Spatial Pythagorean hodographs, quaternions, and rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$

— interspersed with historical vignettes —

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— synopsis —

• motivation — spatial Pythagorean hodographs

• background — Hamilton and the origin of quaternions

• background — fundamentals of quaternion algebra

• basic theory — unit quaternions and rotations in \( \mathbb{R}^3 \)

• historical interlude — vectors versus quaternions

• basic theory — unit quaternions and rotations in \( \mathbb{R}^4 \)

• applications — spatial PH quintic Hermite interpolants and rational rotation-minimizing frames
— bibliography —

- W. R. Hamilton, *Lectures on Quaternions* (1853), and posthumous *Elements of Quaternions* (1866) — extremely difficult reading


- J. Roe (1993), *Elementary Geometry*, Oxford University Press — gentle introduction to quaternions

- A. J. Hanson (2005), *Visualizing Quaternions*, Morgan Kaufmann — applications to computer graphics
Pythagorean-hodograph (PH) curves

\[ r(\xi) = \text{PH curve in } \mathbb{R}^n \iff \text{coordinate components of } r'(\xi) \]

are elements of a “Pythagorean \((n + 1)\)-tuple of polynomials”

PH curves exhibit special algebraic structures in their hodographs

- rational offset curves \( r_d(\xi) = r(\xi) + d \mathbf{n}(\xi) \)
- polynomial arc-length function \( s(\xi) = \int_0^\xi |r'(\xi)| \, d\xi \)
- closed-form evaluation of energy integral \( E = \int_0^1 \kappa^2 \, ds \)
- real-time CNC interpolators, rotation-minimizing frames, etc.
Planar PH curves — complex variable model

\[ x'^2(t) + y'^2(t) = \sigma^2(t) \iff \begin{cases} 
  x'(t) = u^2(t) - v^2(t) \\
  y'(t) = 2 u(t)v(t) \\
  \sigma(t) = u^2(t) + v^2(t) 
\end{cases} \]

rotation invariance of planar PH form: rotate by \( \theta \), \( \mathbf{r}'(t) \to \tilde{\mathbf{r}}'(t) \)

then \( \tilde{\mathbf{r}}'(t) = \tilde{\mathbf{w}}^2(t) \) where \( \tilde{\mathbf{w}}(t) = \tilde{u}(t) + i \tilde{v}(t) = \exp\left(i \frac{1}{2} \theta \right) \mathbf{w}(t) \)
Spatial PH curves — quaternion & Hopf map models

\[ x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \iff \quad \begin{cases} 
    x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\
    y'(t) = 2 \left[ u(t)q(t) + v(t)p(t) \right] \\
    z'(t) = 2 \left[ v(t)q(t) - u(t)p(t) \right] \\
    \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) 
\end{cases} \]


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choose quaternion polynomial \[ \mathcal{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k} \]
spatial Pythagorean hodograph \[ \mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) \]

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choose complex polynomials \[ \alpha(t) = u(t) + i v(t), \quad \beta(t) = q(t) + i p(t) \]

\[ (x'(t), y'(t), z'(t)) = (|\alpha(t)|^2 - |\beta(t)|^2, 2 \text{Re}(\alpha(t)\overline{\beta(t)}), 2 \text{Im}(\alpha(t)\overline{\beta(t)})) \]

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equivalence — identify “i” with “\( \mathbf{i} \)” and set \( \mathcal{A}(t) = \alpha(t) + k \beta(t) \)
Sir William Rowan Hamilton (1805-1865)

- as a child prodigy, acquired varying degrees of proficiency with thirteen different languages
- appointed Professor of Astronomy at Trinity College, Dublin (age 22)
- theoretical prediction of “conical refraction” by biaxial crystals in 1832 — experimentally verified the same year
- Hamiltonian mechanics: systematic derivation of equations of motion for complicated dynamical systems with multiple degrees of freedom — paved way for development of quantum mechanics
- interpretation of complex numbers as “theory of algebraic couples” — search for “theory of algebraic triples” led to discovery of quaternions
- latter career devoted to (failed) effort to establish quaternions as the “new language of science”
there are no three-dimensional numbers

- generalize complex numbers \( x + y \, i \) to 3D numbers \( x + y \, i + z \, j \)
- basis elements \( 1, i, j \) are assumed to be linearly independent
- commutative and associative — exhibit closure under \(+, -, \times, \div\)
- closure \( \implies \) must have \( i \, j = a + b \, i + c \, j \) for some \( a, b, c \in \mathbb{R} \)
- multiply by \( i \) and substitute \( i^2 = -1, \, i \, j = a + b \, i + c \, j \) then gives
  \[
  j = \frac{b - ac - (a + bc) \, i}{1 + c^2}
  \]
- \( \implies \) contradicts the assumed linear independence of \( 1, i, j \)!
Hurwitz’s theorem (1898) on composition algebras

key property — norm of product = product of norms: $|A B| = |A||B|

commutative law — $A B = B A$, associative law — $(A B) C = A (B C)$

there are four possible composition algebras, of dimension $n = 1, 2, 4, 8$

• $\mathbb{R} (n = 1)$, real numbers — product is commutative & associative

• $\mathbb{C} (n = 2)$, complex numbers — product is commutative & associative

• $\mathbb{H} (n = 4)$, quaternions — product is associative, but not commutative

