Ideals of regular functions of a quaternionic variable

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Abstract

In this paper we prove that, for any $n \in \mathbb{N}$, the ideal generated by n slice regular functions $f_1, \dots, f_n$ having no common zeros concides with the entire ring of slice regular functions. The proof required the study of the non-commutative syzygies of a vector of regular functions, that manifest a different character when compared with their complex counterparts.

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1 Introduction

The theory of slice regular functions of a quaternionic variable (often simply called regular functions) has been introduced in [13], [14], and further developed in a series of papers, including in particular [2], where most of the recent developments are discussed. The full theory is presented in the monograph [12], while an extension of the theory to the case of real alternative algebras is discussed in [16], [17] and [18]. The theory of regular functions has been applied to the study of a non-commutative functional calculus, (see for example the monograph [5] and the references therein) and to address the problem of the construction and classification of orthogonal complex structures in open subsets of the space $\mathbb{H}$ of quaternions (see [8]). In many cases, the results one obtains in the theory of regular functions are inspired by complex analysis, though they often require essential modifications, due to the different nature of zeroes and singularities of regular

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functions. Examples of these results, including those on power and Laurent series expansions, can be found in [1], [4], [10], [11], [15], [21], [23], [24]. Recent results of geometric theory of regular functions appear in [6], [7], [9].

In this paper we study the ideals in the (non-commutative) ring of regular functions, and we prove an analogue of a classical result for one (and several) complex variables, namely the fact that if a family of holomorphic functions has no common zeroes, then it generates the entire ring of holomorphic functions. In her doctoral dissertation [22], the author proved that this was the case for regular functions as well (in fact, she showed that this was true for bounded regular functions, an analogue of the corona theorem), under the strong hypothesis that not only the functions could not have common zeroes, but also that the functions could not have zeroes on the same spheres.

Here we show that such a request is not necessary, at least for the case of regular functions (we do not consider the bounded case), by employing some delicate local properties of such functions. We show how to reduce the study of the problem to the case of holomorphic functions, and we then use the coherence of the sheaf of holomorphic functions to show that the local solution to the problem extends to a global one.

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2 Preliminary Results

Let $\mathbb{H}$ denote the non commutative real algebra of quaternions with standard basis $\{1, i, j, k\}$. The elements of the basis satisfy the multiplication rules

$$i^2 = j^2 = k^2 = -1,\ ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,$$

which, if we set 1 as the neutral element, extend by distributivity to all $q = x_0 + x_1 i + x_2 j + x_3 k$ in $\mathbb{H}$. Every element of this form is composed by the *real* part $\mathrm{Re}(q) = x_0$ and the *imaginary* part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The *conjugate* of $q \in \mathbb{H}$ is then $\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q)$ and its *modulus* is defined as $|q|^2 = q\bar{q}$. We can therefore calculate the multiplicative inverse of each $q \neq 0$ as $q^{-1} = \frac{\bar{q}}{|q|^2}$. Notice that for all non real $q \in \mathbb{H}$, the quantity $\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$ is an imaginary unit, that is a quaternion whose square equals -1. Then we can express every $q \in \mathbb{H}$ as $q = x + yI$, where x, y are real (if $q \in \mathbb{R}$, then $y = 0$) and I is an element of the unit 2-dimensional sphere of purely imaginary quaternions,

$$\mathbb{S} = \{q \in \mathbb{H} \mid q^2 = -1\}.$$

In the sequel, for every $I \in \mathbb{S}$ we will denote by L_I the plane $\mathbb{R} + \mathbb{R}I$, isomorphic to $\mathbb{C}$ and, if Ω is a subset of $\mathbb{H}$, by Ω_I the intersection $\Omega \cap L_I$. As explained in [12], the natural domains of definition for slice regular functions are the symmetric slice domains. These domains actually play the role played by domains of holomorphy in the complex case:

Definition 2.1. *Let Ω be a domain in $\mathbb{H}$ that intersects the real axis. Then:*

1. *Ω is called a* slice domain *if, for all $I \in \mathbb{S}$, the intersection Ω_I with the complex plane L_I is a domain of L_I;*
2. *Ω is called a* symmetric domain *if for all $x, y \in \mathbb{R}$, $x + yI \in \Omega$ implies $x + y\mathbb{S} \subset \Omega$.*

We can now recall the definition of slice regularity. From now on, Ω will always be a symmetric slice domain in $\mathbb{H}$, unless differently stated.

Definition 2.2. *A function $f : \Omega \to \mathbb{H}$ is said to be* (slice) regular *if, for all $I \in \mathbb{S}$, its restriction f_I to Ω_I has continuous partial derivatives and satisfies*

$$\overline{\partial}_I f(x+yI) := \frac{1}{2}\Big(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\Big) f_I(x+yI) = 0$$

for all $x + yI \in \Omega_I$.

In the sequel we may refer to the vanishing of $\overline{\partial}_I f$ saying that the restriction f_I is holomorphic on Ω_I.
A basic result in the theory of regular functions, that relates slice regularity and classical holomorphy, is the following, [12, 14]:

Lemma 2.3 (Splitting Lemma)**.** *If f is a regular function on Ω, then for every $I \in \mathbb{S}$ and for every $J \in \mathbb{S}$, J orthogonal to I, there exist two holomorphic functions $F, G : \Omega_I \to L_I$, such that for every $z = x + yI \in \Omega_I$, it holds*

$$f_I(z) = F(z) + G(z)J.$$

One of the first consequences of the previous result is the following version of the Identity Principle, [14]:

Theorem 2.1 (Identity Principle)**.** *Let f be a regular function on Ω. Denote by Z_f the zero set of f, $Z_f = \{q \in \Omega |\, f(q) = 0\}$. If there exists $I \in \mathbb{S}$ such that $\Omega_I \cap Z_f$ has an accumulation point in Ω_I, then f vanishes identically on Ω.*

