\left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}\right) * M_{\widehat{\beta}_n}(q) = \left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}\right) * \big(\widehat{\beta}_n - q\big) * \left(\big(1 - q\overline{\widehat{\beta}}_n\big)^{-*}\frac{\overline{\widehat{\beta}}_n}{|\widehat{\beta}_n|}\right)$$
$$= \left(\left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}\right)\big(\widehat{\beta}_n - T_n(q)\big)\right) * \left(\big(1 - q\overline{\widehat{\beta}}_n\big)^{-*}\frac{\overline{\widehat{\beta}}_n}{|\widehat{\beta}_n|}\right).$$

Hence, if we set

$$\widehat{\beta}_n = T_n(\overline{\beta}_n)$$

we get that

$$\left(f_\beta * \prod_{k=0}^{n-1}{}^{*} M_{\widehat{\beta}_k}\right) * M_{\widehat{\beta}_n}(q)$$

vanishes both at $q = \beta_n$ and $q = \overline{\beta}_n$.
The convergence of the infinite product $B_{\overline{\beta}}(q)$ is guaranteed by the fact that it is the Blaschke product associated with a sequence where each element $T_n(\overline{\beta}_n)$ has the same

modulus of $\beta_n$, and each $\beta_n$ is contained in the zero sequence of a function in $H^p(\mathbb{B})$. Hence

$$f_\beta * B_{\overline{\beta}}(q)$$

is a regular function that has only spherical zeros, and its zero sequence is $\{\beta_n, \overline{\beta}_n\}_{n\in\mathbb{N}}$. Therefore, if we set

$$B^s_\beta(q) = \prod_{n\geq 0}^{*} \left(M_{\beta_n} * M_{\overline{\beta}_n}\right)(q) = \prod_{n\geq 0}^{*} M^s_{\beta_n}(q),$$

then we can write

$$f_\beta * B_{\overline{\beta}}(q) = B^s_\beta * h(q) = B^s_\beta(q)h(q), \tag{6.17}$$

for some function $h$, never vanishing and regular on $\mathbb{B}$. To prove the regularity of $h$, it suffices to observe that since $B^s_\beta$ has real coefficients, is regular and it has exactly the same zeros of $f_\beta * B_{\overline{\beta}}(q)$, then the regular quotient

$$\left(B^s_\beta\right)^{-*} * \left(f_\beta * B_{\overline{\beta}}\right)(q) = \left(B^s_\beta\right)^{-1}(q)\left(f_\beta * B_{\overline{\beta}}\right)(q)$$

is well defined (and regular) on the entire ball $\mathbb{B}$.
Consider the regular conjugate of $B_{\overline{\beta}}$, and $*$-multiply on the right by $B^c_{\overline{\beta}}$ all terms of equation (6.17). We obtain

$$f_\beta * B_{\overline{\beta}} * B^c_{\overline{\beta}}(q) = B^s_\beta(q)h * B^c_{\overline{\beta}}(q).$$

That can also be written as

$$\left(B_{\overline{\beta}}\right)^s(q)f_\beta(q) = B^s_\beta(q)h * B^c_{\overline{\beta}}(q).$$

Now notice that $\left(B_{\overline{\beta}}\right)^s(q) = B^s_\beta(q)$ because they are Blaschke products associated to the same spherical zeros (we only generated them starting from different points on the same 2-spheres). Therefore it results

$$f_\beta(q) = h * B^c_{\overline{\beta}}(q),$$

that, for $f$, means

$$f(q) = B_\alpha(q)h * B^c_{\overline{\beta}}(q) = h * B_\alpha * B^c_{\overline{\beta}}(q).$$

Setting $g(q) = B_\alpha * B^c_{\overline{\beta}}(q)$ leads to the conclusion of the proof. □

Once we extract the zeros of a function in $H^p(\mathbb{B})$, we would like to identify its outer factor and its singular part. In the case of a function that is continuous up to the boundary of $\mathbb{B}$ we obtain the following result.

**Proposition 6.39.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$ and let $f$ be continuous up to the boundary of $\mathbb{B}$. Suppose that $f(q) \neq 0$ for any $q \in \overline{\mathbb{B}}$. Then $f$ is an outer function.*

*Proof.* Let $g \in H^p(\mathbb{B})$ be such that $|\tilde{g}(q)| = |\tilde{f}(q)|$ for almost every $q \in \partial\mathbb{B}$, (then, since $f$ is continuous up to the boundary, $|\tilde{g}(q)| = |f(q)|$ for almost every $q \in \partial\mathbb{B}$). Consider $h(q) = f^{-*} * g(q)$. The function $h$ is regular on $\mathbb{B}$ (since $f$ is non-vanishing on $\mathbb{B}$). Let $T(q)$ be the transformation defined by

$$T(q) = (f^c(q))^{-1} q f^c(q),$$

so that, according to Proposition 1.38,

$$|h(q)| = |f^{-1}(T(q))g(T(q))| = |f(T(q))|^{-1}|g(T(q))| \tag{6.18}$$

for any $q \in \mathbb{B}$. Thanks to Proposition 1.26, since $f$ is non-vanishing on $\overline{\mathbb{B}}$, then also the regular conjugate $f^c$ has the same property. This yields that $T(q)$ is a diffeomorphism of the closed ball onto itself and in particular of the boundary $\partial\mathbb{B}$ onto itself. Then, taking the radial limit of equality (6.18), recalling Proposition 6.19 we get that for any $I \in \mathbb{S}$, for almost every $\theta \in [-\pi, \pi)$,

