argomento / subject: On the structure of Ricci Shrinkers

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讲座. 如何组织用黑板.

$\begin{cases} 55 \text{ minutes} \\ 8 \text{ pages} \end{cases}$

quickly review classification

$R_{ij} + f_{ij} - \frac{g_{ij}}{2} = 0.$

trivial if $f \equiv C$.

($\Leftrightarrow$ Einstein with positive scalar)

Aim: study 4-d Ricci-flow singularity.

4-d. non-compact Ricci-shrinkers.

Examples: ① $(\mathbb{R}^n, \frac{|x|^2}{4})$ $\quad \vec{x} \in \mathbb{R}^n$.

② $(\mathbb{R}^{n-k} \times S^k, \frac{|x|^2}{4})$ $k \geq 2$. $\quad \vec{x} \in \mathbb{R}^{n-k}$

③ $(\mathbb{R}^{n-k} \times N^k, \frac{|x|^2}{4})$ $k \geq 2$. $\quad \vec{x} \in \mathbb{R}^{n-k}$

Classification: Ni-Wallach 2008.

Cao-Chen-Zhu. P.78. 2009

Further examples. ④ $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$.

⑤ FIK (JDG 2003.

$L = H^k$.
$\downarrow$
$\mathbb{CP}^{n-1}$

($-n < k < 0 \Rightarrow$ shrinking.
$-n = k \Rightarrow$ steady
$-n > k \Rightarrow$ expanding)

Choose $n = 2$. tautological bundle over $\mathbb{CP}^1$.
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

Picture. $\rightarrow$ Noncompact Shrinkers are more important.

Cao-Zhou:

min f can always be achieved.

let's say it is achieved at p.

(M, p, g, f) Ricci Shrinker

Regularity.

① ⓛ $r^{-n} d_{GH}(B(x, r), B(0, r)) < \varepsilon$.

$\Rightarrow$. curvature estimate.

②

comparison.

③ volume linear growth.

Return to example
$\Rightarrow$ optimal

④. Sphere Theorem.

Let $\lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of the curvature operator Rm.

Suppose $\lambda_2 > 0$ on the Ricci shrinker (M, g, f), then $M \cong S^n/\Gamma$

for some $\Gamma \subset O(n+1)$ acting freely on S^n.

Cao-Zhou's growth estimate of f:

Let (M, g, f) be a Ricci shrinker. Then there exists a point $p \in M$ where f attains its infimum and f satisfies the quadratic growth estimate:

$$\frac{1}{4}\left(d(x,p) - 5n\right)_+^2 \le f(x) \le \frac{1}{4}\left(d(x,p) + \sqrt{2n}\right)^2, \quad \forall x \in M.$$

~~where~~ $a_+ = \max\{a, 0\}$

$|B(p,r)| \le C r^n$ by Cao-Zhou

$$R_{ij} + f_{ij} - \frac{g_{ij}}{2} = 0$$

$$\Rightarrow \quad R + \Delta f - \frac{n}{2} = 0$$

$$(R + |\nabla f|^2) - f = C.$$

$$\Rightarrow \quad \underbrace{(R + 2\Delta f - |\nabla f|^2) + f - n}_{\mu} = \text{Constant}.$$

Then $$(4\pi)^{-\frac{n}{2}} \int_M e^{-f} \, dv = e^{\mu}$$

$\exists\ C(n)$ s.t.

$$\frac{1}{C}|B(p,1)| < e^{\mu} < C\,|B(p,1)|$$

Thm. $\xi = \xi(m)$ satisfies.

(M, p, g, f) Ricci-shrinker, $x \in M$, $D = d(x,p) + 10n$, $r \in \left(0, \frac{1}{100D}\right)$.

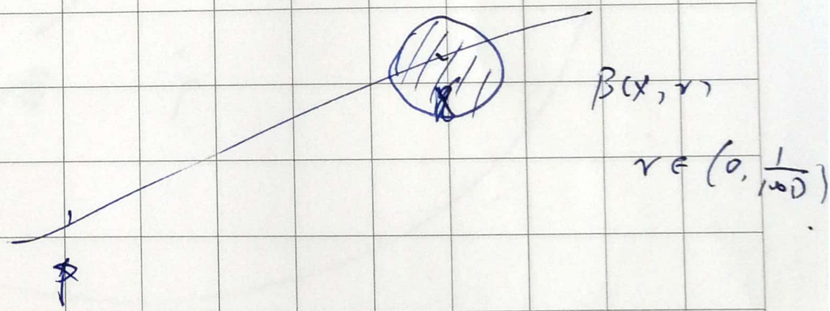
$$r^{-1} d_{GH}\left(B(x,r), B(0,r)\right) < \xi.$$