In the sequel we will use an important extension result (see [1, 2]) that we will present in the following special formulation:

Lemma 2.4 (Extension Lemma)**.** *Let Ω be a symmetric slice domain and choose $I \in \mathbb{S}$. If $f_I : \Omega_I \to \mathbb{H}$ is holomorphic, then setting*

$$f(x+yJ) = \frac{1}{2}[f_I(x+yI) + f_I(x-yI)] + J\frac{I}{2}[f_I(x-yI) - f_I(x+yI)]$$

extends f_I to a regular function $f : \Omega \to \mathbb{H}$. Moreover f is the unique extension and it is denoted by $\mathrm{ext}(f_I)$*.*

The product of two regular functions is not, in general, regular. To guarantee the regularity we have to use a different multiplication operation, the $*$-product. From now on, if F is a holomorphic function, we will use the notation:

$$\hat{F}(z) := \overline{F(\bar{z})}.$$

Definition 2.5. *Let f, g be regular functions on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G, H, K be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ, g_I = H + KJ$. Consider the holomorphic function defined on Ω_I by*

$$f_I * g_I(z) = \Big[F(z)H(z) - G(z)\hat{K}(z)\Big] + \Big[F(z)K(z) + G(z)\hat{H}(z)\Big]\, J. \tag{1}$$

Its regular extension $\mathrm{ext}(f_I * g_I)$ *is called the* regular product *(or* $*$-product*) of f and g and it is denoted by $f * g$.*

Notice that the $*$-product is associative but generally is not commutative. Its connection with the usual pointwise product is stated by the following result.

Proposition 2.6. *Let $f(q)$ and $g(q)$ be regular functions on Ω. Then, for all $q \in \Omega$,*

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if} \quad f(q) \neq 0 \\ 0 & \text{if} \quad f(q) = 0 \end{cases} \tag{2}$$

Corollary 2.7. *If f, g are regular functions on a symmetric slice domain Ω and $q \in \Omega$, then $f * g(q) = 0$ if and only if $f(q) = 0$ or $f(q) \neq 0$ and $g(f(q)^{-1}qf(q)) = 0$.*

The regular product coincides with the pointwise product for the special class of regular functions defined as follows.

Definition 2.8. *A regular function $f : \Omega \to \mathbb{H}$ such that $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$ is called a* slice preserving *regular function.*

Lemma 2.9. *Let f, g be regular functions on a symmetric slice domain Ω. If f is slice preserving, then fg is a regular function on Ω and $f * g = fg = g * f$.*

The following operations are naturally defined in order to study the zero set of regular functions.

Definition 2.10. *Let f be a regular function on a symmetric slice domain Ω. Choose $I, J \in \mathbb{S}$ with $I \perp J$ and let F, G be holomorphic functions from Ω_I to L_I such that $f_I = F + GJ$. If f_I^c is the holomorphic function defined on Ω_I by*

$$f_I^c(z) = \hat{F}(z) - G(z)J. \tag{3}$$

Then the regular conjugate *of f is the regular function defined on Ω by $f^c = \mathrm{ext}(f_I^c)$, and the* symmetrization *of f is the regular function defined on Ω by $f^s = f * f^c = f^c * f$.*

If the regular function $f : \Omega \to \mathbb{H}$ is such that $f_I(z) = F(z) + G(z)J$, with $F, G : \Omega_I \to L_I$ holomorphic functions, then it easy to see that (see, e.g., [12])

$$f_I^s = f_I * f_I^c = f_I^c * f_I = F(z)\hat{F}(z) + G(z)\hat{G}(z). \tag{4}$$

Hence $f^s(\Omega_I) \subseteq L_I$ for every $I \in \mathbb{S}$, i.e., f^s is slice preserving. Moreover if g is a regular function on Ω, then

$$(f * g)^c = g^c * f^c \quad \text{and} \quad (f * g)^s = f^s g^s = g^s f^s. \tag{5}$$

Zeroes of regular functions have a nice geometric property:

Theorem 2.2. *Let f be a regular function on a symmetric slice domain Ω. If f does not vanish identically, then its zero set consists of isolated points or isolated 2-spheres of the form $x + y\mathbb{S}$ with $x, y \in \mathbb{R}$, $y \neq 0$.*

Notice that $f(q)^{-1}qf(q)$ belongs to the same sphere $x + y\mathbb{S}$ as q. Hence each zero of $f * g$ in $x + y\mathbb{S}$ corresponds to a zero of f or to a zero of g in the same sphere.

Lemma 2.11. *Let f be a regular function on a symmetric slice domain Ω and let f^s be its symmetrization. Then for each $S = x + y\mathbb{S} \subset \Omega$ either f^s vanishes identically on S or it has no zeroes in S.*

The *regular reciprocal* f^{-*} of a regular function f defined on a symmetric slice domain Ω can now be defined in $\Omega \setminus Z_{f^s}$ as

$$f^{-*} = (f^s)^{-1} f^c, \tag{6}$$

where Z_{f^s} denotes the zero set of the symmetrization f^s.

Remark 2.12. If f is a regular function defined on a slice symmetric domain of $\mathbb{H}$, then its regular reciprocal $f^{-*} = (f^s)^{-1} f^c$ has a sphere of poles at Z_{f^s} and is a quasi regular function in the sense of [24].

3 Ideals generated by two regular functions

In this section we will prove that if f_1 and f_2 are two regular functions with no common zeroes on a symmetric slice domain Ω, then they generate the entire ring of regular functions on Ω, i.e. there are two regular functions h_1 and h_2 on Ω such that $f_1 * h_1 + f_2 * h_2 = 1$.

We begin with a simple application of the Borsuk-Ulam Theorem.

