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Abstract. Let $\mathcal{X}$ be a category fibered in groupoids over a finite field $\mathbb{F}_q$, and let k be an algebraically closed field containing $\mathbb{F}_q$. Denote by $\phi_k \coloneqq \mathcal{X}_k \to \mathcal{X}_k$ the arithmetic Frobenius of X_k/k and suppose that $\mathcal{M}$ is a stack over $\mathbb{F}_q$ (not necessarily in groupoids). Then there is a natural functor $\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \longrightarrow \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$, where $\mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ is the category of ϕ_k-invariant maps $\mathcal{X}_k \to \mathcal{M}$. A version of Drinfeld's lemma states that if $\mathcal{X}$ is a projective scheme and $\mathcal{M}$ is the stack of quasi-coherent sheaves of finite presentation, then $\alpha_{\mathcal{M},\mathcal{X}}$ is an equivalence.

We extend this result in several directions. For proper algebraic stacks or affine gerbes $\mathcal{X}$, we prove Drinfeld's lemma and deduce that $\alpha_{\mathcal{M},\mathcal{X}}$ is an equivalence for very general algebraic stacks $\mathcal{M}$.

For arbitrary $\mathcal{X}$, we show that $\alpha_{\mathcal{M},\mathcal{X}}$ is an equivalence when $\mathcal{M}$ is the stack of immersions, the stack of quasi-compact separated étale morphisms or any quasi-separated Deligne-Mumford stack with separated diagonal.

1. Introduction

We work over a finite field $\mathbb{F}_q$ of characteristic p, where q is a power of p and fix an algebraically closed field k over $\mathbb{F}_q$. We denote by $\phi_k \colon k \to k$ the power of the absolute Frobenius corresponding to q, that is $a \mapsto a^q$. More generally if $\mathcal{X}$ is a category fibered in groupoids over $\mathbb{F}_q$ we use the notation

$$\mathcal{X}_k = \mathcal{X} \times k \text{ and } \phi_k = \mathrm{id}_{\mathcal{X}} \times \phi_k \colon \mathcal{X}_k \to \mathcal{X}_k$$

The map ϕ_k above is called the arithmetic Frobenius of $\mathcal{X}_k$.

Let $\mathcal{M}$ be a stack in the étale topology over $\mathbb{F}_q$ and $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$. We denote by $\mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ the category whose objects are pairs (u, f) where $u \in \mathcal{X}_k \to \mathcal{M}$ is a map and $f \colon u \circ \phi_k \to u$ is a natural isomorphism. In other words an object of $\mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ is a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}_k \arrow[d, "\phi_k"'] \arrow[dr, "u"] & \\ \mathcal{X}_k \arrow[r, "u"'] & \mathcal{M} \end{tikzcd}$$

When $\mathcal{X} = X$ is a scheme an object of $\mathcal{M}(\mathbf{D}_k(X))$ is a pair (x, f) where $x \in \mathcal{M}(X_k)$ and $f \colon \phi_k^*(x) \to x$ is an isomorphism.

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We also denote by $\mathcal{M}(\mathcal{X})$ the category of maps $\mathcal{X} \to \mathcal{M}$. The aim of this paper is to study the canonical functor

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

We now summarize the results obtained. Fully faithfulness holds in general:

Theorem 1.1. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$ and let $\mathcal{M}$ be a stack (not necessarily in groupoids) in the étale topology and whose* Hom*-sheaves are quasi-separated algebraic spaces. Then the functor*

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

is fully faithful.

Next, we consider the "proper" case, which has been studied by Drinfeld. Here one consider $\mathcal{M} = \mathrm{QCoh}_f$, the stack (not in groupoid) of quasi-coherent sheaves of finite presentation. The key point is the case $\mathcal{X} = \operatorname{Spec} \mathbb{F}_q$, proved in [Laf97, Lemma 4, p. 43]:

Theorem 1.2. *The functor*

$$\mathsf{Vect}(\mathbb{F}_q) \to \mathsf{Vect}(\mathbf{D}_k(\mathbb{F}_q))$$

is an equivalence of categories. A quasi-inverse takes an object (V, σ) where $V \in \mathsf{Vect}(k)$ and $\sigma \colon V \to V$ is a ϕ_k-linear isomorphism to the $\mathbb{F}_q$-vector space

$$V^\sigma = \{v \in V \mid \sigma(v) = v\}$$

Building up from the previous result we prove the following.

