ORTHONORMAL FRAMES ON SPACE CURVES

Right-handed orthonormal frame $(\underset{\sim}{e}_1(\xi), \underset{\sim}{e}_2(\xi), \underset{\sim}{e}_3(\xi))$ on space curve $\underset{\sim}{r}(\xi)$.

$|\underset{\sim}{e}_i(\xi)| = 1 \Rightarrow \underset{\sim}{e}_i(\xi) \cdot \underset{\sim}{e}_i'(\xi) = 0$ so $\underset{\sim}{e}_i'(\xi) \perp \underset{\sim}{e}_i(\xi)$

Regard parameter ξ as "time":

$\underset{\sim}{e}_i'(\xi) = \underset{\sim}{\omega}(\xi) \times \underset{\sim}{e}_i(\xi)$, $\underset{\sim}{\omega}(\xi) =$ frame angular velocity

Can express $\underset{\sim}{\omega}$ in terms of $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ as

$$\underset{\sim}{\omega} = \omega_1 \underset{\sim}{e}_1 + \omega_2 \underset{\sim}{e}_2 + \omega_3 \underset{\sim}{e}_3$$

Frenet frame: tangent, principal normal, binormal $(\underset{\sim}{t}, \underset{\sim}{p}, \underset{\sim}{b})$ is most familiar.

$\underset{\sim}{t} = \dfrac{\underset{\sim}{r}'}{|\underset{\sim}{r}'|}$: direction of motion along $\underset{\sim}{r}(\xi)$

$\underset{\sim}{p} = \dfrac{\underset{\sim}{r}' \times \underset{\sim}{r}''}{|\underset{\sim}{r}' \times \underset{\sim}{r}''|} \times \underset{\sim}{t}$: points to center of curvature

$\underset{\sim}{b} = \dfrac{\underset{\sim}{r}' \times \underset{\sim}{r}''}{|\underset{\sim}{r}' \times \underset{\sim}{r}''|}$: normal to osculating plane

Define parametric speed $\sigma = |\underset{\sim}{r}'|$,

curvature $\kappa = \dfrac{|\underset{\sim}{r}' \times \underset{\sim}{r}''|}{|\underset{\sim}{r}'|^3}$, torsion $\tau = \dfrac{(\underset{\sim}{r}' \times \underset{\sim}{r}'') \cdot \underset{\sim}{r}'''}{|\underset{\sim}{r}' \times \underset{\sim}{r}''|^2}$

Frenet frame angular velocity:

$$\underset{\sim}{w} = \sigma(\tau \underset{\sim}{t} + \kappa \underset{\sim}{b}) = \text{"Darboux vector"}$$

NOTE: $\underset{\sim}{w}$ has no component in direction of $\underset{\sim}{p}$,
ie, no instantaneous rotation of $\underset{\sim}{b}, \underset{\sim}{t}$ about $\underset{\sim}{p}$.
$\Rightarrow (\underset{\sim}{t}, \underset{\sim}{p}, \underset{\sim}{b})$ is "rotation-minimizing" w.r.t. $\underset{\sim}{p}$

Defects of Frenet frame:

1. $(\underset{\sim}{t}, \underset{\sim}{p}, \underset{\sim}{b})$ do not depend rationally on parameter ξ
2. $\underset{\sim}{p}$ & $\underset{\sim}{b}$ usually suffer sudden reversals at inflection points, where $\kappa = 0$.
3. For many applications, prefer frame that is rotation-minimizing w.r.t. $\underset{\sim}{t}$ rather than $\underset{\sim}{p}$ (robotics, animation, 5-axis CNC machining, swept surface constructions, etc.)

Rotation-minimizing "adapted" frame $(\underset{\sim}{t}, \underset{\sim}{u}, \underset{\sim}{v})$ consists of tangent $\underset{\sim}{t}$ and normal-plane vectors $\underset{\sim}{u}, \underset{\sim}{v}$ that exhibit no rotation about $\underset{\sim}{t}$.