• $\mathbb{O} (n = 8)$, octonions — product neither commutative nor associative
fundamentals of quaternion algebra

quaternions are four-dimensional numbers of the form

\[ A = a + a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \quad \text{and} \quad B = b + b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k} \]

that obey the sum and (non-commutative) product rules

\[ A + B = (a + b) + (a_x + b_x) \mathbf{i} + (a_y + b_y) \mathbf{j} + (a_z + b_z) \mathbf{k} \]

\[ AB = (ab - a_x b_x - a_y b_y - a_z b_z) + (ab x + ba_x + a_y b_z - a_z b_y) \mathbf{i} + (ab y + ba_y + a_z b_x - a_x b_z) \mathbf{j} + (ab z + ba_z + a_x b_y - a_y b_x) \mathbf{k} \]

basis elements \(1, \mathbf{i}, \mathbf{j}, \mathbf{k}\) satisfy \(i^2 = j^2 = k^2 = \mathbf{i} \mathbf{j} \mathbf{k} = -1\)

equivalently, \(i j = -j i = k\), \(j k = -k j = i\), \(k i = -i k = j\)
scalar-vector representation of quaternions

set $\mathcal{A} = (a, a)$ and $\mathcal{B} = (b, b)$ — $a$, $b$ and $a$, $b$ are scalar and vector parts

($a$, $b$ and $a$, $b$ also called the real and imaginary parts of $\mathcal{A}$, $\mathcal{B}$)

$$\mathcal{A} + \mathcal{B} = (a + b, a + b)$$

$$\mathcal{A} \mathcal{B} = (ab - a \cdot b, ab + b a + a \times b)$$

(historical note: Hamilton’s quaternions preceded, but were eventually supplanted by, the 3-dimensional vector analysis of Gibbs and Heaviside)

$\mathcal{A}^* = (a, -a)$ is the conjugate of $\mathcal{A}$

modulus: $|\mathcal{A}|^2 = \mathcal{A}^* \mathcal{A} = \mathcal{A} \mathcal{A}^* = a^2 + |a|^2$

note that $|\mathcal{A} \mathcal{B}| = |\mathcal{A}| \cdot |\mathcal{B}|$ and $(\mathcal{A} \mathcal{B})^* = \mathcal{B}^* \mathcal{A}^*$
matrix representation of quaternions

matrix algebra embodies non-commutative nature of quaternion product

quaternion basis elements expressed as complex $2 \times 2$ matrices

$$1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad j \rightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad k \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

(closely related to Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ of quantum mechanics)

general quaternion can be expressed as real skew-symmetric $4 \times 4$ matrix

$$\mathbf{A} = a + a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \rightarrow \begin{bmatrix} a & -a_x & -a_y & -a_z \\ a_x & a & -a_z & a_y \\ a_y & a_z & a & -a_x \\ a_z & -a_y & a_x & a \end{bmatrix}$$
unit quaternions & spatial rotations

any unit quaternion has the form $\mathcal{U} = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \mathbf{n})$

describes a spatial rotation by angle $\theta$ about unit vector $\mathbf{n}$

for any vector $\mathbf{v}$ the quaternion product

$$\tilde{\mathbf{v}} = \mathcal{U} \mathbf{v} \mathcal{U}^*$$

yields the vector $\tilde{\mathbf{v}}$ corresponding to a rotation of $\mathbf{v}$ by $\theta$ about $\mathbf{n}$

here $\mathbf{v}$ is short-hand for a “pure vector” quaternion $\mathcal{V} = (0, \mathbf{v})$

unit quaternions $\mathcal{U}$ form a (non-commutative) group under multiplication
rotate vector $\mathbf{v}$ by angle $\theta$ about unit vector $\mathbf{n}$

decompose $\mathbf{v}$ into components parallel & perpendicular to $\mathbf{n}$

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = (\mathbf{n} \cdot \mathbf{v}) \mathbf{n} + (\mathbf{n} \times \mathbf{v}) \times \mathbf{n}$$