$$\lim_{r \to 1^-} \left|h(re^{I\theta})\right| = \left|\tilde{f}(\widetilde{T}(e^{I\theta}))\right|^{-1} \left|\tilde{g}(\widetilde{T}(e^{I\theta}))\right| = \left|f(e^{K\theta})\right|^{-1} \left|\tilde{g}(e^{K\theta})\right|,$$

for some $K \in \mathbb{S}$. Since $|f|$ and $|\tilde{g}|$ coincide almost everywhere at the boundary, we get that for almost every $\theta \in [-\pi, \pi)$,

$$\lim_{r \to 1^-} \left|h(re^{I\theta})\right| = 1.$$

Therefore

$$||h||_\infty = 1$$

and

$$|h(q)| \leq 1$$

for any $q \in \mathbb{B}$. Recalling formula (6.18), we obtain that $|g(q)| \leq |f(q)|$ for any $q \in \mathbb{B}$, and hence that $f$ is an outer function. □

The previous result yields that the inner factor of every function continuous up to the boundary of $\mathbb{B}$, coincides with its Blaschke product. Hence, for every function continuous up to the boundary of $\mathbb{B}$, we can find its factorization into inner and outer factors.

**Corollary 6.40.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$, and suppose that $f$ is continuous up to the closure $\overline{\mathbb{B}}$. Then*

$$f(q) = E * B(q)$$

*where $E$ is an outer function and $B$ is a Blaschke product.*

### 6.3.2 Case of a preserved slice

Another situation in which we obtain a factorization into inner and outer factors, is the special case of functions that preserve at least one slice.

**Theorem 6.41.** *Let $f \in H^p(\mathbb{B})$ for some $p \in (0, +\infty]$, be such that $f$ maps $\mathbb{B}_I$ to $L_I$ for some $I \in \mathbb{S}$. Then we can decompose $f$ as*

$$f(q) = E * \mathcal{I}(q),$$

*where $E$ is a non vanishing function in $H^p(\mathbb{B})$, such that $|\widetilde{E}| = |\tilde{f}|$ almost everywhere on the boundary $\partial\mathbb{B}$ (and hence such that $||E||_p = ||f||_p$) and $\mathcal{I} \in H^\infty(\mathbb{B})$, $||\mathcal{I}||_\infty = 1$ and $|\widetilde{\mathcal{I}}| = 1$ almost everywhere on the boundary $\partial\mathbb{B}$.*

*Proof.* The restriction of $f$ to $\mathbb{B}_I$ is a (complex) holomorphic function, $f_I(z) = F_I(z)$. Let us define the function $E_I \colon \mathbb{B}_I \to L_I$ to be

$$E_I(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{I\theta}+z}{e^{I\theta}-z}\log|F_I(e^{I\theta})|d\theta\right),$$

namely the outer function of $F_I$. From factorization results in the complex setting, (see for instance [39]), we know that we can write

$$F_I(z) = E_I(z)\mathcal{I}_I(z)$$

where $\mathcal{I}_I(z)$ is the inner function of $F_I$. In particular, since both $E_I$ and $\mathcal{I}_I$ map $L_I$ to itself, we can also write

$$F_I(z) = E_I(z)\mathcal{I}_I(z) = E_I(z) * \mathcal{I}_I(z).$$

Hence

$$f(q) = \text{ext}(f_I)(q) = \text{ext}(F_I)(q) = \text{ext}(E_I * \mathcal{I}_I)(q) = \text{ext}(E_I) * \text{ext}(\mathcal{I}_I)(q)$$

where the last equality is due to the Identity Principle 1.8. Let us set $E(q) = \text{ext}(E_I)(q)$ and $\mathcal{I}(q) = \text{ext}(\mathcal{I}_I)(q)$. Since $E_I(z) \neq 0$ for all $z \in \mathbb{B}_I$, Proposition 1.27 yields that also $E$ is never vanishing on $\mathbb{B}$. To estimate the modulus of $\mathcal{I}$, recall that a function that maps the slice $L_I$ to itself attains its maximum and minimum modulus on the preserved slice $L_I$ (see Proposition 1.13). Hence

$$\max_{J\in\mathbb{S}} |\mathcal{I}(x+yJ)| = \max\{|\mathcal{I}(x+yI)|, |\mathcal{I}(x-yI)|\} = \max\{|\mathcal{I}_I(x+yI)|, |\mathcal{I}_I(x-yI)|\}$$

for all $x, y$ such that $x + yI \in \mathbb{B}_I$, which implies

$$||\mathcal{I}||_\infty = \sup_{q\in\mathbb{B}} |\mathcal{I}(q)| = \sup_{z\in\mathbb{B}_I} |\mathcal{I}_I(z)| = ||\mathcal{I}_I||_\infty.$$

By classical results, the inner function is bounded in modulus by 1 and its uniform norm equals 1, therefore we obtain

$$||\mathcal{I}||_\infty = ||\mathcal{I}_I||_\infty = 1$$

and hence that $\mathcal{I} \in H^\infty(\mathbb{B})$. Moreover, since $|\widetilde{\mathcal{I}}_I|$ equals 1 almost everywhere on the boundary $\partial\mathbb{B}_I$, we have that, for almost every $x + yI$ such that $x^2 + y^2 = 1$,

$$\max\left\{|\widetilde{\mathcal{I}}_I(x+yI)|, |\widetilde{\mathcal{I}}_I(x-yI)|\right\} = \min\left\{|\widetilde{\mathcal{I}}_I(x+yI)|, |\widetilde{\mathcal{I}}_I(x-yI)|\right\} = 1.$$