Then the geodesic ball $B(x, \xi r)$ is strongly convex. Furthermore, the exponential map $Exp_x : B(0, \xi r) \mapsto M$ is a diffeomorphism s.t.

$$r^k \sum_{|\beta|=k} \sup_{w \in B(0,\xi r)} \left| |\partial^\beta h_{ij}(w)| + |\partial^\beta \bar{\ell}| \right| < C_k$$

where $h = (Exp_x)^* g$. $\bar{\ell} = (Exp_x)^*(\bar{f} - f(x))$. $C_k = C(n,k)$.

Local conformal transformation.



$$\hat{g} = \rho^{-2} \bar{g} = r^{-2} e^{\frac{-2\bar{f}}{n-2}} g. \qquad \bar{f} = f - f(x).$$

$$\begin{cases} \widehat{Rc} = \dfrac{1}{n-2}\left\{ d\bar{f} \otimes d\bar{f} + \rho^2 (m-1-\bar{f}-f(q)) e^{\frac{2\bar{f}}{n-2}} \hat{g} \right\}, \\ \hat{\Delta} \bar{f} = e^{\frac{2\bar{f}}{n-2}} \left(\dfrac{n}{2} - \bar{f} - f(q) \right). \end{cases}$$

$$\widehat{Rc}_{ij} = -\frac{1}{2} \hat{g}^{kl} \frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l} + P\left(\frac{\partial \hat{g}_{uv}}{\partial x^w} \right).$$

(Guo-Phong-Song-Sturm).

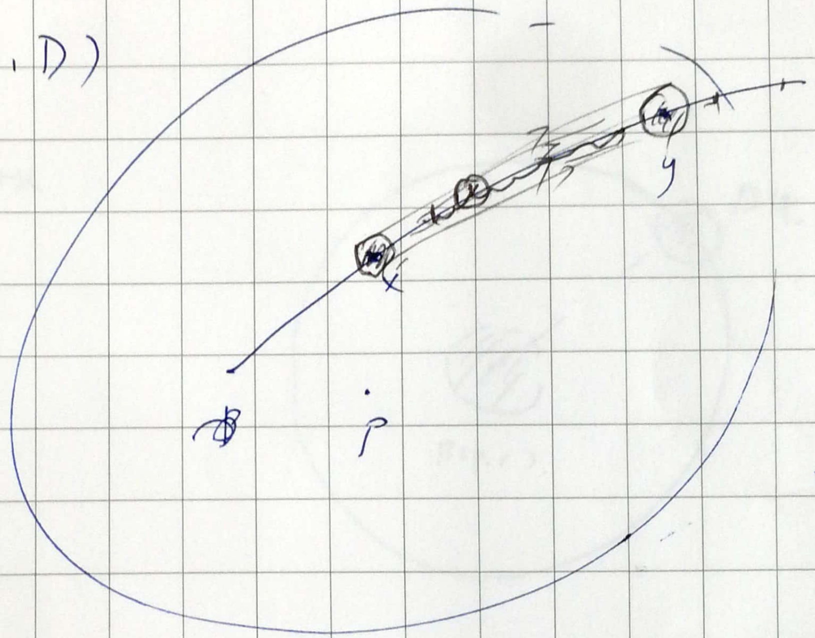
Thm: Fix $\delta \in (0, 0.01)$ small.

Let γ be a minimal geodesic containing in $B(p, D)$ with $|\gamma| = \ell$.

$x, y \in \gamma\left([\delta \ell, (1-\delta)\ell]\right)$ then.