Theorem 1.3 (See 5.6)**.** *Let $\mathcal{X}$ be a quasi-compact category fibered in groupoid over $\mathbb{F}_q$ (see §2 (2)). Suppose that, for all $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}_f(\mathcal{X}_k)$, the k-vector space $\mathrm{Hom}_{\mathcal{X}_k}(\mathcal{F}, \mathcal{G})$ has finite dimension, and all quasi-coherent sheaves on $\mathcal{X}$ are quotient of a direct sum of objects in $\mathrm{QCoh}_f(\mathcal{X})$. Then the functor*

$$\mathrm{QCoh}_f(\mathcal{X}) \to \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence of categories.

Theorem above applies to proper schemes and algebraic stacks over $\mathbb{F}_q$, generalizing Drinfeld's lemma [Laf97, Lemma 3, p. 44], that is the case of projective schemes. It also applies to arbitrary affine gerbes over $\mathbb{F}_q$ (see 5.4 (3)).

The proof follows closely the original one. The new idea is to replace sheafification of graded modules with a more general construction described in [Ton20].

As a consequence of the above theorem we have:

Theorem 1.4. *Let $\mathcal{X}$ be a fibered category as in Theorem 1.3. Assume that either:*

(1) $\mathcal{M}$ is a quasi-compact algebraic stack with quasi-affine diagonal and there exists a representable fpqc covering $V \to \mathcal{X}_k$ from a Noetherian scheme (e.g. $\mathcal{X}$ is of finite type over $\mathbb{F}_q$ or an affine gerbe);

(2) $\mathcal{M}$ is a Noetherian algebraic stack with quasi-affine diagonal;

(3) $\mathcal{M}$ is an affine gerbe over a field.

Then the functor

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence.

Theorem above implies that all affine gerbes over $\mathbb{F}_q$ are trivial (see 5.9), which was known only for affine gerbes of finite type.

Another classical case it has been considered is the case of finite and étale maps. We are going to prove the following:

Theorem 1.5. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$. Then the functor*

$$\mathrm{Et}_s(\mathcal{X}) \to \mathrm{Et}_s(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence of categories.

Here Et_s is the stack (not in groupoid) of representable, étale, quasi-compact and separated maps. The classical case was proved for the stack of finite and étale maps (see [Ked17, Lemma 4.2.6, p. 100]).

As a consequence of the above result we obtain:

Theorem 1.6. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$ and $\mathcal{M}$ be a quasi-separated Deligne-Mumford stack over $\mathbb{F}_q$ with separated diagonal (e.g. a quasi-separated algebraic space). Then the functor*

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence of categories.

2. Notation

(1) By a stack over a ring R we mean a stack (not necessarily in groupoids) for the fpqc topology over Sch/R. If we want to consider a stack in a different topology we will specify the topology.
(2) We call a category $\mathcal{X}$ fibered in groupoids over a ring R *quasi-compact* if it admits a representable fpqc covering from an affine scheme.
(3) We call category $\mathcal{X}$ fibered in groupoids over a ring R *quasi-separated* if the diagonal map is representable and quasi-compact.