$\mathbf{v}_\parallel$ unchanged, but $\mathbf{v}_\perp \to \cos \theta (\mathbf{n} \times \mathbf{v}) \times \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v}$ under a rotation of $\mathbf{v}$ by $\theta$ about $\mathbf{n}$

in terms of quaternions $\mathbf{V} = (0, \mathbf{v})$ and $\mathbf{U} = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \mathbf{n})$ we have

$$\mathbf{U} \mathbf{V} \mathbf{U}^* = (0, (\mathbf{n} \cdot \mathbf{v}) \mathbf{n} + \sin \theta \mathbf{n} \times \mathbf{v} + \cos \theta (\mathbf{n} \times \mathbf{v}) \times \mathbf{n})$$
matrix form of vector rotation in $\mathbb{R}^3$

can write $\tilde{v} = M v$ for $3 \times 3$ matrix $M \in \text{SO}(3)$

$$\begin{bmatrix}
\tilde{v}_x \\
\tilde{v}_y \\
\tilde{v}_z
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33}
\end{bmatrix}
\begin{bmatrix}
v_x \\
v_y \\
v_z
\end{bmatrix}$$

elements of $M$ in terms of rotation angle $\theta$ and axis $\mathbf{n}$

\begin{align*}
m_{11} &= n_x^2 + (1 - n_x^2) \cos \theta, \\
m_{12} &= n_x n_y (1 - \cos \theta) - n_z \sin \theta, \\
m_{13} &= n_z n_x (1 - \cos \theta) + n_y \sin \theta, \\
m_{21} &= n_x n_y (1 - \cos \theta) + n_z \sin \theta, \\
m_{22} &= n_y^2 + (1 - n_y^2) \cos \theta, \\
m_{23} &= n_y n_z (1 - \cos \theta) - n_x \sin \theta, \\
m_{31} &= n_z n_x (1 - \cos \theta) - n_y \sin \theta, \\
m_{32} &= n_y n_z (1 - \cos \theta) + n_x \sin \theta, \\
m_{33} &= n_z^2 + (1 - n_z^2) \cos \theta.
\end{align*}
concatenation of spatial rotations

rotate $\theta_1$ about $n_1$ then $\theta_2$ about $n_2$ \(\rightarrow\) equivalent rotation $\theta$ about $n$

\[
\theta = \pm 2 \cos^{-1}(\cos \frac{1}{2}\theta_1 \cos \frac{1}{2}\theta_2 - n_1 \cdot n_2 \sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2)
\]

\[
n = \pm \frac{\sin \frac{1}{2}\theta_1 \cos \frac{1}{2}\theta_2 n_1 + \cos \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2 n_2 - \sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2 n_1 \times n_2}{\sqrt{1 - (\cos \frac{1}{2}\theta_1 \cos \frac{1}{2}\theta_2 - n_1 \cdot n_2 \sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2)^2}}
\]

sign ambiguity: equivalence of $-\theta$ about $-n$ and $\theta$ about $n$

formulae discovered by Olinde Rodrigues (1794-1851)

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set $U_1 = (\cos \frac{1}{2}\theta_1, \sin \frac{1}{2}\theta_1 n_1)$ and $U_2 = (\cos \frac{1}{2}\theta_2, \sin \frac{1}{2}\theta_2 n_2)$

$U = U_2 U_1 = (\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta n)$ defines angle, axis of compound rotation
spatial rotations do not commute

blue vector is obtained from red vector by the concatenation of two spatial rotations — left: $R_y(\alpha) R_z(\beta)$, right: $R_z(\beta) R_y(\alpha)$ — the end results differ

define $\mathcal{U}_1 = (\cos \frac{1}{2} \alpha, \sin \frac{1}{2} \alpha \mathbf{j})$, $\mathcal{U}_2 = (\cos \frac{1}{2} \beta, \sin \frac{1}{2} \beta \mathbf{k})$ — $\mathcal{U}_1 \mathcal{U}_2 \neq \mathcal{U}_2 \mathcal{U}_1$
the “troubled origins” of vector analysis

The algebraically real part may receive . . . all values contained on the one scale of progression of number from negative to positive infinity; we shall call it therefore the scalar part, or simply the scalar. On the other hand, the algebraically imaginary part, being constructed geometrically by a straight line or radius, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the vector part, or simply the vector . . .

William Rowan Hamilton, *Philosophical Magazine* (1846)

A school of “quaternionists” developed, which was led after Hamilton’s death by Peter Tait of Edinburgh and Benjamin Pierce of Harvard. Tait wrote eight books on the quaternions, emphasizing their applications to physics. When Gibbs invented the modern notation for the dot and cross product, Tait condemned it as a “hermaphrodite monstrosity.” A war of polemics ensued, with luminaries such as Kelvin and Heaviside writing devastating invective against quaternions. Ultimately the quaternions lost, and acquired a taint of disgrace from which they never fully recovered.

A high level of intensity and a certain fierceness characterized much of the debate, and must have led many readers to follow it with interest.