Thanks again to Proposition 1.13, for almost every $x + yJ$ such that $x^2 + y^2 = 1$, we have

$$\max_{J\in\mathbb{S}} \left|\widetilde{\mathcal{I}}(x+yJ)\right| = \min_{J\in\mathbb{S}} \left|\widetilde{\mathcal{I}}(x+y\mathbb{S})\right| = 1$$

namely, for almost every $q \in \partial\mathbb{B}$, $\left|\widetilde{\mathcal{I}}(q)\right| = 1$.
To obtain the desired properties of the modulus of $E$, let us denote by $T$ the transformation

$$T(q) = (E^c(q))^{-1} q E^c(q).$$

The map $T$ is well defined for any $q \in \mathbb{B}$ since $E^c$ is non-vanishing on $\mathbb{B}$. Then, thanks to Proposition 1.38, we can write, for any $q \in \mathbb{B}$,

$$1 \geq |\mathcal{I}(q)| = \left|E^{-*} * f(q)\right| = \left|E^{-1}(T(q))f(T(q))\right| = \left|E(T(q))|^{-1}|f(T(q))\right|. \tag{6.19}$$

Recalling that $|T(q)| = |q|$ for any $q \in \mathbb{B}$ and that $T$ is a diffeomorphism of $\mathbb{B}$ onto itself, it is clear that

$$|E(q)| \geq |f(q)| \quad \text{for any } q \in \mathbb{B}.$$

In order to study the behavior of the modulus $|\widetilde{E}|$ at the boundary, first of all, recall that classical results on outer functions (see e.g. [39]) imply that $E_I \in H^p(\mathbb{B}_I)$. Proposition 6.9 implies then that $E \in H^p(\mathbb{B})$. Let $J \in \mathbb{S}$. Thanks to Proposition 6.19, if we consider the radial limit of formula (6.19), we get that

$$1 = |\widetilde{\mathcal{I}}(e^{I\theta})| = \lim_{r\to 1^-} \left|E^{-*} * f(re^{J\theta})\right| = |\widetilde{E}(\widetilde{T}(e^{J\theta}))|^{-1}|\widetilde{f}(\widetilde{T}(e^{J\theta}))| \tag{6.20}$$

for almost every $\theta \in [-\pi, \pi)$. Let us show that $\widetilde{T}$ maps almost every 2-sphere $x + y\mathbb{S} \subset \partial\mathbb{B}$ onto itself. In fact, let $x, y \in \mathbb{R}$ be such that the function $\widetilde{E^c}$ is defined (and not identically vanishing) on $x + y\mathbb{S}$. Therefore, thanks to the Representation Formula 1.10 and to the fact that $E^c$ preserves $L_I$, there exist $b, c \in L_I$ such that

$$\widetilde{E^c}(x + yJ) = b + Jc \quad \text{for any } J \in \mathbb{S}.$$

Let $x + yK \in x + y\mathbb{S}$, with $K \neq \pm I$. We want to find $J \in \mathbb{S}$, $(J \neq \pm I)$ such that $\widetilde{T}(x + yJ) = x + yK$. Since $b + Jc \neq 0$, this is possible if and only if

$$(b + Jc)^{-1}(x + yJ)(b + Jc) = x + yK,$$

i.e.
$$(x+yJ)(b+Jc)=(b+Jc)(x+yK).$$
Since $x$ and $y$ are real numbers, the previous equality reduces to
$$J(b+Jc)=(b+Jc)K,$$
which is solved by
$$J=(bK+c)(b-cK)^{-1}.$$
Thanks to this property of the transformation $T$, we get that equation (6.20) leads to the equality
$$|\widetilde{E}(q)|=|\tilde{f}(q)| \tag{6.21}$$
for almost every $q\in\partial\mathbb{B}$.
Therefore
$$||E||_p=||\widetilde{E}||_{L^p}=||\tilde{f}||_{L^p}=||f||_p.$$

□

**Remark 6.42.** The function $E$ introduced in the previous theorem is an *outer* function, according to Definition 6.26. In fact, let $g\in H^p(\mathbb{B})$ be such that
$$|\tilde{g}(q)|=|\tilde{f}(q)|$$
for almost every $q\in\partial\mathbb{B}$, and consider the function $E^{-*}*g$, regular on $\mathbb{B}$ since $E$ is non-vanishing. If we set
$$T(q)=(E^c(q))^{-1}\,qE^c(q),$$
then Proposition 6.19 yields that for any $J\in\mathbb{S}$, for almost every $\theta$,
$$\lim_{r\to1^-}|E^{-*}*g(re^{J\theta})|=|\widetilde{E}(\widetilde{T}(e^{J\theta}))|^{-1}|\tilde{g}(\widetilde{T}(e^{J\theta}))|=1$$
where the last equality is due to the fact that $\widetilde{T}$ maps the boundary of $\mathbb{B}$ to itself and
$$|\widetilde{E}(q)|=|\tilde{f}(q)|=|\tilde{g}(q)|$$
for almost every $q\in\partial\mathbb{B}$. Using again the fact that the radial limit $\widetilde{T}$ maps almost every 2-sphere of the form $x+y\mathbb{S}$ contained in $\partial\mathbb{B}$, onto itself, we get that
$$1=||\widetilde{E}^{-*}*\tilde{g}||_{L^\infty}=||E^{-*}*g||_\infty$$
and hence that
$$|E(q)|\geq|g(q)|$$
for every $q\in\mathbb{B}$.

