ON SOME NEW CURVATURE FLOWS IN NON-KÄHLER GEOMETRY

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Geometric PDE and Unified String Theories

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- Electromagnetism: wave equations and Laplacians $\Box$ and Δ;
- Weak and strong interactions: the Yang-Mills equation $d_A^\dagger F = 0,\ d_A F = 0$
- Gravity: Einstein's equation: $R_{ij}(g) = 0$.

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The unification of all the forces of nature into a single consistent quantum theory is one of the grand dreams of theoretical physics. Since the mid 1980's, the main and only candidate for such a theory has been string theories. There were originally 5 string theories, namely the Type I SO(32) theory, the Type II A and the Type II B string theories, and the heterotic string with gauge group either $SO(32)$ or $E_8 \times E_8$. In the mid 1990's, it was realized that these 5 string theories are all equivalent by various dualities, and that they themselves can be unified into a theory called M Theory. A full formulation of M Theory is not yet available, but it is known that it should reduce in the low energy limit to 11-dimensional supergravity.

For our purposes, we need only some mathematical features of the equations resulting from these theories which we discuss next.

THE HETEROTIC STRING AND THE HULL-STROMINGER SYSTEM

Here X is a compact 3-fold, equipped with a non-vanishing holomorphic 3-form Ω, and a holomorphic vector bundle $E \to X$ with $c_1(E) = 0$ for simplicity. Then C. Hull and A. Strominger have independently proposed the following system in a Hermitian metric ω on X, and a Hermitian metric H on E,

$$i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}(Rm \wedge Rm) - \mathrm{Tr}(F \wedge F)) = 0, \qquad \omega^2 \wedge F = 0$$

$$d(\|\Omega\|_\omega \omega^2) = 0$$

- These are the equations for a supersymmetric compactification of the 10-dimensional space time of the heterotic string to $M^{3,1} \times X$, where $M^{3,1}$ is Minkowski space.
- Rm and F are respectively the curvatures of ω and H, viewed as $(1,1)$-forms valued in the bundles $End(T^{1,0}(X))$ and $End(E)$ of endomorphisms of $T^{1,0}(X)$ and E. In coordinates, $R_{\bar{k}j}{}^p{}_q = -\partial_{\bar{k}}(g^{p\bar{m}}\partial_j g_{\bar{m}q})$, $F_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}}(H^{\alpha\bar{\gamma}}\partial_j H_{\bar{\gamma}\beta})$. Thus the first equation is an equation for $(2,2)$-forms.
- A metric ω is said to be balanced in the sense of Michelsohn is ω^2 is closed. Originally, the last equation was written in terms of torsion. The above reformulation of the last equation as the balanced condition for $\|\Omega\|_\omega^{1/2}\omega$ is due to Li and Yau.

- An early solution, which has had an enormous influence, was proposed by Candelas, Horowitz, Strominger, and Witten. It is given by Calabi-Yau 3-folds X, equipped with their Kähler Ricci-flat metrics.
- To see this, assume that ω is a Kähler metric on X. Take $E = T^{1,0}(X)$, and $H_{\bar\alpha\beta} = \omega$. Then

$$i\partial\bar\partial\omega = 0, \quad Rm \wedge Rm - F \wedge F = 0$$

  and the first equation of the Hull-Strominger system is trivially satisfied. The second equation $\omega^2 \wedge F = 0$ becomes the condition that $\omega^2 \wedge Rm = 0$, i.e. ω is Ricci-flat. Hence $\|\Omega\|_\omega$ is constant, and we can then write

$$d(\|\Omega\|_\omega \omega^2) = \|\Omega\|_\omega d(\omega^2) = 0$$

  and the third condition is a consequence of the Kähler condition.
- But for many reasons, including phenomenology, the moduli problem, and non-perturbative effects such as branes and duality, it is desirable to have a more general class of solutions. In particular $i\partial\bar\partial\omega$ may no longer be zero, and the metric ω may have non-vanishing torsion $T = i\partial\omega$.
- Nevertheless, the third condition says that $\|\Omega\|_\omega^{1/2}\omega$ is still in a de Rham $(2,2)$-class, and the second condition can be interpreted as a curvature condition. From this point of view, the Hull-Strominger system is still providing a notion of canonical metric in non-Kähler geometry.