... Gibbs and Heaviside must have appeared to the quaternionists as unwelcome intruders who had burst in upon the developing dialogue between the quaternionists and the scientists of the day to arrive at a moment when success seemed not far distant. Charging forth, these two vectorists, the one brash and sarcastic, the other spouting historical irrelevancies, had promised a bright new day for any who would accept their overtly pragmatic arguments for an algebraically crude and highly arbitrary system. And worst of all, the system they recommended was, not some new system ... but only a perverted version of the quaternion system. Heretics are always more hated than infidels, and these two heretics had, with little understanding and less acknowledgement, wrenched major portions from the Hamiltonian system and then claimed that these parts surpassed the whole.
the sad demise of quaternions


Hamilton’s *Lectures on Quaternions* (1853) “would take any man a twelve-month to read, and near a lifetime to digest . . .” – Sir John Herschel, discoverer of the planet Uranus

Hamilton’s vision of quaternions as the “universal language” of mathematical and physical sciences was never realized — this role is now occupied by vector analysis, distilled from the quaternion algebra by the physicists James Clerk Maxwell (1831-1879) and Josiah Willard Gibbs (1839-1903), and the engineer Oliver Heaviside (1850-1925)
some “dirty secrets” of vector analysis

• there are two fundamentally different types of vector in $\mathbb{R}^3$
  — polar vectors and axial vectors

• a polar vector $v = (v_x, v_y, v_z)$ becomes $(-v_x, -v_y, -v_z)$ under a
  transformation $(x, y, z) \mapsto (-x, -y, -z)$ between right–handed and
  left–handed coordinate systems — also called a true vector

• an axial vector, such as the cross product $a \times b$, is unchanged under
  transformation $(x, y, z) \mapsto (-x, -y, -z)$ — also called a pseudovector

• similar distinction exists between true scalars and pseudoscalars,
  e.g., $(a \times b) \cdot c$ is a pseudoscalar if $a$, $b$, $c$ are true (polar) vectors

• vector analysis in $\mathbb{R}^3$ does not have a natural specialization to $\mathbb{R}^2$
  or generalization to $\mathbb{R}^n$ for $n \geq 4$
spatial rotations as products of reflections

Π = { p | n · p = d } = plane with unit normal n, distance d from origin

reflection of point p in plane Π: p → p + 2(d − n · p)n

any rotation in \( \mathbb{R}^3 \) can be interpreted as a product of reflections in distinct, non–parallel planes \( \Pi_1, \Pi_2 \) — line of intersection \( \Pi_1 \cap \Pi_2 \) is axis of rotation

If \( A = (0, a) \) with \( |a| = 1 \) and \( V = (0, v) \) is a pure vector quaternion, then

\[ A \triangledown A = (0, v - 2(a \cdot v)a) \]

defines the reflection of \( v \) in the plane through origin with unit normal \( a \)

Compound reflections defined by \( A = (0, a) \) & \( B = (0, b) \) with \( |a|, |b| = 1 \)

\[ B (A \triangledown A) B = U \triangledown U^* \]

\[ U = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta n) \] is defined by \( n = b \times a \) and \( \theta = 2 \cos^{-1}(a \cdot b) \)
families of spatial rotations

find $\mathcal{U} = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \, n)$ that rotates $\mathbf{i} = (1, 0, 0) \rightarrow \mathbf{v} = (\lambda, \mu, \nu)$

$n_x^2 (1 - \cos \theta) + \cos \theta = \lambda,$

$n_x n_y (1 - \cos \theta) + n_z \sin \theta = \mu,$

$n_z n_x (1 - \cos \theta) - n_y \sin \theta = \nu.$

$n_x = \pm \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta} \sin \frac{1}{2} \theta,$

$n_y = \frac{\pm \mu \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta - \nu \cos \frac{1}{2} \theta}}{(1 + \lambda) \sin \frac{1}{2} \theta },$

$n_z = \frac{\pm \nu \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta + \mu \cos \frac{1}{2} \theta}}{(1 + \lambda) \sin \frac{1}{2} \theta }.$

general solution, where $\alpha = \cos^{-1} \lambda$ and $\alpha \leq \theta \leq 2\pi - \alpha$
Parameterizes family of spatial rotations mapping unit vectors $\mathbf{i} \rightarrow \mathbf{v}$ by specifying rotation axis $\mathbf{n}$ as a function of rotation angle $\theta$, over restricted domain $\theta \in [\alpha, 2\pi - \alpha]$ where $\alpha$ is angle between $\mathbf{i}$ and $\mathbf{v}$.

Define unit vectors $\mathbf{e}_\perp$, $\mathbf{e}_0$ orthogonal to and in common plane of $\mathbf{i}$ and $\mathbf{v}$

$$\mathbf{e}_\perp = \frac{\mathbf{i} \times \mathbf{v}}{|\mathbf{i} \times \mathbf{v}|} \quad \text{and} \quad \mathbf{e}_0 = \frac{\mathbf{i} + \mathbf{v}}{|\mathbf{i} + \mathbf{v}|}$$