3. General Remarks

Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$ and $\mathcal{M}$ be a stack in the étale topology over $\mathbb{F}_q$. In this section we reinterpret the map

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

We define

$$\mathbf{D}_k(\mathbb{F}_q) = [\operatorname{Spec} k/\underline{\mathbb{Z}}] \qquad \text{and} \qquad \mathbf{D}_k(\mathcal{X}) \coloneqq \mathbf{D}_k(\mathbb{F}_q) \times_{\mathbb{F}_q} \mathcal{X}$$

for a fibered category $\mathcal{X}$ over $\mathbb{F}_q$, where $\underline{\mathbb{Z}}$ is the constant group scheme over $\mathbb{F}_q$ of $\mathbb{Z}$. The action of $\mathbb{Z}$ on $\operatorname{Spec} k$ is the one induced by the Frobenius ϕ_k. Since

$$\underline{\mathbb{Z}} = \bigsqcup_{n \in \mathbb{Z}} \operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathbb{F}_q$$

and therefore a covering in the étale (actually Zariski) topology, a $\underline{\mathbb{Z}}$-torsor $P \to X$ for the fpqc topology over a scheme X is automatically a $\underline{\mathbb{Z}}$-torsors for the étale topology: P become trivial after the étale covering $P \to X$. This means that the quotient $[\operatorname{Spec} k/\underline{\mathbb{Z}}]$ made with respect to the étale topology is a stack in the fpqc topology, or, in other words, it coincide with quotient made with respect to the fpqc topology.

One can check easily that the action of ϕ_k on $\operatorname{Spec} k$ is free, so that $\mathbf{D}_k(\mathbb{F}_q)$ is actually a sheaf (in the fpqc topology). It is an algebraic space in the sense of , but not in the sense of [LMB00, Definition 4.1, p. 25], because $\mathbf{D}_k(\mathbb{F}_q)$ is not quasi-separated.

By definition of $\mathbf{D}_k(\mathcal{X})$, the map $\mathcal{X}_k \to \mathbf{D}_k(\mathcal{X})$ is a $\underline{\mathbb{Z}}$-torsor and the corresponding action of $\mathbb{Z}$ on $\mathcal{X}_k$ is given by the geometric Frobenius ϕ_k. If $\mathcal{X}$ is just a sheaf in the étale topology we can conclude that $\mathbf{D}_k(\mathcal{X}) = [\mathcal{X}_k/\mathbb{Z}]$ is the stack quotient for the étale topology.

In general there is a 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}_k \arrow[dr, "u"] \arrow[d, "\phi_k"'] & \\ \mathcal{X}_k \arrow[r, "u"'] & \mathbf{D}_k(\mathcal{X}) \end{tikzcd}$$

and it induces a functor

$$\operatorname{Hom}_{\mathbb{F}_q}(\mathbf{D}_k(\mathcal{X}), \mathcal{M}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

Lemma 3.1. *If $\mathcal{M}$ is a stack in the étale topology then the above functor is an equivalence.*

Proof. First, one reduces to the case when $\mathcal{X}$ is a stack in the étale topology. Since

$$\operatorname{Hom}_{\mathbb{F}_q}(\mathbf{D}_k(\mathcal{X}), \mathcal{M})) = \operatorname{Hom}_{\mathbb{F}_q}(\mathbf{D}_k(\mathbb{F}_q), \underline{\operatorname{Hom}}_{\mathbb{F}_q}(\mathcal{X}, \mathcal{M}))$$

we can reduce to the case $\mathcal{X} = \operatorname{Spec} \mathbb{F}_q$.

In this case the claim follows from the universal property of the map $\operatorname{Spec} k \longrightarrow [\operatorname{Spec} k/\underline{\mathbb{Z}}] = \mathbf{D}_k(\mathbb{F}_q)$: an object $x \in \mathcal{M}(k)$ together with an isomorphism $\sigma\colon \phi_k^*(x) \to x$ is a descent datum along the groupoid $\operatorname{Spec} k \times_{\mathbf{D}_k(\mathbb{F}_q)} \operatorname{Spec} k = \operatorname{Spec} k \times \underline{\mathbb{Z}} \rightrightarrows \operatorname{Spec} k$. □

The above equivalence motivates the use of the symbol $\mathcal{M}(\mathbf{D}_k(\mathcal{X}))$. In this paper we identify functors $\mathbf{D}_k(\mathcal{X}) \to \mathcal{M}$ and objects of $\mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ when $\mathcal{M}$ is a stack in the étale topology. The projection map $\mathbf{D}_k(\mathcal{X}) \to \mathcal{X}$ induces a functor

$$\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

which is easily checked to be the map $\alpha_{\mathcal{M},\mathcal{X}}$ defined in the introduction.