ANOMALY FLOWS

A major difficulty in solving the Hull-Strominger system is the absence of a $\partial\bar\partial$-lemma, or of an effective parametrization for metrics in a given $(2,2)$-class, In joint work with S. Picard and Xiangwen Zhang, the PI proposed to bypass this problem by considering flows of $(2,2)$-forms. In this specific case, they would be given by the following coupled system

$$\partial_t(\|\Omega\|\omega^2) = i\partial\bar\partial\omega - \frac{\alpha'}{4}(\mathrm{Tr}\,(Rm\wedge Rm) - \mathrm{Tr}(F\wedge F)), \quad H^{-1}\partial_t H = (\omega^2\wedge F)\omega^{-3}$$

for initial data satisfying the conformally balanced condition $d(\|\Omega\|_{\omega_0}\omega_0^2) = 0$. Because the right hand side is closed, the closedness of $\|\Omega\|_\omega\omega^2$ is then preserved for all time. In particular, the stationary points of the system satisfy the full Hull-Strominger system.

Sometimes H and F may not depend on ω and may be considered as known. Similarly, the term $Rm\wedge Rm$ may be viewed as a higher order term. Thus it is often convenient to consider, at least as an intermediate step, a single flow in ω,

$$\partial_t(\|\Omega\|\omega^2) = i\partial\bar\partial\omega - \Phi(\omega, Rm)$$

for a given $(2,2)$-form Φ. We shall refer to all these flows as "Anomaly flows".

THE TYPE II STRING THEORIES

We discuss next solutions of the Type II B string with 05/D5 brane sources and of the Type II A string with O6/D6 brane sources, as formulated by L.S. Tseng and Yau, building on earlier formulations of Grana-Minasian-Petrini-Tomasiello, Tomasiello, and others. In particular, (subspaces of) linearized solutions have been identified by Tseng and Yau with Bott-Chern and Aeppli cohomologies in the case of Type II B, and with their own symplectic cohomology in the case of Type II A, as well as interpolating cohomologies between the two notions. Related boundary value problems have been recently studied by Tseng and Wang.

Here we shall focus on the resulting non-linear partial differential equations.

TYPE II B STRINGS

Let X be compact 3-dimensional complex manifold, equipped with a nowhere vanishing holomorphic 3-form Ω. Let ρ_B be the Poincare dual of a linear combination of holomorphic 2-cycles. We look for a Hermitian metric satisfying the following system

$$d\omega^2 = 0, \qquad i\partial\bar\partial(\|\Omega\|_\omega^{-2}\omega) = \rho_B$$

where $\|\Omega\|_\omega$ is defined by $i\Omega \wedge \bar\Omega = \|\Omega\|_\omega^2 \omega^3$. If we set $\eta = \|\Omega\|_\omega^{-2}\omega$, this system can be recast in a form similar to the Hull-Strominger system,

$$d(\|\Omega\|_\eta \eta^2) = 0, \qquad i\partial\bar\partial\eta = \rho_B$$

TYPE II A STRINGS

Let X be this time a real 6-dimensional symplectic manifold, in the sense that it admits a closed, non-degenerate 2-form ω (but there may be no compatible complex structure, so it may not be a Kähler form). Then the equations are now for a complex 3-form Ω with $\mathrm{Im}\,\Omega = \star\mathrm{Re}\,\Omega$, and

$$d(\mathrm{Re}\,\Omega) = 0, \qquad dd^{\Lambda}(\star\|\Omega\|^2\mathrm{Re}\,\Omega) = \rho_A$$

where ρ_A is the Poincare dual of a linear combination of special Lagrangians. Here $d^{\Lambda} = d\Lambda - \Lambda d$ is the symplectic adjoint.