Rotation axis lies in plane spanned by these vectors, may be written as

$$\mathbf{n}(\theta) = \frac{\sin \frac{1}{2} \alpha \cos \frac{1}{2} \theta \mathbf{e}_\perp \pm \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta} \mathbf{e}_0}{\cos \frac{1}{2} \alpha \sin \frac{1}{2} \theta}.$$

for any $\theta \in (\alpha, 2\pi - \alpha)$ there are two axes $\mathbf{n}$ — in the plane of $\mathbf{e}_\perp$, $\mathbf{e}_0$ with equal inclinations to $\mathbf{e}_\perp$ — about which a rotation by angle $\theta$ maps $\mathbf{i} \rightarrow \mathbf{v}$

- when $\theta = \alpha$ or $2\pi - \alpha$, we have $\mathbf{n} = \mathbf{e}_\perp$ or $-\mathbf{e}_\perp$, and rotation is along great circle between $\mathbf{i}$ and $\mathbf{v}$;

- when $\theta = \pi$, we have $\mathbf{n} = \pm \mathbf{e}_0$, so $\mathbf{i}$ executes either a clockwise or anti-clockwise half-rotation about $\mathbf{e}_0$ onto $\mathbf{v}$;
Spatial rotations of unit vectors $\mathbf{i} \rightarrow \mathbf{v}$. (a) Vectors $\mathbf{e}_\perp$ (orthogonal to $\mathbf{i}$, $\mathbf{v}$) and $\mathbf{e}_0$ (bisector of $\mathbf{i}$, $\mathbf{v}$) — the plane $\Pi$ of $\mathbf{e}_\perp$ and $\mathbf{e}_0$ is orthogonal to that of $\mathbf{i}$ and $\mathbf{v}$. (b) For any rotation angle $\theta \in (\alpha, 2\pi - \alpha)$, where $\alpha = \cos^{-1}(\mathbf{i} \cdot \mathbf{v})$, there are two possible rotations, with axes $\mathbf{n}$ inclined equally to $\mathbf{e}_\perp$ in the plane $\Pi$. (c) Sampling of the family of spatial rotations $\mathbf{i} \rightarrow \mathbf{v}$, shown as loci on the unit sphere. (d) Axes $\mathbf{n}$ for these rotations, lying in the plane $\Pi$. 
theory versus practice — a “philosophical” interlude

theoretical astronomer —
don’t believe an observation until there’s a theory to explain it

observational astronomer —
don’t believe a theory until there’s an observation to confirm it

“In theory, there is no difference between theory and practice. In practice, there is.”

Yogi Berra
Yankees baseball player, aspiring philosopher
famous sayings of Yogi Berra, sportsman-philosopher

- Baseball is ninety percent mental, and the other half is physical.
- Always go to other people’s funerals — otherwise they won’t come to yours.
- It was impossible to get a conversation going, everyone was talking too much.
- You better cut the pizza into four pieces, because I’m not hungry enough to eat six.
- You got to be very careful if you don’t know where you are going, because you might not get there.
- Nobody goes there anymore. It’s too crowded.
quaternion model for spatial PH curves

quaternion polynomial \( \mathcal{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k} \)

maps to \( \mathbf{r}'(t) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) = \left[ u^2(t) + v^2(t) - p^2(t) - q^2(t) \right] \mathbf{i} \\
+ 2 \left[ u(t)q(t) + v(t)p(t) \right] \mathbf{j} + 2 \left[ v(t)q(t) - u(t)p(t) \right] \mathbf{k} \)

rotation invariance of spatial PH form: rotate by \( \theta \) about \( \mathbf{n} = (n_x, n_y, n_z) \)

define \( \mathcal{U} = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \mathbf{n}) \) — then \( \mathbf{r}'(t) \rightarrow \tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t) \mathbf{i} \tilde{\mathcal{A}}^*(t) \)

where \( \tilde{\mathcal{A}}(t) = \mathcal{U} \mathcal{A}(t) \) (can interpret as rotation in \( \mathbb{R}^4 \))
matrix form of \( \tilde{A}(t) = \mathcal{U}A(t) \)

\[
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{p} \\
\tilde{q}
\end{bmatrix} = \begin{bmatrix}
cos \frac{1}{2} \theta & -n_x \sin \frac{1}{2} \theta & -n_y \sin \frac{1}{2} \theta & -n_z \sin \frac{1}{2} \theta \\
n_x \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta & -n_z \sin \frac{1}{2} \theta & n_y \sin \frac{1}{2} \theta \\
n_y \sin \frac{1}{2} \theta & n_z \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta & -n_x \sin \frac{1}{2} \theta \\
n_z \sin \frac{1}{2} \theta & -n_y \sin \frac{1}{2} \theta & n_x \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta
\end{bmatrix} \begin{bmatrix}
u \\
v \\
p \\
q
\end{bmatrix}
\]

matrix \( \in \text{SO}(4) \)

in general, points have non-closed orbits under rotations in \( \mathbb{R}^4 \)
“strange events” in $\mathbb{R}^4$ defy geometric intuition!