In what follows we prove some general properties of the functor $\alpha_{\mathcal{M},\mathcal{X}}$. We keep the notation from above.

We first observe a general procedure to reduce problems to categories fibered in groupoids.

Remark 3.2. If $\mathcal{Y}$ is a fibered category over some site one defines $\mathcal{Y}^{\mathrm{Cart}}$ as the sub fibered category of $\mathcal{Y}$ whose objects are the same, but whose maps are the Cartesian arrows of $\mathcal{Y}$ (see [Vis05, Def 3.31]). In terms of sections, given an object T in the site, one has that $\mathcal{Y}^{\mathrm{Cart}}(T)$ is the groupoid associated with $\mathcal{Y}(T)$.

Thus $\mathcal{Y}^{\mathrm{Cart}}$ is a category fibered in groupoid and any map $\mathcal{X} \to \mathcal{Y}$ from a category fibered in groupoids has image in $\mathcal{Y}^{\mathrm{Cart}}$. More precisely $\mathcal{Y}^{\mathrm{Cart}}(\mathcal{X}) = \mathrm{Hom}(\mathcal{X}, \mathcal{Y}^{\mathrm{Cart}})$ is the groupoid associated with $\mathcal{Y}(\mathcal{X}) = \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$.

Moreover if $\mathcal{Y}$ is a stack (or prestack) then so is $\mathcal{Y}^{\mathrm{Cart}}$. Indeed there is a Cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Iso}}_{\mathcal{Y}}(\xi,\eta) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{Y}}(\xi,\eta) \times \underline{\mathrm{Hom}}_{\mathcal{Y}}(\eta,\xi) \\ \downarrow & & \downarrow \\ T & \xrightarrow{(\mathrm{id},\mathrm{id})} & \underline{\mathrm{End}}_{\mathcal{Y}}(\xi) \times \underline{\mathrm{End}}_{\mathcal{Y}}(\eta) \end{array}$$

which shows that if $\mathcal{Y}$ is a prestack then so in $\mathcal{Y}^{\mathrm{Cart}}$. On the other hand descent data is expressed in terms of isomorphisms, thus are the same for $\mathcal{Y}$ or $\mathcal{Y}^{\mathrm{Cart}}$.

From the properties of $(-)^{\mathrm{Cart}}$ we easily deduce the following result.

Lemma 3.3. *If the functors*

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X})) \text{ and } \alpha_{\mathcal{M}^{\mathrm{Cart}},\mathcal{X}} \colon \mathcal{M}^{\mathrm{Cart}}(\mathcal{X}) \to \mathcal{M}^{\mathrm{Cart}}(\mathbf{D}_k(X))$$

are respectively fully faithful and essentially surjective then they are both equivalences.

Lemma 3.4. *Assume that $\mathcal{M}$ is a stack in the étale topology. If for all affine schemes X over $\mathbb{F}_q$ the functor $\alpha_{\mathcal{M},X} \colon \mathcal{M}(X) \to \mathcal{M}(\mathbf{D}_k(X))$ is fully faithful (resp. an equivalence) then $\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ is so for all categories $\mathcal{X}$ fibered in groupoids over $\mathbb{F}_q$.*