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All these present the same feature of a cohomological condition together with a curvature-type condition. In joint work with T. Fei, S. Picard, and X.W. Zhang, the PI proposes to study them by analogous Anomaly flows,

- Type II B string:

$$\partial_t(\|\Omega\|_\eta \eta^2) = i\partial\bar{\partial}\eta - \rho_B, \quad d(\|\Omega\|_{\eta_0}\eta_0^2) = 0$$

- Type II A string:

$$\partial_t(\mathrm{Re}\,\Omega) = dd^{\Lambda}(\star\|\Omega\|^2 \mathrm{Re}\,\Omega) - \rho_A, \quad d(\mathrm{Re}\,\Omega_0) = 0$$

Because the right hand side is closed, the closedness of the initial condition is preserved, and the system is solved if the flow converges.

- It would require too much background to explain why notions such as complex structures, existence of a non-vanishing top form such as Ω, and torsion forms are required in string theory (while such notions have appeared before in solutions of Yang-Mills or Einstein's equations, they were just optional).
- In essence, they can be traced back to supersymmetry and supergravity theories, which are theories containing gravity which are supersymmetric. In such theories, the graviton, which is a metric G_{MN}, has a supersymmetric partner, which is 1-form spinor-valued field χ_M, called the gravitino.
- Supersymmetry is generated by spinor fields ξ, and their action on χ_M is given by $\delta\chi_M = D_M\xi$, where D_M is not necessarily the Levi-Civita connection, but may differ from it by a torsion field H.
- A supersymmetric vacuum is invariant under supersymmetric changes, and hence requires a spinor ξ with $D_M\xi = 0$, i.e., ξ is covariantly constant with respect to some connection with torsion H.
- The geometric data, e.g. holonomy, appearing in the string equations described above, are consequences of the existence of such a spinor.

SOME APPLICATIONS OF ANOMALY FLOWS

All these equations suggest that we may be seeing at this moment only the tip of an iceberg, and that a lot more needs to be worked out. In particular, for none of these equations do we have at this moment a good guess on what could be necessary and sufficient conditions for their long-time behavior or convergence. Nor do we know what singularities may develop, which will necessarily depend on the initial data.

We shall see shortly that, despite its very different original motivation, the Anomaly flow in the case of the heterotic string will already turn out to be somewhere between the Ricci flow and the Kähler-Ricci flow, and closer to the former than the latter, so that the answers to any of these questions is likely to be complicated. Nevertheless we can discuss three distinct applications which can illustrate their flexibility and potential for the future:

- The convergence of the flow on toric fibrations and a new proof of the Fu-Yau theorem ;
- The onvergence of the flow in the conformally Kähler case, and a new proof of Yau's theorem on Calabi-Yau metrics;
- The solution of a problem of Ustinovskyi on a Hermitian curvature flow.

THE FU-YAU SOLUTION AND ANOMALY FLOWS

- A toric fibration of Calabi and Eckmann

  Let $(Y, \hat{\omega}_Y)$ be a Calabi-Yau surface, admitting a nowhere vanishing holomorphic form Ω_Y. Let $\omega_1, \omega_2 \in H^2(M, \mathbf{Z})$ satisfy $\omega_1 \wedge \hat{\omega}_Y = \omega_2 \wedge \hat{\omega}_Y = 0$. From this data, building on earlier ideas of Calabi and Eckmann, Goldstein and Prokushkin constructed a toric fibration $\pi : X \to Y$, with a $(1,0)$-form θ satisfying

  $$\partial\theta = 0, \quad \bar{\partial}\theta = \pi^*(\omega_1 + i\omega_2)$$

  Furthermore, the form $\Omega = \Omega_Y \wedge \theta$ is a holomorphic nowhere vanishing $(3,0)$-form on X, and for any scalar function u on Y, the $(1,1)$-form

  $$\omega_u = \pi^*(e^u \hat{\omega}_Y) + i\theta \wedge \bar{\theta}$$

  is a conformally balanced metric on X.