- an elastic sphere can be turned inside out without tearing the material!
- a prisoner may escape from a locked room without penetrating its walls!
- rigid motions change “left-handed objects” into “right-handed objects”!
- a knot in a length of string can be untied without ever moving its ends!

**Early 20th Century:** can existence of a fourth dimension, imperceptible to human senses, explain mysterious psychic and paranormal phenomena?


→ strange phenomena arise from “extra maneuvering freedom” in $\mathbb{R}^4$
elementary geometry of four dimensions

lines, planes, and hyperplanes of \( \mathbb{R}^4 \) are the sets of points linearly dependent upon two, three, and four points of \( \mathbb{R}^4 \) in “general position”

alternatively, lines, planes, and hyperplanes are point sets satisfying three, two, and one linear equations in the Cartesian coordinates of \( \mathbb{R}^4 \)

hyperplane = a copy of familiar Euclidean space \( \mathbb{R}^3 \) — separates \( \mathbb{R}^4 \) into two disjoint regions (as with a plane in \( \mathbb{R}^3 \), and a line in \( \mathbb{R}^2 \))

generic incidence relations for \( \mathbb{R}^4 \):

- two hyperplanes intersect in a plane
- three hyperplanes intersect in a line
- four hyperplanes intersect in a point
  \[ \implies \] two planes intersect in a point
“absolutely orthogonal” planes in $\mathbb{R}^4$

two planes $\Pi_1, \Pi_2 \in \mathbb{R}^4$ with intersection point $p$ are absolutely orthogonal if every line through $p$ on $\Pi_1$ is orthogonal to every line through $p$ on $\Pi_2$.

pairs of “absolutely orthogonal” planes are a strictly four-dimensional phenomenon — they have no analog in $\mathbb{R}^3$.

through each point $p$ of a given plane $\Pi_1 \in \mathbb{R}^4$, there is a unique plane $\Pi_2 \in \mathbb{R}^4$ that is absolutely orthogonal.

pairs of absolutely orthogonal planes in $\mathbb{R}^4$ play an important role in characterizing four-dimensional rotations.
characterization of rotations in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$

$\mathbb{R}^2: (x + i y) \to e^{i\theta}(x + i y) \quad \text{— one parameter, rotation angle } \theta$

$\mathbb{R}^3: \mathbf{v} \to \mathbf{U}\mathbf{v}\mathbf{U}^* \quad \text{where } \mathbf{U} = (\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta \mathbf{n})$

$\quad \text{— three parameters, rotation axis } \mathbf{n} \text{ and angle } \theta$

$\mathbb{R}^4: \mathbf{V} \to \mathbf{U}_1\mathbf{V}\mathbf{U}_2^* \quad \text{— two unit quaternions } \mathbf{U}_1, \mathbf{U}_2 \Rightarrow \text{ six parameters}$

*stationary set* of rotation in $\mathbb{R}^n = \text{ set of points that do not move}$

*simple rotation* in $\mathbb{R}^n \quad \text{— the stationary set is of dimension } n - 2$

$\Rightarrow \text{ in } \mathbb{R}^2 \text{ and } \mathbb{R}^3, \text{ every rotation is simple}$

simple rotation in $\mathbb{R}^4 \quad \text{— stationary set is a plane through the origin, and unique absolutely orthogonal plane rotates on itself}$
a new possibility in $\mathbb{R}^4$ — double rotations

if $\Pi_1, \Pi_2 \in \mathbb{R}^4$ are absolutely orthogonal planes through the origin, each may rotate upon itself about the other, and these rotations commute — i.e., the outcome is independent of their order

the stationary set of such a double rotation is the single common point of $\Pi_1, \Pi_2$ — i.e., the origin

of the six parameters describing a general rotation in $\mathbb{R}^4$, four define the absolutely orthogonal planes $\Pi_1, \Pi_2$ and two specify the rotation angles $\theta_1, \theta_2$ associated with them

under a continuous double rotation — with angular speeds $\omega_1, \omega_2$ associated with the absolutely orthogonal planes $\Pi_1, \Pi_2$ — points in $\mathbb{R}^4$ have closed orbits if and only if the ratio $\omega_2/\omega_1$ is a rational number
spatial PH quintic Hermite interpolants

spatial PH quintic interpolating end points $p_i, p_f$ & derivatives $d_i, d_f$

$$\mathbf{r}'(t) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t), \quad \mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2$$