Proof. The association

$$\mathcal{N} \colon \mathrm{Sch}/\mathbb{F}_q \to (\text{groupoids}),\ \mathcal{N}(U) = \mathcal{M}(\mathbf{D}_k(U))$$

defined a fibered category over $\mathbb{F}_q$. If $\delta \colon \mathbf{D}_k(\mathbb{F}_q) \to \operatorname{Spec} \mathbb{F}_q$ is the canonical map, it is easy to see that $\mathcal{N} = \delta_*\delta^*\mathcal{M}$. Moreover, since by definition $\delta^*\mathcal{X} = \mathcal{X} \times \mathbf{D}_k(\mathbb{F}_q) = \mathbf{D}_k(\mathcal{X})$, we have

$$\mathrm{Hom}_{\mathbb{F}_q}(\mathcal{X}, \mathcal{N}) \simeq \mathrm{Hom}_{\mathbf{D}_k(\mathbb{F}_q)}(\delta^*\mathcal{X}, \delta^*\mathcal{M}) \simeq \mathrm{Hom}_{\mathbb{F}_q}(\mathbf{D}_k(\mathcal{X}), \mathcal{M}) \simeq \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

and, applying $\mathrm{Hom}(\mathcal{X}, -)$ to the unit map $\mathcal{M} \to \delta_*\delta^*\mathcal{N}$, we exactly obtain $\alpha_{\mathcal{M},\mathcal{X}}$.

In particular it is enough to prove that $\mathcal{M} \to \mathcal{N}$ is fully faithful (resp. an equivalence) and, the hypothesis, means that $\mathcal{M}(X) \to \mathcal{N}(X)$ is fully faithful (resp. an equivalence) for all affine schemes X. Since $\mathcal{M}$ and $\mathcal{N}$ are stacks in the étale topology this ends the proof. □

Lemma 3.5. *If $\mathcal{M}$ a stack in the étale topology and a prestack in the fpqc topology, then the functor $\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$ is faithful*

Proof. The functors $\mathcal{X}_k \to \mathbf{D}_k(\mathcal{X})$ and $\mathcal{X}_k \to \mathcal{X}$ are both representable fpqc coverings. It follows that $\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{X}_k)$, $\mathcal{M}(\mathbf{D}_k(\mathcal{X})) \to \mathcal{M}(\mathcal{X}_k)$ and thefore the map in the statement are all faithful. □

Now we want to understand how to reduce the study of $\alpha_{\mathcal{M},\mathcal{X}}$ locally on $\mathcal{X}$.

Situation 3.6. We consider the following data and assumptions:

- a map $\psi\colon V \to X$ of algebraic spaces;
- the stack $\mathcal{M}$ is a prestack in the fpqc topology (e.g. an algebraic stack by [Sta18, 0APL]);
- the map $V \to X$ is an étale (resp. fppf, fpqc) covering and $\mathcal{M}$ is a stack in the étale (resp. fppf, fpqc) topology.

Consider the diagram

$$\begin{tikzcd}
\mathcal{M}(X) \arrow[r, "\psi^*"] \arrow[d, "\alpha_{\mathcal{M},X}"] & \mathcal{M}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha_{\mathcal{M},V}"] & \mathcal{M}(V \times_X V) \arrow[d, "\alpha_{\mathcal{M},V\times_X V}"] \\
\mathcal{M}(\mathbf{D}_k(X)) \arrow[r, "\psi^*"] & \mathcal{M}(\mathbf{D}_k(V)) \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M}(\mathbf{D}_k(V \times_X V))
\end{tikzcd}$$

The two rows above are exact: indeed $\mathbf{D}_k(V \times_X V) \rightrightarrows \mathbf{D}_k(V) \to \mathbf{D}_k(X)$ is just $V \times_X V \rightrightarrows V \to X$ base changed along $\mathbf{D}_k(\mathbb{F}_q) \to \operatorname{Spec} \mathbb{F}_q$. Notice moreover that the functors $\alpha_{\mathcal{M},*}$ are faithful thanks to 3.5.

The following results all follow by diagram chasing.