## 6.4 The space $H^2(\mathbb{B})$

In analogy with the complex case, the space $H^2(\mathbb{B})$ is special. Indeed the 2-norm turns out to be induced by an inner product (compare with [3]).

**Proposition 6.43.** *Let $f \in H^2(\mathbb{B})$ and let $f(q) = \sum_{n\geq 0} q^n a_n$ be its power series expansion. Then the 2-norm of $f$,*

$$||f||_2 = \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{I\theta}\right|^2 d\theta\right)^{\frac{1}{2}},$$

*coincides with*

$$\Big(\sum_{n\geq 0} |a_n|^2\Big)^{\frac{1}{2}}.$$

*Proof.* We can express the square of the modulus of $f$ in terms of its power series expansion as follows

$$\begin{aligned}\left|f(re^{I\theta})\right|^2 &= \overline{f(re^{I\theta})}f(re^{I\theta}) = \sum_{n\geq 0}\overline{r^n e^{In\theta}a_n}\sum_{m\geq 0} r^m e^{Im\theta}a_n \\ &= \sum_{n\geq 0} r^n\overline{a_n}e^{-In\theta}\sum_{m\geq 0} r^m e^{Im\theta}a_n = \sum_{n,m\geq 0} r^{n+m}\overline{a_n}e^{I(m-n)\theta}a_m.\end{aligned}$$

Thanks to the uniform convergence on compact sets of the series expansion, when we integrate in $\theta$ we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(re^{I\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n,m\geq 0} r^{n+m}\overline{a_n}e^{I(m-n)\theta}a_m d\theta \\ &= \sum_{n,m\geq 0} r^{n+m}\overline{a_n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{I(m-n)\theta}d\theta\right)a_m.\end{aligned}$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{I(m-n)\theta}d\theta = \delta_{m,n},$$

where $\delta_{k,j}$ is the Kronecker delta defined on $\mathbb{N}\times\mathbb{N}$ by

$$\delta_{k,j} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(re^{I\theta})\right|^2 d\theta = \sum_{n\geq 0} r^{2n}\overline{a_n}a_n = \sum_{n\geq 0} r^{2n}|a_n|^2. \tag{6.22}$$

Notice that the last quantity in equation (6.22) does not depend on $I \in \mathbb{S}$. This yields that

$$\begin{aligned}||f||_2^2 &= \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{I\theta})\right|^2 d\theta = \sup_{I\in\mathbb{S}} \lim_{r\to 1^-} \sum_{n\geq 0} r^{2n}|a_n|^2 \\ &= \lim_{r\to 1^-} \sum_{n\geq 0} r^{2n}|a_n|^2 = \sum_{n\geq 0} |a_n|^2.\end{aligned}$$

□

**Remark 6.44.** As a consequence of the previous proposition we have that the 2-norm of $f \in H^2(\mathbb{B})$ is the same on each slice $L_I$,

$$||f||_2 = ||f_I||_2 \quad \text{for all } I \in \mathbb{S}.$$

Proposition 6.43 suggests us a way to define an inner product on the space $H^2(\mathbb{B})$. In fact, if $f, g \in H^2(\mathbb{B})$, let $f(q) = \sum_{n\geq 0} q^n a_n$, $g(q) = \sum_{n\geq 0} q^n b_n$ be their power series expansions. Following the proof of the previous result, we obtain that

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{g(re^{I\theta})} f(re^{I\theta}) d\theta = \sum_{n\geq 0} \overline{b_n} a_n \quad \text{for any } I \in \mathbb{S}.$$

Therefore it is natural to define, in $H^2(\mathbb{B})$, the inner product

$$\langle f, g\rangle = \sum_{n\geq 0} \overline{b_n} a_n.$$

The choice of conjugating the term on the left in the product $\overline{b_n} a_n$ is motivated by the proof of Proposition 6.43. We could have set the opposite order ( i.e. $\overline{a_n} b_n$) but with our choice we get "right linearity" in the first argument and "left anti-linearity" in the second one:

$$\langle f\alpha, g\rangle = \langle f, g\rangle\alpha, \quad \langle f, g\alpha\rangle = \overline{\alpha}\langle f, g\rangle \quad \text{for all } \alpha \in \mathbb{H}.$$

**Remark 6.45.** Thanks to Proposition 6.43, we can show also using power series, that the $N_2$ norm of $f(q) = (1-q)^{-1}$ is finite while $||f||_2$ does not. First notice that we can express $|f(q)|^2$ by means of its power series expansion as follows.

$$\begin{aligned}|f(q)|^2 &= \sum_{n\geq 0} r^n e^{-In\theta} \sum_{m\geq 0} r^m e^{Im\theta} = \sum_{n\geq 0} r^{2n} + \sum_{n>m\geq 0} r^{n+m}\left(e^{I(n-m)\theta} + e^{I(m-n)\theta}\right) \\ &= \sum_{n\geq 0} r^{2n} + \sum_{n\geq 0, k\geq 1} r^{2n+k}\left(e^{Ik\theta} + e^{-Ik\theta}\right) \\ &= \sum_{n\geq 0} r^{2n} + \sum_{n\geq 0} r^{2n} \sum_{k\geq 1} r^k \left(e^{Ik\theta} + e^{-Ik\theta}\right) \\ &= \sum_{n\geq 0} r^{2n} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{Ik\theta}.\end{aligned}$$