$\to$ three quadratic equations in three quaternion unknowns $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$

$$\mathbf{r}'(0) = \mathcal{A}_0 \mathbf{i} \mathcal{A}_0^* = d_i \quad \text{and} \quad \mathbf{r}'(1) = \mathcal{A}_2 \mathbf{i} \mathcal{A}_2^* = d_f$$

$$\int_0^1 \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) \, dt = \frac{1}{5} \mathcal{A}_0 \mathbf{i} \mathcal{A}_0^* + \frac{1}{10}(\mathcal{A}_0 \mathbf{i} \mathcal{A}_1^* + \mathcal{A}_1 \mathbf{i} \mathcal{A}_0^*)$$

$$+ \frac{1}{30}(\mathcal{A}_0 \mathbf{i} \mathcal{A}_2^* + 4 \mathcal{A}_1 \mathbf{i} \mathcal{A}_1^* + \mathcal{A}_2 \mathbf{i} \mathcal{A}_0^*)$$

$$+ \frac{1}{10}(\mathcal{A}_1 \mathbf{i} \mathcal{A}_2^* + \mathcal{A}_2 \mathbf{i} \mathcal{A}_1^*) + \frac{1}{5} \mathcal{A}_2 \mathbf{i} \mathcal{A}_2^* = p_f - p_i$$

two-parameter family of solutions for given data $p_i, p_f$ and $d_i, d_f$

total arc length $S$ depends only on difference of the parameters

$\Longrightarrow$ one–parameter family of interpolants with identical arc lengths

Hermite interpolants of extremal arc length are helical PH curves
examples of spatial PH quintic Hermite interpolants

One-parameter families of PH quintic interpolants, of identical arc length
special classes of spatial PH curves

helical polynomial space curves

satisfy \( \mathbf{a} \cdot \mathbf{t} = \cos \alpha \) (\( \mathbf{a} = \) axis, \( \alpha = \) pitch angle) and \( \kappa / \tau = \tan \alpha \)

all helical polynomial curves are PH curves (implied by \( \mathbf{a} \cdot \mathbf{t} = \cos \alpha \))

all spatial PH cubics are helical, but not all PH curves of degree \( \geq 5 \)

“double” Pythagorean–hodograph (DPH) curves

\( \mathbf{r}'(t) \) and \( \mathbf{r}'(t) \times \mathbf{r}''(t) \) both have Pythagorean structures — have rational Frenet frames \( (t, \mathbf{n}, \mathbf{b}) \) and curvatures \( \kappa \)

all helical polynomial curves are DPH — not just PH — curves

all DPH quintics are helical, but not all DPH curves of degree \( \geq 7 \)
rational rotation–minimizing frame (RRMF) curves

Frenet frame
rotation–minimizing frame

orthonormal frame \((t, u, v)\) on space curve, \(t = \text{curve tangent}\)

frame angular velocity \(\omega\) satisfies \(\omega \cdot t = 0\)

\((u, v)\) have no instantaneous rotation about \(t\)
derivatives \(u'\) and \(v'\) are always parallel to \(t\)

RRMF curve — spatial PH curve with rational rotation-minimizing frame

constraints on coefficients in quaternion & Hopf map representations

applications — animation, motion planning, robotics, swept surfaces
Frenet frame (center) & rotation-minimizing frame (right) on space curve

motion of an ellipsoid oriented by Frenet & rotation-minimizing frames
sufficient & necessary conditions for RRMF quintics

quaternion form \( \mathbf{r}'(t) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) \), \( \mathcal{A}(t) = \mathcal{A}_0 (1 - t)^2 + \mathcal{A}_1 (1 - t) t + \mathcal{A}_2 t^2 \)

\[
\mathcal{A}_0 \mathbf{i} \mathcal{A}_2^* + \mathcal{A}_2 \mathbf{i} \mathcal{A}_0^* = 2 \mathcal{A}_1 \mathbf{i} \mathcal{A}_1^*
\]

Hopf map form \( \mathbf{r}'(t) = (|\alpha(t)|^2 - |\beta(t)|^2, 2 \Re (\alpha(t) \overline{\beta}(t)), 2 \Im (\alpha(t) \overline{\beta}(t))) \)
\( \alpha(t) = \alpha_0 (1 - t)^2 + \alpha_1 (1 - t) t + \alpha_2 t^2 \), \( \beta(t) = \beta_0 (1 - t)^2 + \beta_1 (1 - t) t + \beta_2 t^2 \)

\[
\Re (\alpha_0 \overline{\alpha}_2 - \beta_0 \overline{\beta}_2) = |\alpha_1|^2 - |\beta_1|^2, \quad \alpha_0 \overline{\beta}_2 + \alpha_2 \overline{\beta}_0 = 2 \alpha_1 \overline{\beta}_1
\]

geometric Hermite interpolation algorithm for rigid-body motion planning
Clifford algebra: extensions from $\mathbb{R}^2$, $\mathbb{R}^3$ to $\mathbb{R}^n$
and from Euclidean to Minkowski space

H. I. Choi, D. S. Lee, and H. P. Moon, Clifford algebra, spin representation, and rational

consider “$n$–dimensional numbers” $x = x_1e_1 + \cdots + x_ne_n$
where $x_1, \ldots, x_n \in \mathbb{R}$ and $(e_1, \ldots, e_n) =$ orthonormal basis for $\mathbb{R}^n$