Lemma 3.1. *Let f be a regular function on a symmetric slice domain Ω, let $q = x + yL \in \Omega$ be a point which is not a spherical zero for f, and for every $I \in \mathbb{S}$ let $f(x+yI) = F^I(x+yI) + G^I(x+yI)J$, for some J orthogonal to I. Then there exists at least one imaginary unit $I \in \mathbb{S}$ such that either $F^I(x+yI)F^I(x-yI) \neq 0$ or $G^I(x+yI)G^I(x-yI) \neq 0$.*

Proof. Denote by $\langle \, , \, \rangle$ the scalar product in $\mathbb{R}^4$, and note that we can write $F^I(x+yI) = \mathrm{Re}(f(x+yI)) + \langle f(x+yI), I\rangle I$. The function $\psi : \mathbb{S} \to \mathbb{R}^2$ defined by $\psi(I) = (\mathrm{Re}(f(x+yI)), \langle f(x+yI), I\rangle)$ is a continuous function from the 2-dimensional sphere $\mathbb{S}$ to $\mathbb{R}^2$. Thus, by the Borsuk-Ulam Theorem, we know that there is at least one choice of I for which $\psi(I) = \psi(-I)$, that implies $F^I(x+yI) = F^I(x-yI)$. If for this value of I it is $F^I(x+yI) \neq 0$ the lemma is proved. If, on the other hand, $F^I(x+yI) = F^I(x-yI) = 0$, then either $G^I(x+yI)$ or $G^I(x-yI)$ does not vanish (since f does not have a spherical zero in q). If both $G^I(x+yI) \neq 0$ and $G^I(x-yI) \neq 0$ we obtain the thesis. Otherwise suppose, for instance, $G^I(x+yI) = 0$ and $G^I(x-yI) \neq 0$. To find $M \in \mathbb{S}$ such that $G^M(x+yM)G^M(x-yM) \neq 0$, we will use the Representation Formula to write, for any $M \in \mathbb{S}$, $M \perp J$

$$\begin{aligned}
f(x+yM) &= \frac{1}{2}\left(f(x+yI) + f(x-yI)\right) + \frac{MI}{2}\left(f(x-yI) - f(x+yI)\right) \\
&= \frac{1+MI}{2} G^I(x-yI)J \\
&= \frac{1 - \langle M, I\rangle + M \times I}{2}\left(\mathrm{Re}(G^I(x-yI)) + \mathrm{Im}(G^I(x-yI))\right) J.
\end{aligned}$$

Since, by the Splitting Lemma,

$$f(x+yM) = F^M(x+yM) + G^M(x+yM)J$$

then

$$\mathrm{Re}(G^M(x+yM)) = \frac{1 - \langle M, I\rangle}{2}\,\mathrm{Re}(G^I(x-yI)).$$

Now, if $\mathrm{Re}(G^I(x-yI)) \neq 0$, we get that for any $M \in \mathbb{S}$, $M \perp J$, $M \neq \pm I$,

$$\mathrm{Re}(G^M(x \pm yM)) \neq 0$$

and hence

$$G^M(x+yM)G^M(x-yM) \neq 0.$$

In the case $\mathrm{Re}(G^I(x-yI)) = 0$, then $\mathrm{Re}(G^I(x-yI)I) \neq 0$ since $G^I(x-yI) \neq 0$. Performing the same computation for the regular function $\tilde{f}(x+yM) = f(x+yM)I$ we get that, with the usual splitting

$$\tilde{f}(x+yM) = \tilde{F}^M(x+yM) + \tilde{G}^M(x+yM)J,$$

the following inequalities hold

$$\tilde{G}^M(x+yM) \neq 0 \ \text{ and } \ \tilde{G}^M(x-yM) \neq 0,$$

for any $M \in \mathbb{S}$, $M \perp J$, $M \neq \pm I$. If we choose $M = \pm IJ$, and take into account that

$$\tilde{f}(x+yM) = F^M(x+yM)I + G^M(x+yM)JI$$

we reach the conclusion

$$F^M(x+yM)I = \tilde{G}^M(x+yM)J \neq 0 \ \text{ and } \ F^M(x-yM)I = \tilde{G}^M(x-yM)J \neq 0.$$

□

In the final part of this section we will prove that two regular functions having no common zeroes locally generate the entire ring of regular functions.

Theorem 3.2. *Let $q = x+yL \in \mathbb{H}$ and let f_1, f_2 be two functions, regular in a symmetric slice neighborhood Ω of q and not simultaneously vanishing at q. Then it is possible to find $I \in \mathbb{S}$ and a symmetric domain W in Ω_I containing $x+yI$ and $x-yI$, such that the equation*

$$f_1 * h_1 + f_2 * h_2 = 1. \tag{7}$$

restricted to W has local holomorphic solutions $h_1, h_2 : W \to \mathbb{H}$ at any point of W.

Proof. Let $q = x+yL$. Notice that, by hypothesis, $x+y\mathbb{S}$ cannot be a spherical zero for both f_1 and f_2. By the Splitting Lemma, for any $I \in \mathbb{S}$, we can represent, for $\ell = 1,2$, the functions f_ℓ via functions holomorphic in a domain in L_I containing $(x+y\mathbb{S}) \cap L_I$ as

$$f_\ell(z) = f_{\ell|I}(z) = F_\ell(z) + G_\ell(z)J,$$

where $J \in \mathbb{S}$ is orthogonal to I. Similarly, the functions h_ℓ that we are looking for can be written as

$$h_\ell(z) = h_{\ell|I}(z) = H_\ell(z) + K_\ell(z)J,$$

for suitable holomorphic functions H_ℓ and K_ℓ. Let us apply Lemma 3.1 to f_1, and choose I such that, without loss of generality, $F_1(x+yI)F_1(x-yI) \neq 0$. As a consequence we get that also $\overline{F_1(x+yI)}\,\overline{F_1(x-yI)} \neq 0$, $f_1(x+yI)f_1(x-yI) \neq 0$ and $f_1^c(x+yI)f_1^c(x-yI) \neq 0$. Notice that the previous inequalities hold outside the discrete subset of Ω_I consisting of all zeroes of $F_1(z)$ and