Hence

$$N_2(f)^2 = \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} r\Big(\int_{-\pi}^{\pi} \sum_{n\geq 0} r^{2n} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{Ik\theta} \sin^2\theta d\theta\Big) d\sigma_2(S_r)$$

$$= \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} r \sum_{n\geq 0} r^{2n} \sum_{k=-\infty}^{+\infty} r^{|k|} \Big(\int_{-\pi}^{\pi} e^{Ik\theta} \sin^2\theta d\theta\Big) d\sigma_2(S_r).$$

Moreover

$$\int_{-\pi}^{\pi} e^{Ik\theta} \sin^2\theta d\theta = \int_{-\pi}^{\pi} e^{Ik\theta} \Big(\frac{e^{I\theta} - e^{-I\theta}}{2I}\Big)^2 d\theta = \frac{-1}{4} \int_{-\pi}^{\pi} e^{Ik\theta} \big(e^{I2\theta} + e^{-I2\theta} - 2\big) d\theta$$

$$= \pi\delta_{k,0} - \frac{\pi}{2}\delta_{k,-2} - \frac{\pi}{2}\delta_{k,2}.$$

Hence we get

$$N_2(f)^2 = \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} \Big(r \frac{1}{1-r^2} \Big(-\frac{\pi}{2} r^2 + \pi - \frac{\pi}{2} r^2\Big)\Big) d\sigma_2(S_r)$$

$$= \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} \Big(r \frac{\pi(1-r^2)}{1-r^2}\Big) d\sigma_2(S_r)$$

$$= \sup_{0<r<1} \frac{1}{2\pi^2 r^3} \int_{S_r} r\pi d\sigma_2(S_r) = \sup_{0<r<1} \frac{1}{2\pi r^2} \int_{S_r} d\sigma_2(S_r) = 1.$$

To show that $f(q)$ does not have finite 2-norm, just recall that its power series expansion has every coefficient equal to 1, hence

$$||f||_2^2 = \sum_{n\geq 0} 1 = +\infty.$$

## 6.5 The Corona Problem in the quaternionic setting

The Corona Theorem is a well celebrated result in the theory of bounded holomorphic functions of one complex variable (see for instance [24]). Let $\mathbb{D}$ denote as usual the open unit disc of the complex plane, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let us recall that the space $H^\infty(\mathbb{D})$ is defined as

$$H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and bounded}\}. \tag{6.23}$$

It is a well known fact that $H^\infty(\mathbb{D})$, provided with the uniform norm, is a Banach algebra (endowed with the operations of pointwise addition and multiplication) and it is interesting to study its maximal ideal space $\mathcal{M}$. The Corona Theorem establishes in fact a very strong property of this space. One of the possible statements of the theorem is the following.

**Theorem 6.46** (Corona Theorem). *Let $f_1, \dots, f_n \in H^\infty(\mathbb{D})$, and suppose that there exists $\varepsilon > 0$ such that*

$$|f_1(z)| + \cdots + |f_n(z)| \geq \varepsilon > 0 \quad \text{for all } z \in \mathbb{D}. \tag{6.24}$$

*Then there exist $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ such that*

$$f_1(z)g_1(z) + \cdots + f_n(z)g_n(z) = 1 \quad \text{for all } z \in \mathbb{D}. \tag{6.25}$$

Equation (6.25) implies that the ideal generated by $f_1, \dots, f_n$ in $H^\infty(\mathbb{D})$ contains the identity element of $H^\infty(\mathbb{D})$ and hence it coincides with the entire algebra $H^\infty(\mathbb{D})$. It is possible to prove (see for instance [17, 24]) that this result is equivalent to the fact that the family

$$\mathcal{F} = \{(z-a)_{H^\infty(\mathbb{D})} \,|\, a \in \mathbb{D}\}$$

of ideals generated by linear polynomials, is dense in $\mathcal{M}$ (with respect to the weak-$*$ topology). Since $\mathcal{F}$ is isomorphic to the unit disc itself, we can say that $\mathbb{D}$ is dense in $\mathcal{M}$, and hence that the "corona" $\mathcal{M} \setminus \mathbb{D}$ has empty interior. One of the simplest proofs of Theorem 6.46 is due to Wolff and can be found in [22].
To state a quaternionic weak version of the Corona Theorem, we will use a stronger version of a slightly different formulation of hypothesis (6.24).

**Proposition 6.47.** *Let $f_1, \dots, f_n \in H^\infty(\mathbb{D})$. Then there exists $\varepsilon > 0$ such that*

$$|f_1(z)| + \cdots + |f_n(z)| \geq \varepsilon > 0 \quad \text{for all } z \in \mathbb{D} \tag{6.26}$$

*if and only if there exists $\delta > 0$ such that for all $z \in \mathbb{D}$ there exists $i \in \{1, \dots, n\}$ such that*

$$|f_i(z)| \geq \delta.$$

*Proof.* If condition (6.26) is satisfied, then we can chose $\delta = \frac{\varepsilon}{n}$ and we get one implication. On the other hand, if the second condition is verified, we can set $\varepsilon = \delta$ and then conclude. □

### 6.5.1 The space of bounded regular functions

Also in the quaternionic setting, the space of bounded regular functions $H^\infty(\mathbb{B})$ is a Banach algebra.