- It is in this set-up that Fu and Yau (2006) found the first non-Kähler solution of the Hull-Strominger system. They showed that, with ω_u as ansatz, the full Hull-Strominger system can be reduced to a scalar Monge-Ampère equation on the base of the form

  $$\frac{\det(e^u \hat{\omega}_Y + 4\alpha' i\partial\bar{\partial}u)}{\det \hat{\omega}_Y} = (e^{2u} - 4\alpha' e^u |\nabla u|^2) - \mu$$

  with μ a given function with average 0.

Fu and Yau solved this Monge-Ampère equation by the method of continuity. Compared to the Monge-Ampère equation solved earlier by Yau for the Calabi conjecture, this equation has an essential new difficulty, namely the right hand side depends on ∇u, and may slide down to 0 as the method of continuity deforms the equation. Thus compared to the earlier work, they need a new estimate, either a lower bound for the Monge-Ampère determinant, or a very sharp upper bound for $|\nabla u|^2$. This they managed to achieve starting from an initial data satisfying $\|e^{-u}\|_{L^4} = M^{-1}$ sufficiently small. They also needed a test function G which was very finely tuned

$$G = 1 - 4\alpha' e^{-u}|\nabla u|^2 + 4\alpha' e^{-\epsilon u} - 4\alpha' e^{-\epsilon \inf u}$$

The desired lower bound is obtained from the maximum principle applied to G.

Using the Anomaly flow, we obtain a different proof:

Theorem 1 (P.,S. Picard, X.W. Zhang) Consider the Anomaly flow on the fibration $X \to Y$ constructed above, with initial data $\omega(0) = \pi^*(M\hat{\omega}_Y) + i\theta\bar{\theta}$, where A is a positive constant. Fix $H = \pi^*(H_Y)$ on the bundle $E = \pi^*(E_Y)$ satisfying the Hermitian-Yang-Mills equation $\omega_Y^2 \wedge F(H_Y) = 0$ on Y.

Then $\omega(t)$ is of the form $\pi^*(e^u\hat{\omega}_Y) + i\theta \wedge \bar{\theta}$ and, assuming an integrability condition on the data (which is necessary), there exists $M_0 > 0$, so that for all $M \geq M_0$, the flow exists for all time, and converges to a metric ω_∞ with (ω_∞, H) satisfying the Hull-Strominger system.

General remarks about the proof

- Indeed, for pull-back $\pi^*(E_Y)$ bundles of a bundle E_Y on the base Y, then the system can be shown to descend to a system on the base Y, for an unknown metric of the form $\hat{\omega}_u = e^u \hat{\omega}_Y$. If H_Y is a metric on E_Y which is Hermitian-Einstein with respect to $\hat{\omega}_Y$, and hence with respect to $\hat{\omega}_u$ for any u. Then the Anomaly flow descends to the following flow on the base Y,

$$\partial_t \hat{\omega}_u = -\frac{1}{2\|\Omega\|_{\hat{\omega}_u}}\left(\frac{R(\hat{\omega}_u)}{2} - |T(\hat{\omega}_u)|^2 - \frac{\alpha'}{4}\sigma_2(iRic_{\hat{\omega}(u)}) + 2\alpha' \frac{i\partial\bar{\partial}(\|\Omega\|_{\hat{\omega}_u}\rho)}{\hat{\omega}_u^2} - 2\frac{\mu}{\hat{\omega}_u^2}\right)\hat{\omega}_u$$

 where ρ and μ are fixed and given forms which depend on the geometric data $(Y, \Omega_Y, \omega_1, \omega_2)$.

- We assume that $|\alpha' Ric_\omega| << 1$, so that the diffusion operator

$$\Delta_F = F^{p\bar{q}}\nabla_p \nabla_{\bar{q}}, \qquad F^{p\bar{q}} = g^{p\bar{q}} + \alpha' \|\Omega\|^3_\omega \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2}(Rg^{p\bar{q}} - R^{p\bar{q}})$$

 is positive definite. Of course, an important and difficult step will be to prove that this condition is preserved along the flow.