$\hat{F}_1(z)$, hence in a symmetric domain U in Ω_I containing both $x+yI$ and $x-yI$. Using (1), it is immediate to see that (7) can be rewritten as a system of two equations for holomorphic functions in L_I, namely, omitting the variable z,

$$\begin{cases} F_1H_1 - G_1\hat{K}_1 + F_2H_2 - G_2\hat{K}_2 = 1 \\ F_1K_1 + G_1\hat{H}_1 + F_2K_2 + G_2\hat{H}_2 = 0. \end{cases} \tag{8}$$

Since F_1 does not vanish on U, there exist H_1, K_1, H_2, K_2, holomorphic in U, which define a solution of the first equation of (8). In general, the functions H_1, K_1, H_2, K_2 will not satisfy system (8). However, one can modify the solution to the first equation by adding an element of the syzygies of (F_1, G_1, F_2, G_2) and try to solve the system. Since the latter functions have no common zeroes on U, their syzygies (see, e.g., [3]) are generated by the columns of the following matrix

$$A = \begin{pmatrix} G_1 & F_2 & G_2 & 0 & 0 & 0 \\ -F_1 & 0 & 0 & F_2 & G_2 & 0 \\ 0 & -F_1 & 0 & -G_1 & 0 & G_2 \\ 0 & 0 & -F_1 & 0 & -G_1 & -F_2 \end{pmatrix}.$$

Hence the general solution to the first equation of (8) is given by

$$\begin{pmatrix} H_1 + \hat{\beta}_1G_1 + \hat{\beta}_2F_2 + \hat{\beta}_3G_2 \\ -\hat{K}_1 - \hat{\beta}_1F_1 + \hat{\beta}_4F_2 + \hat{\beta}_5G_2 \\ H_2 - \hat{\beta}_2F_1 - \hat{\beta}_4G_1 + \hat{\beta}_6G_2 \\ -\hat{K}_2 - \hat{\beta}_3F_1 - \hat{\beta}_5G_1 - \hat{\beta}_6F_2 \end{pmatrix} \tag{9}$$

where $\beta_1, \dots, \beta_6$ are arbitrary holomorphic functions in U. Consider now the matrix B of holomorphic functions defined by

$$B = \begin{pmatrix} \hat{F}_1 & 0 & 0 & -\hat{F}_2 & -\hat{G}_2 & 0 \\ \hat{G}_1 & \hat{F}_2 & \hat{G}_2 & 0 & 0 & 0 \\ 0 & 0 & \hat{F}_1 & 0 & \hat{G}_1 & \hat{F}_2 \\ 0 & -\hat{F}_1 & 0 & -\hat{G}_1 & 0 & \hat{G}_2 \end{pmatrix}. \tag{10}$$

In order to obtain a solution of (8) we now need to request that the vector

$$\begin{pmatrix} K_1 + \beta_1\hat{F}_1 - \beta_4\hat{F}_2 - \beta_5\hat{G}_2 \\ \hat{H}_1 + \beta_1\hat{G}_1 + \beta_2\hat{F}_2 + \beta_3\hat{G}_2 \\ K_2 + \beta_3\hat{F}_1 + \beta_5\hat{G}_1 + \beta_6\hat{F}_2 \\ \hat{H}_2 - \beta_2\hat{F}_1 - \beta_4\hat{G}_1 + \beta_6\hat{G}_2 \end{pmatrix} \tag{11}$$

belongs to the syzygies of (F_1, G_1, F_2, G_2). That is, setting $H = {}^t(K_1, \hat{H}_1, K_2, \hat{H}_2)$, we need to find $\beta = {}^t(\beta_1, \dots, \beta_6)$ and $\alpha = {}^t(\alpha_1, \dots, \alpha_6)$ vectors of holomorphic functions such that

$$H + B\beta = A\alpha,$$

namely such that

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = H. \tag{12}$$

Our next goal is to establish that the rank of the (4×12)-matrix $(A, -B)$ equals 4 on the entire U. Recalling that F_1 and $\hat{F}_1$ do not vanish in U, an easy computation shows that both A and B have rank 3: in fact in both matrices the first three columns are a maximal subset of linearly independent columns on U. Denote by $A^1, \dots, A^6$ the columns of A and by $B^1, \dots, B^6$ the columns of B. The rank of $(A, -B)$ is not maximum at a point $z \in U$ if and only if all the determinants of the six (4×4)-matrices

$$M_1 = (A^1, A^2, A^3, B^1), \ \dots \ , \ M_6 = (A^1, A^2, A^3, B^6)$$

vanish at z. Namely, the rank of $(A, -B)$ is 3 where (in U) the following system is satisfied

$$\begin{cases} F_1^2(F_1\hat{F}_1 + G_1\hat{G}_1) = 0 \\ F_1^2(F_1\hat{F}_2 + G_2\hat{G}_1) = 0 \\ F_1^2(F_1\hat{G}_2 - F_2\hat{G}_1) = 0 \\ F_1^2(G_1\hat{F}_2 - G_2\hat{F}_1) = 0 \\ F_1^2(F_2\hat{F}_1 + G_1\hat{G}_2) = 0 \\ F_1^2(F_2\hat{F}_2 + G_2\hat{G}_2) = 0 \end{cases} \tag{13}$$