**Proposition 6.48.** *The space $H^\infty(\mathbb{B})$, endowed with pointwise addition and pointwise regular multiplication, is a real Banach algebra with respect to the uniform norm $||\cdot||_\infty$.*

*Proof.* The set of regular functions on $\mathbb{B}$, denoted by $\mathcal{R}(\mathbb{B})$, endowed with the operations of pointwise addition and $*$-product, is a (non commutative) ring with identity (see 1.19). In order to show that $(H^\infty(\mathbb{B}), +, *)$ is a subring of $(\mathcal{R}(\mathbb{B}), +, *)$, first notice that if $f$ and $g$ are bounded regular functions, then also $f - g$ does. Let us denote by $Z_f$

the zero set of $f$, $Z_f = \{q \in \mathbb{B} \,|\, f(q) = 0\}$. Since when $f(q) = 0$ also $f * g(q) = 0$, we get

$$||f * g||_\infty = \sup_{q\in\mathbb{B}} |f * g(q)| = \sup_{q\in\mathbb{B}\setminus Z_f} |f * g(q)|.$$

Hence, Proposition 1.20 yields that, for all $f, g \in \mathcal{R}(\mathbb{B})$,

$$\begin{aligned} ||f * g||_\infty &= \sup_{q\in\mathbb{B}\setminus Z_f} |f * g(q)| = \sup_{q\in\mathbb{B}\setminus Z_f} |f(q)g(f(q)^{-1}qf(q))| \\ &\leq \sup_{q\in\mathbb{B}\setminus Z_f} |f(q)| \sup_{q\in\mathbb{B}\setminus Z_f} |g(f(q)^{-1}qf(q))| \leq ||f||_\infty ||g||_\infty \end{aligned} \tag{6.27}$$

where last inequality is due to the fact that for all $q \in \mathbb{H}$ such that $f(q) \neq 0$ we have $|q| = |f(q)^{-1}qf(q)|$. Therefore we obtain that the regular product of two bounded regular functions is still a bounded regular function, and hence $(H^\infty(\mathbb{B}), +, *)$ is a subring of $(\mathcal{R}(\mathbb{B}), +, *)$. Also, it is easy to prove that $H^\infty(\mathbb{B})$ is a real vector space, and as a consequence a real algebra. To see that $H^\infty(\mathbb{B})$ is a Banach space, observe that $\mathbb{H}$ is a complete metric space (since it is isomorphic to $\mathbb{R}^4$) and hence the same holds for $H^\infty(\mathbb{B})$ with respect to the uniform norm. Finally, equation (6.27) gives us the condition for $(H^\infty(\mathbb{B}), +, *, ||\cdot||_\infty)$ to be a Banach algebra. □

### 6.5.2 A weak version of the Corona Theorem

Let us prove a weak version of the Corona Theorem for regular functions.

**Theorem 6.49.** *Let $f_1, \ldots, f_n \in H^\infty(\mathbb{B})$. If there exists $\varepsilon > 0$ such that for all sphere $x + y\mathbb{S} \subset \mathbb{B}$ there exists $i \in \{1, \ldots, n\}$ such that*

$$|f_i(q)| \geq \varepsilon \quad \textit{for all} \quad q \in x + y\mathbb{S}, \tag{6.28}$$

*then we can find $n$ bounded regular functions $g_1, \ldots, g_n \in H^\infty(\mathbb{B})$ such that*

$$f_1 * g_1(q) + \cdots + f_n * g_n(q) = 1$$

*for all $q \in \mathbb{B}$.*

*Proof.* Consider the symmetrization functions $f_1^s, \ldots, f_n^s$. Notice that if $f_i$ vanishes somewhere in $\mathbb{B}$ then

$$\inf_{q\in\mathbb{B}} |f_i(q)| = \inf_{q\in\mathbb{B}} |f_i^c(q)| = \inf_{q\in\mathbb{B}} |f_i^s(q)| = 0$$

for all $i \in \{1, \ldots n\}$. Hence, using Proposition 1.20 omitting the case where $f_i$ vanishes, we can write

$$\begin{aligned} \inf_{q\in\mathbb{B}} |f_i^s(q)| &= \inf_{q\in\mathbb{B}} |f_i * f_i^c(q)| = \inf_{q\in\mathbb{B}} |f_i(q)||f_i^c(f_i(q)^{-1}qf_i(q))| \\ &\geq \inf_{q\in\mathbb{B}} |f_i(q)| \inf_{q\in\mathbb{B}} |f_i^c(f_i(q)^{-1}qf_i(q))| = \inf_{q\in\mathbb{B}} |f_i(q)| \inf_{q\in\mathbb{B}} |f_i^c(q)|. \end{aligned}$$

Corollary 2.4 yields then

$$\inf_{q\in\mathbb{B}} |f_i^s(q)| \geq \inf_{q\in\mathbb{B}} |f_i(q)|^2.$$

for all $i = 1, \dots, n$. Hence, by hypothesis (6.28) we obtain that for all sphere $x + y\mathbb{S}$ contained in $\mathbb{B}$, there exists $i \in \{1, \dots, n\}$ such that

$$|f_i^s(q)| \geq \varepsilon^2 > 0$$

for every $q \in x + y\mathbb{S}$. In particular it holds also that

$$|f_1^s(q)| + \cdots + |f_n^s(q)| \geq \varepsilon^2 > 0 \quad \text{for all} \quad q \in \mathbb{B}.$$