Uniform equivalence of the metrics $\omega(t)$

This is equivalent to a uniform estimate for the conformal factor u in $\hat{\omega}_u = e^u \hat{\omega}$, exploiting the fact that the quantity $\int_X \|\Omega\|_\omega \omega^2$ is conserved along the flow. For our purposes, we shall need the following precise version in terms of M. Assume that the flow exists for $t \in [0, T)$ and starts with $\hat{\omega}(0) = M\hat{\omega}$. Then there exists M_0 so that, for $M \geq M_0$, we have

$$\sup_{X \times [0,T)} e^u \leq C_1 M, \qquad \sup_{X \times [0,T)} e^{-u} \leq \frac{C_2}{M}$$

where C_1, C_2 depend only on $(X, \hat{\omega})$, μ, ρ, and α'.

A first step is to extract an L^2 gradient inequality from the flow, e.g.,

$$\frac{k}{2} \int_Y e^{(k+1)u} (\hat{\omega}_Y + \alpha' e^{-2u} \rho) \wedge i\partial u \wedge \bar{\partial} u + \frac{\partial}{\partial t} \Big(\frac{1}{k+1} \int_Y e^{k+1)u} \hat{\omega}_Y^2 \Big)$$

$$\leq (\|\mu\|_{C^0} + 2|\alpha'| \cdot \|\rho\|_{C^2}) \int_X (e^{ku} + e^{(k-1)u}) \hat{\omega}_Y^2$$

The above precise bounds can then be obtained with some care from Moser iteration.

Estimates for the torsion

There exists M_0 with the following property. If the flow is started with $\omega(0) = M\hat{\omega}$ and $M \geq M_0$, and if

$$|\alpha' Ric_{\hat{\omega}}| \leq 10^{-6}$$

along the flow, then there exists a constant C_3 depending only on $Y, \hat{\omega}, \mu, \rho, \alpha'$ so that

$$|T|^2 \leq \frac{C_3}{M^{4/3}} << 1$$

Estimates for the curvature

Start the flow with $\omega(0) = M\hat{\omega}$. There exists $M_0 > 1$ such that, for every $M \geq M_0$, if

$$\|\Omega\|^2 \leq \frac{C_2^2}{M^2}, \qquad |T|^2 \leq \frac{C_3}{M^{4/3}}$$

along the flow, then

$$|\alpha' \, Ric_{\hat{\omega}}| \leq \frac{1}{M^{1/2}}$$

Higher order estimates

Under similar conditions, we can establish estimates for the higher order derivatives of the torsion and the curvature. An interesting new technical point is the usefulness of test functions of the form

$$G = (|\alpha' Ric_{\hat{\omega}}| + \tau_1)|\nabla Ric_{\hat{\omega}}|^2 + (|T|^2 + \tau_2)|\nabla T|^2$$

Closing the loop of estimates

If we start with an initial data satisfying $|\alpha' Ric_{\hat{\omega}}| \leq 10^{-6}$, the estimates for the torsion imply that $|T|^2 \leq C_3 M^{-4/3}$ and hence $|\alpha' Ric_{\hat{\omega}}| \leq M^{-1/2}$. But this implies that for $M >> 1$, the flow does not leave the region $|\alpha' Ric_{\hat{\omega}}| \leq 10^{-6}$, and hence all these estimates hold for all time. This implies the existence for all time of the flow, and an additional argument establishes its convergence in C^∞.