Lemma 3.7. *Assume Situation 3.6. Let $\xi, \eta \in \mathcal{M}(X)$ and $\omega\colon \alpha_{\mathcal{M},X}(\xi) \to \alpha_{\mathcal{M},X}(\eta)$ be a morphism. If $\psi^*(\omega)$ comes from a morphism $\psi^*(\xi) \to \psi^*(\eta)$ then ω comes from a morphism $\xi \to \eta$.*

Lemma 3.8. *Assume Situation 3.6. Assume moreover that $\alpha_{\mathcal{M},V}$ and $\alpha_{\mathcal{M},V\times_X V}$ are fully faithful. Consider a 2-commutative diagram as in the outer diagram of*

$$\begin{tikzcd}
 & V \arrow[dr] \arrow[drr, bend left] & & \\
\mathbf{D}_k(V) \arrow[ur] \arrow[dr] & & X \arrow[r, dashed] & \mathcal{M} \\
 & \mathbf{D}_k(X) \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Then there exists a dashed arrow as in the above diagram and it is unique up to a unique isomorphism.

We will often use the above lemma in the following form.

Lemma 3.9. *Assume Situation 3.6. Assume moreover that $\alpha_{\mathcal{M},Y}$ is fully faithful for all schemes Y over $\mathbb{F}_q$. Consider commutative diagrams*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
\mathbf{D}_k(X) \arrow[r] & \mathcal{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & V \arrow[d] \\
\mathbf{D}_k(X) \arrow[r] & X
\end{tikzcd}$$

where B is a sheaf and the second one is Cartesian, so that $A \simeq \mathbf{D}_k(V)$. Assume moreover that the induced map $\mathbf{D}_k(V) \to B$ factors as $\mathbf{D}_k(V) \to V \to B$. Then there exists a $X \to \mathcal{M}$, unique up to a unique isomorphism, fitting in the following commutative diagram

$$\begin{tikzcd} A = \mathbf{D}_k(V) \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & B \arrow[d] \\ \mathbf{D}_k(X) \arrow[r] & X \arrow[r] & \mathcal{M} \end{tikzcd}$$

If moreover the original diagram was Cartesian, then so is the above diagram on the right.

Proof. The first claim follows from 3.8, so only the last one needs to be proved.

One checks easily that there is a Cartesian diagram:

$$\begin{tikzcd} V_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_k \arrow[d] \\ X_k \arrow[r] & \mathcal{M}_k \end{tikzcd}$$

which is the pullback of

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ X \arrow[r] & \mathcal{M} \end{tikzcd}$$

along $\operatorname{Spec} k \longrightarrow \operatorname{Spec} \mathbb{F}_q$, whence the claim. ◻

Corollary 3.10. *Assume Situation 3.6. If $\alpha_{\mathcal{M},V}$ is fully faithful then so is $\alpha_{\mathcal{M},X}$. If $\alpha_{\mathcal{M},V}$ and $\alpha_{\mathcal{M},V\times_X V}$ are equivalences then so is $\alpha_{\mathcal{M},X}$.*

Remark 3.11. If $\mathcal{X}$ is a category fibered in groupoids over $\mathbb{F}_q$ and $\mathcal{M}$ is an affine scheme then $\alpha_{\mathcal{M},\mathcal{X}}$ is an isomorphism (of sets). Indeed by 3.4 we can assume that $\mathcal{X} = X = \operatorname{Spec} A$ is affine as well. If $\mathcal{M} = \operatorname{Spec} B$ then $\mathcal{M}(X) = \operatorname{Hom}_{\mathbb{F}_q}(B, A)$ as algebras, while

$$\mathcal{M}(\mathbf{D}_k(X)) = \operatorname{Hom}_{\mathbb{F}_q}(B, (A \otimes_{\mathbb{F}_q} k)^{\phi_k})$$

It is easily checked by choosing a basis of A over $\mathbb{F}_q$ that the map $A \to (A \otimes_{\mathbb{F}_q} k)^{\phi_k}$ is an isomorphism, thus we see that $\alpha_{\mathcal{M},X}$ is an isomorphism.

4. The case of immersions

In this section we consider as $\mathcal{M}$ the stack of immersions.

Definition 4.1. We denote by Emb the fibered category of immersions over $\mathbb{F}_q$.

The aim of this section is to prove the following.