Moreover we proved in 6.13 that if $f_i \in H^\infty(\mathbb{B})$, then also $f_i^s \in H^\infty(\mathbb{B})$. Recall that symmetrization functions always have real coefficients and hence they are slice preserving. Then, for each $I \in \mathbb{S}$ and for each $i \in \{1, \dots, n\}$, the restriction of $f_i^s$ to the slice $L_I$ is a holomorphic function. By the Corona Theorem for holomorphic functions 6.46 we get that there exist $n$ bounded holomorphic functions $h_1, \dots, h_n \in H^\infty(\mathbb{B}_I)$ such that

$$f_1^s(z)h_1(z) + \cdots + f_n^s(z)h_n(z) = 1 \quad \text{for all} \quad z \in \mathbb{B}_I.$$

Since on the slice $L_I$ the pointwise multiplication of two (complex) holomorphic functions coincide with their regular product, we can write that

$$f_1^s * h_1(z) + \cdots + f_n^s * h_n(z) = 1 \quad \text{for all } z \in \mathbb{B}_I.$$

At this point, thanks to the Identity Principle 1.8, if we consider the unique regular extension of $f_1^s * h_1(z) + \cdots + f_n^s * h_n(z)$, we obtain that

$$\operatorname{ext}(f_1^s * h_1 + \cdots + f_n^s * h_n)(q) = 1 \quad \text{for all } q \in \mathbb{B}.$$

Moreover, again by the Identity Principle 1.8, we get that

$$\begin{aligned}\operatorname{ext}(f_1^s * h_1 + \cdots + f_n^s * h_n)(q) &= \operatorname{ext}(f_1^s * h_1)(q) + \cdots + \operatorname{ext}(f_n^s * h_n)(q)\\ &= f_1^s * \operatorname{ext}(h_1)(q) + \cdots + f_n^s * \operatorname{ext}(h_n)(q).\end{aligned}$$

Therefore we have that

$$f_1 * f_1^c * \operatorname{ext}(h_1)(q) + \cdots + f_n * f_n^c * \operatorname{ext}(h_n)(q) = 1 \quad \text{for all } q \in \mathbb{B}.$$

Now, Corollary 2.4 guarantees that $f_i^c$ is bounded for every $i \in \{1, \dots, n\}$ (because $f_i$ does), and Proposition 1.12 implies that $\operatorname{ext}(h_i)$ is bounded for every $i \in \{1, \dots, n\}$ (because $h_i$ does). Hence, setting $g_i = f_i^c * \operatorname{ext}(h_i)$ for every $i = 1, \dots, n$, leads us to conclude. □

It is evident that there are many challenging open questions left to answer. Certainly the search for a more general version of the Corona Theorem in the quaternionic setting is one of those. In fact it would be very interesting if, endowing $\mathbb{C}^2$ with additional structure, we could find a solution of the Corona Problem in dimension 2. Another aspect that would be fascinating to investigate is the study of certain classes of operators on quaternionic Hardy spaces. It is not difficult to see that it is possible to define the *shift* operator on $H^2(\mathbb{B})$, but we would like also to address the problem of defining a *composition* operator. We plan to further deepen the study of quaternionic Hardy spaces that we began in this Thesis in the next future.

# Bibliography


[1] M. Abate, *Iteration theory of holomorphic mappings on Taut manifolds*, Mediterranean Press, Rende, 1989.

[2] L. Aizenberg, *Multidimensional analogues of Bohr's theorem on power series*, Proc. Amer. Math. Soc. **128** (2000), 1147-1155.

[3] D. Alpay, F. Colombo, I. Sabadini, *Schur functions and their realizations in the slice hyperholomorphic setting*, Integral Equations Operator Theory **72** (2012), 253289.

[4] D. Alpay, F. Colombo, I. Sabadini, *Pontryagin De Branges Rovnyak spaces of slice hyperholomorphic functions*, to appear in J. Anal. Math., (2013).

[5] C. Bisi, G. Gentili, *Möbius transformations and the Poincaré distance in the quaternionic setting*, Indiana Univ. Math. J., **58** (2009), 2729-2764.

[6] H. Bohr, *A theorem concerning power series* Proc. Lond. Math. Soc. (2), **13** (1914), 1-5.

[7] R. B. Burckel, *An introduction to classical complex analysis, Vol. 1*, Birkhäuser , Basel, 1979.

[8] R. B. Burckel, D. E. Marshall, D. Minda, P. Poggi-Corradini, T. J. Ransford, *Area, capacity and diameter versions of Schwarz's lemma*, Conform. Geom. Dyn., **12** (2008), 133-152.

[9] C. Carathéodory, *Sur quelques applications du théorème de Landau-Picard*, C.R. Acad. Sci. Paris **144** (1907), 1203-1206.

[10] F. Colombo, G. Gentili, I. Sabadini, *A Cauchy kernel for slice regular functions*, Ann. Global Anal. Geom. **37** (2010), 361-378.

[11] F. Colombo, G. Gentili, I. Sabadini, D. Struppa, *Extension results for slice regular functions of a quaternionic variable*, Adv. Math., **222** (2009), 1793-1808.

[12] F. Colombo, I. Sabadini, F. Sommen, D.C. Struppa, *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, vol. 39, Birkhäuser, Boston, 2004.

[13] F.Colombo, I. Sabadini, D. C. Struppa, *Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions*, Progress in Mathematics, vol. 289, Birkhäuser/Springer, Basel, 2011.

[14] C. G. Cullen, *An integral theorem for analytic intrinsic functions on quaternions*, Duke Math. J. **32** (1965), 139-148.

[15] C. Della Rocchetta, G. Gentili, G. Sarfatti, *The Bohr theorem for slice regular functions* Math. Nachr. **285** (2012), 2093-2105.