$e_ie_i = \sigma_i = \pm 1$ and $e_je_k = -e_ke_j$ if $j \neq k$

$\Rightarrow \quad x^2 = \sigma_1x_1^2 + \cdots + \sigma_nx_n^2$

$\sigma_1, \ldots, \sigma_n$ define signature of Clifford algebra

write $\mathcal{C}_p,q$ if $\sigma_1 = \cdots = \sigma_p = +1$, $\sigma_{p+1} = \cdots = \sigma_n = -1$

$\mathcal{C}_{n,1}$ equivalent to Minkowski space $\mathbb{R}^{n,1}$ (special relativity theory)
graded algebra — e.g., general element of $\mathcal{C}_3$ is the multivector

$$a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_{23}e_2e_3 + a_{31}e_3e_1 + a_{12}e_1e_2 + a_{123}e_1e_2e_3$$

- 1 is grade zero element (scalar)
- $e_1, e_2, e_3$ are grade one elements (vectors)
- $e_2e_3, e_3e_1, e_1e_2$ are grade two elements (bivectors)
- $e_1e_2e_3$ is highest grade element (pseudoscalar)

subspace of even grade multivectors = sub–algebra $\mathcal{C}_n^+$ of $\mathcal{C}_n$

e.g., complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ isomorphic to $\mathcal{C}_2^+$ and $\mathcal{C}_3^+

(1, i) \leftrightarrow (1, e_1e_2) \quad \text{and} \quad (1, i, j, k) \leftrightarrow (1, e_2e_3, e_1e_2, e_3e_1)$
• inner product $a \cdot b$ reduces grade

• outer product $a \wedge b$ increases grade

• geometric product $ab = a \cdot b + a \wedge b$

• applies to arbitrary multivectors (mixed grade)

• applications — concise description and analysis of reflections and rotations in $\mathbb{R}^n$
geometric product form of spatial PH curves


Spatial Pythagorean hodograph can be represented by unit vector \( \mathbf{n} \) and vector polynomial \( \mathbf{a}(t) \) using geometric product of Clifford algebra

\[
\mathbf{r}'(t) = \mathbf{a}(t) \mathbf{n} \mathbf{a}(t)
\]

\( \mathbf{r}'(t) \) generated by reflecting unit vector \( \mathbf{n} \) in \( \mathbf{a}(t) \) and scaling it by \( |\mathbf{a}(t)|^2 \)

If \( \mathbf{n} = \mathbf{i} \), for example, components of \( \mathbf{r}'(t) \) and \( \mathbf{a}(t) \) are related by

\[
\begin{align*}
x'(t) &= a_x^2(t) - a_y^2(t) - a_z^2(t), \\
y'(t) &= 2a_x(t)a_y(t), \\
z'(t) &= 2a_x(t)a_z(t)
\end{align*}
\]

Can interpolate general first–order Hermite data, provided \( \mathbf{n} \) is not specified a priori — regard as a free parameter, a feasible instance of which is to be determined by the algorithm

Parameters \( \phi_0, \phi_2 \) in quaternion model equivalent to two degrees of freedoms associated with unit vector \( \mathbf{n} \)
Minkowski Pythagorean-hodograph curves in $\mathbb{R}^{2,1}$


medial axis of planar domain $\mathcal{D} = \text{locus of centers of maximal disks}$
(touching domain boundary $\partial \mathcal{D}$ in at least two points) inscribed in $\mathcal{D}$

medial axis transform or MAT $(x(t), y(t), r(t)) = \text{medial axis locus}$
$(x(t), y(t)) \text{ plus function } r(t) \text{ specifying radii of maximal disks}$

MAT is a Minkowski Pythagorean-hodograph (MPH) curve in $\mathbb{R}^{2,1}$ if

$$x''^2(t) + y''^2(t) - r''^2(t) = \sigma^2(t)$$

MAT = MPH curve $\iff$ domain boundary $\partial \mathcal{D}$ can be exactly recovered as a (piecewise) rational curve
• quaternions offer natural language for describing & manipulating spatial rotations

• the quaternion model allows simple and intuitive constructions of spatial Pythagorean-hodograph curves

• the historical legacy of quaternions (origins of vector analysis) has been sadly neglected

• complexity of rotations in $\mathbb{R}^4$ provides cautionary evidence against extending geometric intuition from $\mathbb{R}^2$ and $\mathbb{R}^3$

• Clifford algebra formulations: extension to higher dimensions, and from Euclidean $\rightarrow$ Minkowski space
ANY QUESTIONS ??

It is better to ask a simple question, and perhaps seem like a fool for a moment, than to be a fool for the rest of your life.

old Chinese proverb

Please note —

Answers to all questions will be given exclusively in the form of Yogi Berra quotations.
some famous Yogi Berra responses

• If you ask me anything I don’t know, 
  *I’m not gonna answer.*

• I wish I knew the answer to that, because 
  *I’m tired of answering that question.*

Concerning future research directions . . .

• *The future ain’t what it used to be.*