Taking into account that F_1 is nonvanishing in U, the rank equals 3 if and only if the following six equations are contemporarily satisfied:

$$F_1\hat{F}_1 + G_1\hat{G}_1 = 0 \tag{14}$$

$$F_1\hat{F}_2 + G_2\hat{G}_1 = 0 \tag{15}$$

$$F_1\hat{G}_2 - F_2\hat{G}_1 = 0 \tag{16}$$

$$G_1\hat{F}_2 - G_2\hat{F}_1 = 0 \tag{17}$$

$$F_2\hat{F}_1 + G_1\hat{G}_2 = 0 \tag{18}$$

$$F_2\hat{F}_2 + G_2\hat{G}_2 = 0 \tag{19}$$

Equation (14) and (19) can be written in U as the quaternionic equations $f_1^s(z) = 0$ and $f_2^s(z) = 0$. We will now investigate the meaning of the other terms. Using (1) and the fact that U is symmetric, we get

$$\begin{aligned} (f_1^c * f_2)_I(z) &= (F_2(z)\hat{F}_1(z) + G_1(z)\hat{G}_2(z)) - (G_1(z)\hat{F}_2(z) - G_2(z)\hat{F}_1(z))J \\ (f_2^c * f_1)_I(z) &= (F_1(z)\hat{F}_2(z) + G_2(z)\hat{G}_1(z)) + (G_1(z)\hat{F}_2(z) - G_2(z)\hat{F}_1(z))J \\ (f_1^c * f_2)_I(\bar{z}) &= \overline{(F_1(z)\hat{F}_2(z) + G_2(z)\hat{G}_1(z))} + \overline{(F_1(z)\hat{G}_2(z) - F_2(z)\hat{G}_1(z))}J \\ (f_2^c * f_1)_I(\bar{z}) &= \overline{(F_2(z)\hat{F}_1(z) + G_1(z)\hat{G}_2(z))} + (F_1(z)\hat{G}_2(z) - F_2(z)\hat{G}_1(z))J. \end{aligned}$$

Hence if the matrix $(A, -B)$ has rank 3 at $z \in U$, then equations (15)-(18) imply that $(f_1^c * f_2)_I(z) = (f_2^c * f_1)_I(z) = (f_1^c * f_2)_I(\bar{z}) = (f_2^c * f_1)_I(\bar{z}) = 0$. Consequently if $(A, -B)$ has rank 3 at $z \in U$, then we have

$$\begin{cases} f_1^s(z) = 0 \\ f_1^c * f_2(z) = 0 \\ f_2^c * f_1(z) = 0 \\ f_1^c * f_2(\bar{z}) = 0 \\ f_2^c * f_1(\bar{z}) = 0 \\ f_2^s(z) = 0 \end{cases} \tag{20}$$

Consider at first the case in which f_1 has a zero in $x+y\mathbb{S}$. If $q = x+yL$, let $Z = \{x_n + y_n\mathbb{S}\}_{n\in\mathbb{N}} \subset \Omega$ be the set of spheres, different from $x+y\mathbb{S}$, containing zeroes of $f_1^c * f_2$. Then $V = U \setminus Z$ is a symmetric domain in L_I containing $x+yI$ (and $x-yI$) on which $f_1^c * f_2$ never vanishes. To prove this fact, thanks to Lemma 2.11 and to (2.6), we obtain

$$0 = f^s(x+yI) = f_1^c * f_1(x+yI) = f_1^c(x+yI)f_1(f_1^c(x+yI)^{-1}(x+yI)f_1^c(x+yI)).$$

As we pointed out $f_1^c(x+yI) \neq 0$, and therefore

$$f_1(f_1^c(x+yI)^{-1}(x+yI)f_1^c(x+yI)) = 0.$$

Since, by hypothesis, f_2 cannot vanish at a zero of f_1, we get

$$f_1^c * f_2(x+yI) = f_1^c(x+yI)f_2(f_1^c(x+yI)^{-1}(x+yI)f_1^c(x+yI)) \neq 0.$$

An analogous computation shows that $f_1^c * f_2(x-yI) \neq 0$, thus proving our assertion concerning V. As a consequence one of the holomorphic determinants appearing in (13) (namely one of those related to (17) or (18)) cannot vanish at $x+yI$ and one cannot vanish at $x-yI$: let us call this functions N_1 and N_2. Let $Z_N \subset \Omega_I$ be the set of common zeroes of N_1 and N_2. Then $W = V \setminus Z_N$ is a symmetric domain in Ω_I containing (and hence a neighborhood of) $x+yI$ and $x-yI$. Since for every point p of W one of the holomorphic determinants $N_1(z), N_2(z)$ is non vanishing at $p \in W$, using the classical Rouché - Capelli method it is now possible to find a local holomorphic solution $H = {}^t(K_1, \hat{H}_1, K_2, \hat{H}_2)$ of the system (12) in a neighborhood $A_p \subseteq W \subseteq \Omega_I$ of p. This solution is such that the holomorphic functions $h_1(z) = H_1(z) + K_1(z)J$ and $h_2(z) = H_2(z) + K_2(z)J$ are local holomorphic solutions of the restriction of equation (7) to A_p, i.e., it is such that

$$f_1 * h_1(z) + f_2 * h_2(z) = 1$$

for all $z \in A_p$. In the remaining case in which f_1 has no zeroes in $x+y\mathbb{S}$, then $f_1^s = F_1\hat{F}_1 + G_1\hat{G}_1 \neq 0$ at both $x+yI$ and $x-yI \in x+y\mathbb{S}$. Setting $N = F_1^2 f_1^s = F_1^2(F_1\hat{F}_1 + G_1\hat{G}_1)$ we proceed exactly as in the previous case and conclude. □

To extend the local result of Theorem 3.2 to Ω, and then to state it in a global version identifying a solution of (7) in the entire domain $\Omega \subseteq \mathbb{H}$, we will apply results from the theory of sheaves of germs of holomorphic functions in one complex variable. As a first step we prove a preliminary lemma to study the structure of the sheaf of the syzygies of a pair of regular functions (f_1, f_2) restricted to a suitable complex plane L_I.