CALABI-YAU METRICS AND ANOMALY FLOWS

The special case $\alpha' = 0$ of the Anomaly flow is of independent geometric interest as well. In n-dimensions, it is given by

$$\partial_t(\|\Omega\|_\omega \omega^{n-1}) = i\partial\bar{\partial}\omega^{n-2}$$

Its stationary points are Kähler Ricci-flat metrics, and hence it can perhaps be put to good use in determining whether a conformally balanced manifold is actually Kähler or not. We have

Theorem 2 Assume that the initial data $\omega(0)$ satisfies $\|\Omega\|_{\omega(0)}\omega(0)^{n-1} = \hat{\chi}^{n-1}$, where $\hat{\chi}$ is a Kähler metric. Then the flow exists for all time $t > 0$, and as $t \to \infty$, the solution $\omega(t)$ converges smoothly to a Kähler, Ricci-flat, metric ω_∞. If we define the metric χ_∞ by

$$\omega_\infty = \|\Omega\|_{\chi_\infty}^{-2/(n-2)} \chi_\infty,$$

then χ_∞ is the unique Kähler Ricci-flat metric in the cohomology class $[\hat{\chi}]$, and $\|\Omega\|_{\chi_\infty}$ is an explicit constant.

In any case, we have obtained another proof of Yau's theorem, with different estimates, as we shall see below.

In this case, the Anomaly flow is actually conformally equivalent to a flow of metrics $t \to \chi(t)$ in the Kähler class of $\hat{\chi}$. Indeed, given a solution $\omega(t)$ of the Anomaly flow, we can define the Hermitian metric $\chi(t)$ by

$$\|\Omega\|_{\omega(t)}\omega^{n-1}(t) = \chi^{n-1}(t)$$

Then the Anomaly flow with $\alpha' = 0$ can be rewritten as

$$\partial_t \chi^{n-1}(t) = i\partial\bar{\partial}(\|\Omega\|^{-2}_{\chi(t)}\chi^{n-2}(t))$$

and hence, upon carrying out the differentiations and assuming that $\chi(t)$ is Kähler,

$$(n-1)\partial_t\chi \wedge \chi^{n-2} = (i\partial\bar{\partial}\|\Omega\|^{-2}_{\chi(t)}) \wedge \chi^{n-2}$$

This last equation is satisfied if $\chi(t)$ flows according to

$$(n-1)\partial_t\chi = i\partial\bar{\partial}\|\Omega\|^{-2}_{\chi(t)}$$

But this flow preserves the Kähler condition, and hence all the steps of the previous derivation are justified. In particular, by the uniqueness of solutions of parabolic equations, the solution $\omega(t)$ of the Anomaly flow is indeed given by its formula above in terms of $\chi(t)$, with $\chi(t)$ Kähler.

The flow for the Kähler metrics $\chi(t)$ can be written explicitly in terms of potentials. Setting $\chi(t) = \hat{\chi} + i\partial\bar{\partial}\varphi(t) > 0$, we find

$$\partial_t \varphi = e^{-f} \frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n}, \quad \varphi(z, 0) = 0$$

where the scalar function $f \in C^\infty(X, \mathbf{R})$ is defined by $(n-1)e^{-f} = \|\Omega\|_{\hat{\chi}}^{-2}$. We note that this flow is of Monge-Ampère type, but without the log as in the solution by Kähler-Ricci flow by H.D. Cao, and without the inverse power of the determinant, as in the recent equation proposed by H.D. Cao and J. Keller and Collins, Hisamoto, and Takahashi. Because of this, we need a new way of obtaining C^2 estimates. It turns out that the test function

$$G(z, t) = \log \mathrm{Tr}\, h - A(\varphi - \frac{1}{[\hat{\chi}^n]} \int_X \varphi \hat{\chi}^n) + B[\frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n}]^2$$

can do the job. We observe that it differs from the standard test function $\hat{G}(z, t) = \log \mathrm{Tr}\, h - A\varphi$ used in the study of Monge-Ampère equations by terms involving the square of the Monge-Ampère determinant. The lack of concavity in the second derivatives also puts this equation beyond the reach of recent techniques of subsolutions, as given in the elliptic case by B. Guan and G. Szekelyhidi, and in the parabolic case by P. and D. To. Thus the study of the Anomaly flow can help develop new tools for the study of non-linear partial differential equations.

PERIODIC POINTS IN THE ANOMALY FLOW

Theorem 3 The periodic points in the Anomaly flow with $\alpha' = 0$ are precisely the Kähler Ricci-flat metrics.