Theorem 4.2. *Let $\mathcal{X}$ be a category fibered in groupoids. Then the functor*

$$\operatorname{Emb}(\mathcal{X}) \to \operatorname{Emb}(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence of categories and it preserves quasi-compact (resp. open, closed) immersions

We first show that Emb is a stack for the fppf topology.

Proposition 4.3. *The fibered category* Emb *is a stack in the fppf topology and a prestack in the fpqc topology. If $\mathcal{X}$ is a category fibered in groupoids over $\mathbb{F}_q$, then* $\mathrm{Emb}(\mathcal{X})$ *is the category of immersions in $\mathcal{X}$, that is functors $\mathcal{Z} \to \mathcal{X}$ representable by immersions.*

Proof. The category Emb is a sub-fibered category of $\mathrm{Sh}_{\mathbb{F}_q}$, the stack of fpqc sheaves over $\mathbb{F}_q$. In particular it is a prestack in the fpqc topology.

Effective descent for the fppf topology instead follows from [Sta18, 04SK] and [Sta18, 02YM] ☐

We need some preparatory lemmas. The following is called the Moore determinant [Goss98, Corollary 1.3.7]. We present here a direct proof of the result.

Lemma 4.4. *The following relation holds in the ring $\mathbb{F}_q[x_0, \dots, x_r]$:*

$$\det \begin{pmatrix} x_0 & x_1 & \dots & x_r \\ x_0^q & x_1^q & \dots & x_r^q \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{q^r} & x_1^{q^r} & \dots & x_r^{q^r} \end{pmatrix} = \omega \prod_{(a_0:\dots:a_r)\in\mathbb{P}^r(\mathbb{F}_q)} (a_0x_0 + \dots + a_rx_r)$$

where $\omega \in \mathbb{F}_q{}^*$ and, in the product on the right, one chooses a representative of an element of $\mathbb{P}^r(\mathbb{F}_q)$.

Proof. Denote by $N = N(x_0, \dots, x_r)$ the matrix in the statement and by $F = F(x_0, \dots, x_r)$ its determinant. We are going to prove the equality on $\overline{\mathbb{F}_q}$.

Let $a = (a_0, \dots, a_r) \in \mathbb{F}_q{}^{r+1}$ be a non zero element and set

$$L_a = a_0x_0 + \dots + a_rx_r \in \mathbb{F}_q[x_0, \dots, x_r]$$

Notice that

$$N(x_0, \dots, x_r)\cdot a = (L_a, L_a^q, \dots, L_a^{q^r})$$

where the right hand side should be thought of as a vertical vector. In particular it is clear that, if $(u_0, \dots, u_r) \in \overline{\mathbb{F}_q}^{r+1}$ is such that $L_a(u_0, \dots, u_r) = 0$ then $N(u_0, \dots, u_r)\cdot a = 0$ and therefore $F(u_0, \dots, u_r) = 0$. In other words the zero locus $\{L_a = 0\} \subseteq \overline{\mathbb{F}_q}^{r+1}$ is contained in the zero locus $\{F = 0\} \subseteq \overline{\mathbb{F}_q}^{r+1}$. We can therefore conclude that L_a divides F in $\overline{\mathbb{F}_q}[x_0, \dots, x_r]$.

Notice that given $a, b \in \mathbb{F}_q{}^{r+1}$ we have that L_a and L_b generates the same ideal in $\overline{\mathbb{F}_q}[x_0, \dots, x_n]$ if and only if $a = \lambda b$ for some $\lambda \in \mathbb{F}_q$. Using the factorization into primes we can conclude that the product P in the statement divides F. But

$$\deg P = \#\mathbb{P}^r(\mathbb{F}_q) = \frac{q^{r+1}-1}{q-1} = 1 + q + \dots + q^r$$

Using the inductive determinant formula it is easy to see that $\deg F = \deg P$, which ends the proof. ☐

Corollary 4.5. *Let L be a field over $\mathbb{F}_q$ and $\mu_0, \dots, \mu_r \in L$. Then $\mu_0, \dots, \mu_r$ are $\mathbb{F}_q$-linear independent if and only if the matrix*

$$\begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_r \\ \mu_0^q & \mu_1^q & \dots & \mu_r^q \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0^{q^r} & \mu_1^{q^r} & \dots & \mu_r^{q^r} \end{pmatrix}$$

is invertible.

