QUINTIC RRMF CURVES

$$\mathcal{A}(\xi) = u(\xi) + v(\xi)\mathbf{i} + p(\xi)\mathbf{j} + q(\xi)\mathbf{k} = \alpha(\xi) + \mathbf{k}\,\beta(\xi)$$

where $\alpha(\xi) = u(\xi) + v(\xi)\mathbf{i}$, $\beta(\xi) = q(\xi) + p(\xi)\mathbf{i}$

Quaternion & Hopf map forms:

$$\mathbf{r}' = \mathcal{A}\mathbf{i}\mathcal{A}^* = (|\alpha|^2 - |\beta|^2, 2\,\mathrm{Re}(\alpha\bar\beta), 2\,\mathrm{Im}(\alpha\bar\beta))$$

RRMF condition: $a(\xi), b(\xi)$ with $\gcd(a,b) = 1$ exist, such that

$$\frac{uv' - u'v - pq' + p'q}{u^2+v^2+p^2+q^2} = \frac{ab' - a'b}{a^2+b^2}$$

Set $w(\xi) = a(\xi) + b(\xi)\mathbf{i}$, equivalent to:

$$\frac{\mathrm{scal}(\mathcal{A}\mathbf{i}\mathcal{A}'^*)}{|\mathcal{A}|^2} = \frac{\mathrm{Im}(\bar\alpha\alpha' + \bar\beta\beta')}{|\alpha|^2 + |\beta|^2} = \frac{\mathrm{Im}(\bar w w')}{|w|^2}$$

Simplest non-trivial case:

$$\deg(\mathcal{A}) = \deg(\alpha,\beta) = \deg(w) = 2$$

Use Hopf map form, with

$$\alpha(\xi) = \alpha_0(1-\xi)^2 + \alpha_1 2(1-\xi)\xi + \alpha_2\xi^2$$
$$\beta(\xi) = \beta_0(1-\xi)^2 + \beta_1 2(1-\xi)\xi + \beta_2\xi^2$$
$$w(\xi) = w_0(1-\xi)^2 + w_1 2(1-\xi)\xi + w_2\xi^2$$

Then for some real γ, we have:

$$\operatorname{Im}(\bar{\alpha}\alpha' + \bar{\beta}\beta') = \gamma \operatorname{Im}(\bar{w}w') \ \& \ |\alpha|^2 + |\beta|^2 = \gamma |w|^2$$

Comparing coefficients gives

$$|\alpha_0|^2 + |\beta_0|^2 = \gamma |w_0|^2$$
$$\bar{\alpha}_0\alpha_1 + \bar{\beta}_0\beta_1 = \gamma \bar{w}_0 w_1$$
$$(*) \quad \bar{\alpha}_0\alpha_2 + \bar{\beta}_0\beta_2 + 2(|\alpha_1|^2 + |\beta_1|^2) = \gamma(\bar{w}_0 w_2 + 2|w_1|^2)$$
$$\bar{\alpha}_1\alpha_2 + \bar{\beta}_1\beta_2 = \gamma \bar{w}_1 w_2$$
$$|\alpha_2|^2 + |\beta_2|^2 = \gamma |w_2|^2$$

Equations (*) must be satisfied for some w_0, w_1, w_2.

Simplify by "canonical form" transformation: fix coordinate system so that $\Gamma'(0) = (1,0,0)$.

Corresponds to $A_0 = 1$, $(\alpha_0, \beta_0) = (1,0)$, $w_0 = 1$, $\gamma = 1$

Equations (*) simplify to:

$$\alpha_1 = w_1$$
$$\alpha_2 + 2(|\alpha_1|^2 + |\beta_1|^2) = w_2 + 2|w_1|^2$$
$$\bar{\alpha}_1\alpha_2 + \bar{\beta}_1\beta_2 = \bar{w}_1 w_2$$
$$|\alpha_2|^2 + |\beta_2|^2 = |w_2|^2$$

need to eliminate w_1, w_2

3 complex & 1 real equations = 7 constraints

$\alpha_1, \alpha_2, \beta_1, \beta_2, \omega_1, \omega_2$: 12 real variables

$\Rightarrow$ canonical form RRMF quintics have $12-7=5$ degrees of freedom.

Proposition 1 : In canonical form, solutions to (*) can be parameterized by 2 complex variables ξ, η and one real variable c as:

$$(\alpha_0, \alpha_1, \alpha_2) = (1, \xi, |\xi|^2 - |\eta|^2 + ic)$$
$$(\beta_0, \beta_1, \beta_2) = (0, \eta, 2\bar{\xi}\eta)$$
$$(\omega_0, \omega_1, \omega_2) = (1, \xi, |\xi|^2 + |\eta|^2 + ic)$$

Proposition 2 : In canonical form, $A(\xi) = (1-\xi)^2 + A_1 2(1-\xi)\xi + A_2 \xi^2$ generates an RRMF quintic if and only if A_2 is given in terms of A_1 and c by

$$A_2 = (c - A_1 \underset{\sim}{i} A_1^*)\underset{\sim}{i} \qquad (\dagger)$$

Proposition 3 : In arbitrary coordinates, $A(\xi) = A_0(1-\xi)^2 + A_1 2(1-\xi)\xi + A_2 \xi^2$ generates an RRMF quintic if and only if A_0, A_1, A_2 satisfy

$$\operatorname{vect}(A_2 \underset{\sim}{i} A_0^*) = A_1 \underset{\sim}{i} A_1^*.$$

Proof: $(A_0, A_1, A_2) \to (1, A_1, A_2)$ by multiplication with $\frac{A_0^*}{|A_0|^2}$. Replace A_1, A_2 in $(+)$ by

$$\frac{A_1 A_0^*}{|A_0|^2}, \quad \frac{A_2 A_0^*}{|A_0|^2}$$ & simplify.

The condition $\operatorname{vect}(A_2 \mathbf{i} A_0^*) = A_1 \mathbf{i} A_1^*$ gives an elegant, symmetric, constructive characterization for RRMF quintics.

Fix A_0 & A_2 $\to$ obtain one-parameter family of solutions for A_1.

Complex polynomial $w(\xi) = a(\xi) + b(\xi)\mathbf{i}$ (needed for mapping ERF to rational RMF) is a by-product of the construction procedure.