The proof is a simple application of monotonicity formulas for the Anomaly flow due to T. Fei and S. Picard. First, a direct calculation gives

$$\partial_t \|\Omega\|_\omega = \frac{1}{2(n-1)}[R - \frac{1}{n-2}|T|^2 - \frac{2(n-3)}{n-2}|\tau|^2]$$

This implies for any $\alpha > 2$,

$$\partial_t \int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!} = \int_X \|\Omega\|_\omega^{\alpha-1} \frac{\alpha-2}{2(n-1)}[R - \frac{1}{n-2}|T|^2 - \frac{2(n-3)}{n-2}|\tau|^2]\frac{\omega^n}{n!}$$

and hence, in view of the definition of the scalar curvature,

$$R\frac{\omega^n}{n!} = i\partial\bar\partial \log \|\Omega\|_\omega^2 \wedge \frac{\omega^{n-1}}{(n-1)!}$$

we find

$$\partial_t \int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!} = -\frac{(\alpha-1)(\alpha-2)}{2(n-1)} \int_X \|\Omega\|_\omega^{\alpha-1} i\partial \log \|\Omega\|_\omega^2 \wedge \bar\partial \log \|\Omega\|^2 \wedge \frac{\omega^{n-1}}{(n-1)!}$$
$$-\frac{\alpha-2}{2(n-1)(n-2)} \int_X \|\Omega\|_\omega^{\alpha-1}(|T|^2 + 2(n-3)|\tau|^2)\frac{\omega^n}{n!} \leq 0$$

This implies that $\int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!}$ is monotone decreasing.

- In particular, if there is a periodic point, the quantity $\int_X \|\Omega\|^\alpha \omega^n$ must be constant in time.
- But then $\partial \log \|\Omega\|_\omega$ and T must vanish identically.
- This implies that ω is Kähler and $\|\Omega\|_\omega$ is constant. The Ricci curvature of ω is then identically 0.
- Thus ω is Kähler and Ricci-flat, as was to be proved.

So far, we have used only the original formulation of the Anomaly flow as a flow of $(n-1, n-1)$ forms in an n-dimensional complex manifold, from which it is easy to derive the flows for the volume form needed in the previous calculations. But it is now important to express $\partial_t \omega$ more explicity. We can actually write, setting $\omega = i g_{\bar k j} dz^j \wedge d\bar z^k$, $T = i\partial\omega = \frac{1}{2} T_{\bar k pq} dz^q \wedge dz^p \wedge d\bar z^k$,

$$\partial_t g_{\bar k j} = \frac{1}{(m-1)\|\Omega\|_\omega} \{ -\tilde R_{\bar k j} - \frac{1}{2} T_{\bar k pq} \bar T_j{}^{pq} + T_{\bar k js} \bar\tau^s + \tau^{\bar r} \bar T_{j\bar k \bar r} + \tau_j \bar\tau_{\bar k} + \frac{1}{2(m-2)} (|T|^2 - 2|\tau|^2) g_{\bar k j} \}$$

This formula is clearly reminiscent of the Ricci flow, up to the factor of $\|\Omega\|_\omega$.

But it turns out that this factor can be eliminated by introducing the following rescaled metric

$$\eta = \|\Omega\|_\omega \omega$$

Changing now to $\eta = i g_{\bar{k}j}(\eta) dz^j \wedge d\bar{z}^k$, we find the following flow

$$\partial_t g_{\bar{k}j}(\eta) = -\frac{1}{n-1}(\tilde{R}_{\bar{k}j}(\eta) + \frac{1}{2} T_{\bar{k}pq}(\eta) \bar{T}_j{}^{pq}(\eta)).$$

This actually coincides exactly with the Hermitian curvature flow that Ustinovkyi had identified as the one preserving the Griffiths and the dual Nakano-positivity of the tangent bundle !

Applying his result that such fixed points must be conformally balanced together with Theorem 3, we obtain then an answer to Ustinovskyi's question, namely any periodic point of this Hermitian curvature flow must be a Kähler Ricci-flat metric.