Lemma 3.3. *Let $f_1, f_2 : \Omega \to \mathbb{H}$, $I \in \mathbb{S}$ and $W \subseteq \Omega_I$ be as in Theorem 3.2. If $\mathcal{O}$ is the sheaf of germs of holomorphic functions on W and if $\mathcal{K}$ is the sheaf of germs of solutions of equation*

$$f_1 * h_1 + f_2 * h_2 = 0 \tag{21}$$

restricted to W, then

$$\mathcal{K} \cong \mathcal{O}^8/\mathcal{O}^6.$$

Proof. As it appears in the proof (and using the same notations) of Theorem 3.2, $\mathcal{K}$ corresponds to the sheaf of germs of the local solutions of the system

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{22}$$

It easy to see that the matrix $(A, -B)$ has rank 4, both matrices A and B have rank 3, and A^1, A^2, A^3 and B^1, B^2, B^3 are maximal sets of linearly independent columns of A and B respectively. Therefore each element of $\mathcal{K}$ can be written in terms of 8 (germs of) holomorphic functions. Since with our choice of I, the matrix (A^1, A^2, A^3, B^ℓ) is invertible for some $\ell \in \{1, 2, 3\}$, say $\ell = 1$, we can write for arbitrary germs of holomorphic functions $\alpha_4, \alpha_5, \alpha_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$,

$$\begin{cases} \alpha_1 = \alpha_1(\alpha_4, \alpha_5, \alpha_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) \\ \alpha_2 = \alpha_2(\alpha_4, \alpha_5, \alpha_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) \\ \alpha_3 = \alpha_3(\alpha_4, \alpha_5, \alpha_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) \\ \beta_1 = \beta_1(\alpha_4, \alpha_5, \alpha_6, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6). \end{cases} \tag{23}$$

We therefore have a surjective map

$$\varphi : \mathcal{O}^8 \to \mathcal{K}.$$

The germ in $\mathcal{O}^8$ associated to the solution of (22) corresponding to the vector ${}^t(\alpha, \beta)$ belongs to $\ker \varphi$ if and only if

$$A\alpha = B\beta = 0,$$

which, in view of the rank of A and B, implies

$$\begin{cases} \alpha_1 = \alpha_1(\alpha_4, \alpha_5, \alpha_6) \\ \alpha_2 = \alpha_2(\alpha_4, \alpha_5, \alpha_6) \\ \alpha_3 = \alpha_3(\alpha_4, \alpha_5, \alpha_6) \\ \beta_1 = \beta_1(\beta_4, \beta_5, \beta_6) \\ \beta_2 = \beta_2(\beta_4, \beta_5, \beta_6) \\ \beta_3 = \beta_3(\beta_4, \beta_5, \beta_6). \end{cases} \tag{24}$$

As a consequence, the kernel of φ is isomorphic to $\mathcal{O}^6$ and $\mathcal{K}$ is isomorphic to $\mathcal{O}^8/\mathcal{O}^6$. □

Theorem 3.4. *Let $q \in \mathbb{H}$ and let f_1, f_2 be two functions, regular in a symmetric slice neighborhood Ω of q and not simultaneously vanishing at q. Then it is possible to find a symmetric slice neighborhood Σ of q and functions h_1, h_2 regular on Σ such that, on Σ,*

$$f_1 * h_1 + f_2 * h_2 = 1. \tag{25}$$

Proof. By Theorem 3.2 there exist $I \in \mathbb{S}$, a symmetric domain W in Ω_I and an an open covering $\mathcal{A} = \{A_t\}_{t \in T}$ of W whose elements A_t are such that the equation

$$f_1 * h_1 + f_2 * h_2 = 1 \tag{26}$$

restricted to $W \subseteq \Omega_I$ has a solution on each of them. With the same notation used in the proof of Theorem 3.2, equation (26) induces by restriction a local holomorphic solution to the complex system

$$\begin{cases} F_1 H_1 - G_1 \hat{K}_1 + F_2 H_2 - G_2 \hat{K}_2 = 1 \\ F_1 K_1 + G_1 \hat{H}_1 + F_2 K_2 + G_2 \hat{H}_2 = 0. \end{cases} \tag{27}$$

on W. Since the sheaf $\mathcal{K}$ of solutions to equation (21) is a coherent sheaf in view of Lemma 3.3 (see [19]), classical arguments show that system (27) has a global holomorphic solution on W. This is equivalent to the existence of globally defined holomorphic functions h_1, h_2 on W such that

$$f_1 * h_1 + f_2 * h_2 = 1$$

on W. Thanks to the Extension Lemma 2.4 the functions h_1, h_2 uniquely extend to the symmetric slice domain

$$\Sigma = \bigcup_{u+vI \in W} (u + v\mathbb{S})$$

as regular functions that satisfy

$$f_1 * h_1 + f_2 * h_2 = 1$$

everywhere in Σ. □

We are now ready to prove

Theorem 3.5. *Let f_1, f_2 be regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$, with no common zeroes in Ω. Then there exist h_1 and h_2 regular functions on Ω such that*

$$f_1 * h_1 + f_2 * h_2 = 1$$

on Ω.