[16] C. Della Rocchetta, G. Gentili, G. Sarfatti, *A Bloch-Landau theorem for slice regular functions*, in Advances in Hypercomplex Analysis, ed. by G. Gentili, I. Sabadini, M. V. Shapiro, F. Sommen, D. C. Struppa, Springer INdAM Series, Springer, Milan, 2013, pp. 55-74.

[17] P. L. Duren, *Theory of $H^p$ spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.

[18] P. L. Duren, W. Rudin, *Distortion in several variables*, Complex Variables Theory Appl. **5** (1986), 323-326.

[19] S. R. Finch, *Mathematical constants*, Encyclopedia of Mathematics and its Applications, 94, Cambridge University Press, Cambridge, 2003.

[20] R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comment. Math. Helv. **7** (1934),307-330.

[21] R. Fueter, *Über Hartogs'schen Satz*, Comm. Math. Helv. **12** (1939), 75-80.

[22] T. W. Gamelin, *Wolff's proof of the corona theorem*, Israel J. Math., **37** (1980), 113-119.

[23] R. J. Gardner, *Geometric tomography*, (Second edition), Encyclopedia of Mathematics and its Applications, 58. Cambridge University Press, Cambridge, 2006.

[24] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, Vol 96. Academic Press, New York-London, 1981.

[25] G. Gentili, S. Salamon, C. Stoppato *Twistor transforms of quaternionic functions and orthogonal complex structures*, to appear in J. Eur. Math. Soc., arXiv:1205.3513v1 [math.DG], (2012).

[26] G. Gentili, G. Sarfatti, *Landau-Toeplitz theorems for slice regular functions*, Preprint, www.math.unifi.it/users/sarfatti/LTPreprint.pdf, (2013).

[27] G. Gentili, C. Stoppato *Zeros of regular functions and polynomials of a quaternionic variable*, Mich. Math. J., **56** (2008), 655-667.

[28] G. Gentili, C. Stoppato *The open mapping theorem for regular quaternionic functions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **VIII** (2009), 805-815.

[29] G. Gentili, C. Stoppato, *The zero sets of slice regular functions and the open mapping theorem*, in Hypercomplex Analysis and Applications, ed. by I. Sabadini, F. Sommen, Trends in Mathematics, Birkhäuser, Basel, 2011, pp. 95-107.

[30] G. Gentili, C. Stoppato, D. C. Struppa *Regular functions of a quaternionic variable*, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.

[31] G. Gentili, D. C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C. R. Math. Acad. Sci. Paris, **342** (2006), 741-744.

[32] G. Gentili, D. C. Struppa, *A new theory of regular functions of a quaternionic variable*, Adv. Math., **216** (2007), 279-301.

[33] G. Gentili, D. C. Struppa. *On the multiplicity of zeroes of polynomials with quaternionic coefficients*, Milan J. Math., **76** (2008), 15-25.

[34] G. Gentili, D. C. Struppa. *On the real part of slice regular functions*, Preprint, https://neo.math.unifi.it/users/gentili/lavoripdf/GentiliStruppa12.pdf, (2012).

[35] G. Gentili, D. C. Struppa, F. Vlacci, *The fundamental theorem of algebra for Hamilton and Cayley numbers*, Math. Z., **259** (2008), 895-902.

[36] G. Gentili, I. Vignozzi, *The Weierstrass factorization theorem for slice regular functions over the quaternions*, Ann. Global Anal. Geom., **40** (2011), 435-466.

[37] K. Gürlebeck, J. Morais, *Bohr type theorem for monogenic power series*, Comput. Methods Funct. Theory, **9** (2009), 633-651.

[38] K. Gürlebeck, J. Morais, *On the development of Bohr's phenomen in the context of quaternionic analysis and related problems*, in Algebraic structures in partial differential equations related to complex and Clifford analysis, Ho Chi Minh City Univ. Educ. Press, Ho Chi Minh City, 2010, pp. 9-24.

[39] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Series in Modern Analysis Prentice-Hall, Englewood Cliffs, N.J., 1962.

[40] T. Lachand-Robert, É. Oudet *Bodies of constant width in arbitrary dimension*, Math. Nachr. **280** (2007), 740-750.

[41] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, Vol. 131, Springer, New York, 1991.

[42] E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Springer, Berlin, 1929.

[43] E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Math. Z., **30** (1929), 608-634.

[44] E. Landau, O. Toeplitz, *Über die größte Schwankung einer analytischen Funktion in einem Kreise*, Arch. der Math. und Physik, **11** (1907), 302-307.

[45] S. Lang, *Complex analysis*, Graduate Texts in Mathematics, Vol. 103, Springer, New York, 1999.

[46] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966

[47] C. Stoppato, *Poles of regular quaternionic functions*, Complex Var. and Elliptic Equat. **54** (2009), 1001-1018.

[48] C. Stoppato, *Regular Moebius transformations of the space of quaternions*, Ann. Global Anal. Geom., **39** (2010), 387-401.

[49] A. Sudbery, *Quaternionic analysis*, Math. Proc. Camb. Philos. Soc., **85** (1979), 199-224.

[50] F. Vlacci, *Regular composition for slice-regular functions of quaternionic variable*, in Advances in Hypercomplex Analysis, ed. by G. Gentili, I. Sabadini, M. V. Shapiro, F. Sommen, D. C. Struppa, Springer INdAM Series, Springer, Milan, 2013, pp. 141-147.