Euler-Rodrigues frame (ERF) = a <u>rational</u> adapted frame defined on any spatial PH curve:

$$(\underset{\sim}{e}_1(\xi), \underset{\sim}{e}_2(\xi), \underset{\sim}{e}_3(\xi)) = \frac{(\mathcal{A}(\xi)\,\underset{\sim}{i}\,\mathcal{A}^*(\xi), \mathcal{A}(\xi)\,\underset{\sim}{j}\,\mathcal{A}^*(\xi), \mathcal{A}(\xi)\,\underset{\sim}{k}\,\mathcal{A}^*(\xi))}{|\mathcal{A}(\xi)|^2}$$

$\underset{\sim}{e}_1(\xi)$ = tangent; $\underset{\sim}{e}_2(\xi)$ & $\underset{\sim}{e}_3(\xi)$ span normal plane. All rationally dependent on ξ.

If $\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\underset{\sim}{i} + p(\xi)\,\underset{\sim}{j} + q(\xi)\,\underset{\sim}{k}$ and

$\sigma(\xi) = |\mathcal{A}(\xi)|^2 = u^2(\xi) + v^2(\xi) + p^2(\xi) + q^2(\xi)$,

the ERF angular velocity $\underset{\sim}{\omega} = \omega_1 \underset{\sim}{e}_1 + \omega_2 \underset{\sim}{e}_2 + \omega_3 \underset{\sim}{e}_3$ has components

$$\omega_1 = \frac{2(uv' - u'v - pq' + p'q)}{\sigma}$$

$$\omega_2 = \frac{2(up' - u'p + vq' - v'q)}{\sigma}$$

$$\omega_3 = \frac{2(uq' - u'q - vp' + v'p)}{\sigma}$$

NOTE: $\sigma' = 2\,\mathrm{scal}(\mathcal{A}'\mathcal{A}^*)$, $\sigma\underset{\sim}{w} = 2\,\mathrm{vect}(\mathcal{A}'\mathcal{A}^*)$

$$\Rightarrow \sigma'^2 + \sigma^2|\underset{\sim}{w}|^2 = 4|\mathcal{A}|^2|\mathcal{A}'|^2$$

ERF is not rotation-minimizing w.r.t. $\underset{\sim}{e}_1 = \underset{\sim}{t}$, since $w_1 \neq 0$.

If $\underset{\sim}{r}(\xi)$ admits a rational rotation-minimizing frame $(\underset{\sim}{t}, \underset{\sim}{u}, \underset{\sim}{v})$ it must be obtainable from the ERF $(\underset{\sim}{t}, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ by a rational normal-plane rotation:

$$\begin{bmatrix} \underset{\sim}{u}(\xi) \\ \underset{\sim}{v}(\xi) \end{bmatrix} = \frac{1}{a^2(\xi)+b^2(\xi)} \begin{bmatrix} a^2(\xi)-b^2(\xi) & -2a(\xi)b(\xi) \\ 2a(\xi)b(\xi) & a^2(\xi)-b^2(\xi) \end{bmatrix} \begin{bmatrix} \underset{\sim}{e}_2(\xi) \\ \underset{\sim}{e}_3(\xi) \end{bmatrix}$$

where $a(\xi), b(\xi)$ = polynomials with $\gcd(a,b) = 1$.

Corresponds to normal-plane rotation by angle

$$\theta(\xi) = -2\tan^{-1}\frac{b(\xi)}{a(\xi)}$$

which induces angular velocity component

$$\theta' = \frac{2(a'b - ab')}{a^2+b^2} \quad \text{in } \underset{\sim}{e}_1 = \underset{\sim}{t} \text{ direction}$$

For a rational rotation-minimizing frame (RRMF) curve, θ' must exactly cancel ω_1 component of ERF angular velocity.

$\underset{\sim}{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$ with $\mathcal{A} = u + v\underset{\sim}{i} + p\underset{\sim}{j} + q\underset{\sim}{k}$ defines an RRMF curve $\Leftrightarrow$ there exist polynomials $a(\xi), b(\xi)$ with $\gcd(a,b) = 1$ such that:

$$\frac{uv' - u'v - pq' + p'q}{u^2+v^2+p^2+q^2} = \frac{ab' - a'b}{a^2+b^2}$$

* There are no non-trivial (i.e., non-planar) cubic RRMF curves.
* There exist non-trivial quintic RRMF curves (a proper subset of all spatial PH quintics).