Proof. By Theorem 3.4 there exists an open covering $\mathcal{U} = \{U_t\}_{t \in T}$ of Ω whose elements U_t are symmetric slice domains, such that the equation

$$f_1 * h_1 + f_2 * h_2 = 1 \tag{28}$$

has a solution on each of them. Consider now, for an arbitrary $J \in \mathbb{S}$, the slice Ω_J of Ω. With the same notation used in the proof of Theorem 3.2, equation (28) induces by restriction a local holomorphic solution to the complex system

$$\begin{cases} F_1 H_1 - G_1 \hat{K}_1 + F_2 H_2 - G_2 \hat{K}_2 = 1 \\ F_1 K_1 + G_1 \hat{H}_1 + F_2 K_2 + G_2 \hat{H}_2 = 0. \end{cases} \tag{29}$$

on Ω_J. This local solution on Ω_J is generated, via the Extension Lemma (as pointed out in the proof of Theorem 3.4), by a germ of local solution (locally defined on some slice Ω_I) which belongs to $\mathcal{O}^8$. Thanks to the Extension Lemma and Lemma 3.3, the sheaf $\mathcal{K}$ of local syzygies of f_1, f_2 restricted to Ω_J is coherent, and then classical arguments (see [19]) show that system (29) has a global holomorphic solution on Ω_J. This is equivalent to the existence of globally defined holomorphic functions h_1, h_2 on Ω_J such that

$$f_1 * h_1 + f_2 * h_2 = 1$$

on Ω_J. Thanks to the Extension Lemma 2.4 the functions h_1, h_2 uniquely extend to Ω as regular functions that satisfy

$$f_1 * h_1 + f_2 * h_2 = 1$$

everywhere on Ω. □

4 Ideals of regular functions

In this section we show how the proof of Theorem 3.5 can be extended to the case of $n(\geq 2)$ regular functions with no common zeroes.

Lemma 4.1. *Let $q = x + yL \in \mathbb{H}$ and let $f_1, \ldots, f_n$ be n regular functions in a slice symmetric neighborhood Ω of q and not simultaneously vanishing at q. Then there exist $I \in \mathbb{S}$ and a symmetric domain $U \subseteq \Omega_I$, containing $x + yI$, such that, if $f_\ell = F_\ell + G_\ell J$ is the splitting of f_ℓ on Ω_I, for $\ell = 1, \ldots, n$, then:*

1. *the rank of the $\left(2n \times \binom{2n}{2}\right)$-matrix A whose columns are the standard generators of the syzygies of the vector $(F_1, G_1, \ldots, F_n, G_n)$ equals $2n-1$ on U;*
2. *the rank of the $\left(2n \times \binom{2n}{2}\right)$-matrix B whose columns are the standard generators of the syzygies of the vector $(-\hat{G}_1, \hat{F}_1, \ldots, -\hat{G}_n, \hat{F}_n)$ equals $2n-1$ on U;*
3. *the rank of the $\left(2n \times 2\binom{2n}{2}\right)$-matrix $(A, -B)$ equals $2n$ on U.*

Proof. By hypothesis there exists ℓ such that f_ℓ does not have a spherical zero $x + y\mathbb{S}$ containing $q = x + yL$. We can suppose $\ell = 1$. Thanks to Lemma 3.1, as in the proof of Theorem 3.2, we can find $I \in \mathbb{S}$ such that, without loss of generality, $F_1 \neq 0$ and $\hat{F}_1 \neq 0$ on a symmetric domain U in Ω_I containing $x + yI$ (and $x - yI$). We can reorder the columns of A in such a way that all the elements in the subdiagonal are nonzero multiples of F_1 and all entries underneath the subdiagonal vanish. Moreover the matrix $\left(A^{2n}, A^{2n+1}, \ldots, A^{\binom{2n}{2}}\right)$ has a row of zeros. This guarantees that A has rank $2n-1$ on U. The same argument holds for B since $\hat{F}_1$ does not vanish on U.

To prove the third assertion, we will proceed by contradiction. Suppose that the rank of $(A, -B)$ equals $2n-1$ at $z \in U$. Then each column of $-B$ is a linear combination of the first $2n-1$ columns of A, i.e. it belongs to the syzygies of $(F_1, G_1, \ldots, F_n, G_n)$ and analogously each column of A is a syzygy of $(-\hat{G}_1, \hat{F}_1, \ldots, -\hat{G}_n, \hat{F}_n)$. By multiplying each column of B by $(F_1, G_1, \ldots, F_n, G_n)$ and each column of A by $(-\hat{G}_1, \hat{F}_1, \ldots, -\hat{G}_n, \hat{F}_n)$, we get $\binom{2n}{2}$ equations that, as in the case of $n = 2$, imply

$$\begin{cases} f^s_\sigma = 0 \\ f^c_\gamma * f_\delta(z) = 0 \\ f^c_\gamma * f_\delta(\bar{z}) = 0 \end{cases} \tag{30}$$

for any $\sigma, \gamma, \delta \in \{1, \ldots, n\}$, $\gamma \neq \delta$. Recall that f_1^c is non-vanishing on U; hence if f_1 does not vanish on the sphere containing z then $f_1^s(z) \neq 0$ and we find a contradiction. The case in which f_1 has a zero on the sphere containing z can be treated following the lines of the proof of Theorem 3.2 and using the hypothesis that $f_1, \ldots, f_n$ do not vanish simultaneously at q.

□

The previous lemma allows us to prove the following local result, using the same arguments of the case $n = 2$.

Theorem 4.2. *Let $q = x + yL \in \mathbb{H}$ and let $f_1, \ldots, f_n$ be n functions, regular on a symmetric slice neighborhood Ω of q and not simultaneously vanishing at q. Then it is possible to find $I \in \mathbb{S}$ and a symmetric domain W in Ω_I containing $x + yI$ and $x - yI$, such that the equation*

$$f_1 * h_1 + \cdots + f_n * h_n = 1. \tag{31}$$

restricted to W has local holomorphic solutions $h_1, \dots, h_n : W \to \mathbb{H}$ at any point of W.

We now need to investigate the structure of the sheaf of local solutions of equation (31) restricted to a suitable slice L_I.