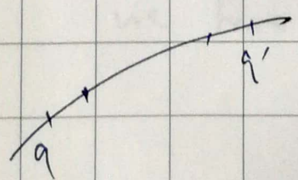
$$r^{-1} d_{GH}\left(B(x, r), B(y, r)\right) \le C \cdot \left(\frac{d(x,y)}{\ell}\right)^{\frac{1}{4m+2}}.$$

for $C = C(m, D)$



Baby-Energy comparison

$$\Delta_f r \le \frac{n-1}{r} + (m-1)\sqrt{\tfrac{1}{2}} + 2D^2.$$



$$\Delta_f (d + d_{q'})\left(\gamma(t)\right) \ge 0$$

$$\Rightarrow \quad \left|\Delta_f d_q\right|_{\substack{\gamma(t) \\ t \in [\delta\ell, (1-\delta)\ell]}} \le \frac{m-1}{\delta \ell} + (m-1)\sqrt{\tfrac{1}{2}} + 2D^2,$$

$$0 \ge \Delta_f |\nabla d_q|^2(\gamma(t)) \ge 2|\mathrm{Hess}\, d_q|^2 + 2\langle \nabla \Delta d_q, \gamma'(t)\rangle - |$$

$$\Rightarrow \int_{\delta\ell}^{\ell - \delta\ell} |\mathrm{Hess}\, d_q|^2 \, dt \le \frac{2(m-1)}{\delta \ell} + 2(m-1)\sqrt{\tfrac{1}{2}} + 4D^2 + (1-2\delta)\cdot\tfrac{1}{2}\ell.$$

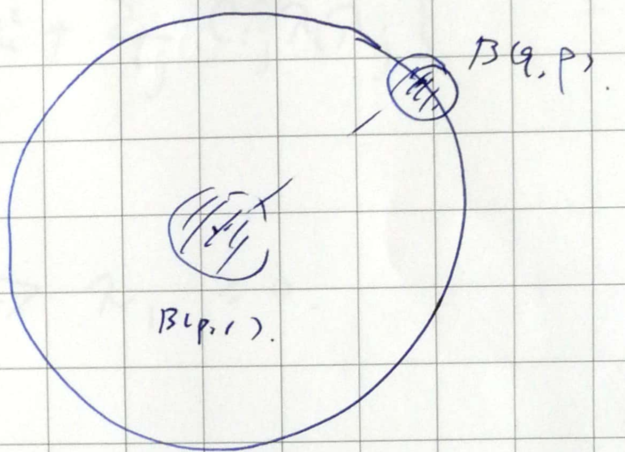
Thm. Suppose (M^n, p, g, f) is a Ricci shrinker, $r > 1$. Then

$$\frac{1}{C} r \le \frac{|B(q,r)|}{|B(p,1)|} \le C r^n$$

Optimal

$$\inf_{\rho \in (0, r^{-1})} \rho^{-n} |B(q,\rho)| \ge \frac{1}{C} |B(p,1)| \ge \frac{1}{C} e^{\mu}.$$

compared with Ricci-flat case



Key: uniform Sobolev constant.

For each compactly supported locally Lipschitz function u, we have

$$\left\{ \int u^{\frac{2n}{n-2}} \, dV \right\}^{\frac{n-2}{n}} \le C \cdot e^{-\frac{2\mu}{n}} \int \{ 4|\nabla u|^2 + R u^2 \} \, dV.$$

Thm : Suppose (M^n, p, g, f) is a Ricci shrinker satisfying $\mu \geq -A$, and $\lambda > 0$. Then.

$$\int_M |Rm|^2 e^{-\lambda f} dv \leq I$$

for some $I = I(n, A, \lambda)$.

This is useful for applying maximum principle of tensor.

$$\Delta_f \lambda_1 \leq \lambda_1 - \left\{ 2\lambda_1^2 + \sum_{i,j} C_{ij}^2 \lambda_i \lambda_j \right\},$$

$$|C_{ij}| \leq 2.$$

if $\lambda_2 \geq 0$.

$$\left.\begin{array}{l} \Rightarrow \Delta_f \lambda_1 \leq \lambda_1 \\ \lambda_1 \in L^2(e^{-f} dv) \end{array}\right\} \Rightarrow \lambda_1 \geq 0.$$

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(Guo-Phong-Song-Sturm, Li-Li-Wang).

Thm.: (M_i, p_i, g_i, f_i) is a sequence of Ricci-shrinkers satisfying

$$|B(p_i, 1)| \geq v_0 > 0.$$

Then by taking subsequence if necessary, we have

$$(M_i, p_i, g_i, f_i) \xrightarrow{p.\ \hat{C}^\infty,\ \text{Cheeger-Gromov}} (M_\infty, p_\infty, g_\infty, f_\infty).$$

Where ~~M_∞ is a~~ the limit is a Riemannian-conifold

Ricci-shrinker. i.e.,

$$M_\infty = R \cup S$$

s.t. ① R is geodesic convex

② $\dim_{\mathcal{M}} S \leq n-4$.