Lemma 4.3. *Let $f_1, \dots, f_n : \Omega \to \mathbb{H}$, $I \in \mathbb{S}$ and $W \subseteq \Omega_I$ be as in Theorem 4.2. If $\mathcal{O}$ is the sheaf of germs of holomorphic functions on W and if $\mathcal{K}$ is the sheaf of germs of solutions of equation*

$$f_1 * h_1 + \cdots + f_n * h_n = 0 \tag{32}$$

restricted to W, then

$$\mathcal{K} \cong \mathcal{O}^{4n^2-4n}/\mathcal{O}^{4n^2-6n+2}.$$

Proof. As for the case $n = 2$, $\mathcal{K}$ corresponds (with the same notation of Lemma 4.1) to the sheaf of germs of the local solutions of the system of $2n$ equations in $2\binom{2n}{2}$ unknowns

$$\begin{pmatrix} A, & -B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{33}$$

Lemma 4.1 yields that we can express locally $2n$ unknowns as holomorphic functions in terms of $2\binom{2n}{2} - 2n = 4n^2 - 4n$ germs of holomorphic functions. We therefore obtain a surjective map

$$\varphi : \mathcal{O}^{4n^2-4n} \to \mathcal{K}.$$

The germ in $\mathcal{O}^{4n^2-4n}$ associated to the solution of (33) corresponding to the vector ${}^t(\alpha, \beta)$ belongs to $\ker \varphi$ if and only if

$$A\alpha = B\beta = 0,$$

which, recalling that the rank of A and B equals $2n - 1$, implies (analogously to what happens in the case $n = 2$) that the kernel of φ is isomorphic to $\mathcal{O}^{4n^2-6n+2}$. Hence we conclude that $\mathcal{K}$ is isomorphic to $\mathcal{O}^{4n^2-4n}/\mathcal{O}^{4n^2-6n+2}$. $\square$

With the same arguments used to prove Theorem 3.4, properties of sheaves of germs of holomorphic functions, and the Extension Lemma, lead us to prove the next result.

Theorem 4.4. *Let $q \in \mathbb{H}$ and let $f_1, \dots, f_n$ be n functions, regular on a symmetric slice neighborhood Ω of q and not simultaneously vanishing at q. Then it is possible to find a symmetric slice neighborhood Σ of q and functions $h_1, \dots, h_n$, regular on Σ, such that, on Σ,*

$$f_1 * h_1 + \cdots + f_n * h_n = 1. \tag{34}$$

The global results can be proved using the same techniques of the case $n = 2$.

Theorem 4.5. *Let $f_1, \dots, f_n$ be regular functions on a symmetric slice domain $\Omega \subseteq \mathbb{H}$, with no common zeroes in Ω. Then there exist $h_1, \dots, h_n$ regular functions on Ω such that*

$$f_1 * h_1 + \cdots + f_n * h_n = 1$$

on Ω.

We conclude the paper with a short description of the syzygies of regular functions. In the complex case, if $f_1, \dots, f_n$ are holomorphic functions of one complex variable with no common zeroes, then their syzygies are generated by $\binom{n}{2}$ vectors of holomorphic functions which can be constructed as follows: let e_ℓ, $\ell = 1, \dots, n$, be the standard basis of $\mathbb{R}^n$. The generators of the syzygies are then

$$f_r e_t - f_t e_r = (0, \dots, 0, -f_t, 0, \dots, 0, f_r, 0, \dots, 0)$$

for $1 \leq r < t \leq n$, a fact which we have repeatedly used in the previous section. It is therefore natural to ask if a similar situation occurs for regular functions without common zeroes. Since the $*$-multiplication is not commutative, the immediate analogue of these syzygies does not work in this context. Natural syzygies would on the other hand be the vectors

$$\mathrm{syz}(r,t) := (f_t^c * f_r^s)e_t - (f_r^c * f_t^s)e_r = (0, \dots, 0, -f_r^c * f_t^s, 0, \dots, 0, f_t^c * f_r^s, 0, \dots, 0)$$

for $1 \leq r < t \leq n$. In fact, in view of Lemma 2.9 ,

$$f_r * (-f_r^c * f_t^s) + f_t * (f_t^c * f_r^s) = 0$$

for all $1 \leq r < t \leq n$. In the case of $n = 2$, this identifies one syzygy, which is consistent with Lemma 3.3. When $n > 2$, as in the case of holomorphic functions, there are $\binom{n}{2}$ syzygies, though Lemma 4.3 immediately implies the following proposition.

Proposition 4.6. *Let $f_1, \dots, f_n$ be regular functions on a slice symmetric domain Ω of $\mathbb{H}$ with no common zeroes. Then their syzygies are locally generated by $n-1$ vectors of regular functions.*

To understand this phenomenon, we note that for any three indices $1 \leq p < r < t \leq n$, we have

$$\mathrm{syz}(r,t) * f_p^s = \mathrm{syz}(p,t) * f_r^s - \mathrm{syz}(p,r) * f_t^s. \tag{35}$$

Let us fix a sphere $S = x + y\mathbb{S} \subseteq \Omega$. If one of the functions f_p, f_r, f_t never vanishes on S, assume f_p, then (35) immediately shows that $\mathrm{syz}(r,t)$ is a combination with regular coefficients of $\mathrm{syz}(p,t)$ and $\mathrm{syz}(p,r)$

$$\mathrm{syz}(r,t) = \mathrm{syz}(p,t) * f_r^s * (f_p^s)^{-1} - \mathrm{syz}(p,r) * f_t^s * (f_p^s)^{-1}. \tag{36}$$

If all f_p, f_r, f_t have a zero on S, without loss of generality, we can assume that f_p has the lesser order (for the notion of order of a zero see, e.g., [12]). Then again (36) can be used to represent $\mathrm{syz}(r,t)$ locally.

Remark 4.7. It therefore appears that the reason why we can reduce to $n-1$ the number of syzygies is a consequence of Remark 2.12, namely the fact that a (isolated, non real) zero of a regular function f generates a sphere of zeroes for f^s and a sphere of poles for its reciprocal f^{-*}.

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