Proof of Theorem 4.2. The last claims hold because those properties of morphisms are local in the target for the fpqc topology (see [Sta18, 02L3], [Sta18, 02L6], [Sta18, 02L8]).

By 3.4 we can assume that $\mathcal{X} = \operatorname{Spec} A$ is affine.

1). Fully faithfulness. Faithfulness follows from 3.5 and 4.3. Thus let $U_i \to \mathcal{X}$, $i = 1, 2$ be immersions and

$$\lambda_D \colon \mathbf{D}_k(U_1) = U_1 \times_{\mathcal{X}} \mathbf{D}_k(\mathcal{X}) \longrightarrow U_2 \times_{\mathcal{X}} \mathbf{D}_k(\mathcal{X}) = \mathbf{D}_k(U_2)$$

be a map over $\mathbf{D}_k(\mathcal{X})$. We have to show that λ_D descends to a map $\lambda \colon U_1 \to U_2$.

Consider the projection $p \colon U_1 \times_{\mathcal{X}} U_2 \to U_1$. By construction its base change along $\mathbf{D}_k(\mathcal{X}) \to \mathcal{X}$ is an immersion with a section, thus an isomorphism. By fpqc descent it follows that also $p \colon U_1 \times_{\mathcal{X}} U_2 \to U_1$ is an isomorphism, which allows to define the map λ.

2). Essential surjectivity. Let us first show how to conclude assuming the result true for closed immersions. Let $Z \to U \to \mathbf{D}_k(A)$ be an immersion, where $Z \to U$ is a closed immersion and $U \subseteq \mathbf{D}_k(A)$ is an open subset. Consider the reduced closed substack $C = \mathbf{D}_k(A) \backslash U \to \mathbf{D}_k(A)$, so that there exists a closed immersion $Q \to \operatorname{Spec} A$ inducing $C \to \mathbf{D}_k(A)$. By construction $V = (\operatorname{Spec} A) \backslash Q \to \operatorname{Spec} A$ induces $U \to \mathbf{D}_k(A)$. In particular $U = \mathbf{D}_k(V)$. Applying again the result for closed immersion on $Z \to U = \mathbf{D}_k(V)$ we get the result.

Let us now focus on the case of closed immersions. Since closed immersions form a substack of Emb, using again 3.4 we just have to show that a closed embedding $Z \to \mathbf{D}_k(A)$ comes from A. This closed embedding is given by an ideal I of A_k such that $\phi_k(I) = I$. We have to prove that I is generated by elements of A. Let $\{\mu_i\}_{i \in J}$ be a basis of k over $\mathbb{F}_q$. Given $f \in I \subseteq A_k$ we can write

$$f = \sum_{r=0}^{n} \mu_{i_r} a_r$$

where $a_r \in A$. Set

$$M \coloneqq \begin{pmatrix} \mu_{i_0} & \mu_{i_1} & \dots & \mu_{i_n} \\ \mu_{i_0}^q & \mu_{i_1}^q & \dots & \mu_{i_n}^q \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i_0}^{q^n} & \mu_{i_1}^{q^n} & \dots & \mu_{i_n}^{q^n} \end{pmatrix}$$

then we have a matrix equation:

$$\begin{pmatrix} f \\ \phi_k(f) \\ \cdots \\ \phi_k^n(f) \end{pmatrix} = M \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ a_n \end{pmatrix}$$

in which M is invertible by 4.5. The relations obtained inverting M in the equation above allow to conclude that

$$\langle a_0, \dots, a_n \rangle_k = \langle f, \phi_k(f), \dots, \phi_k^n(f) \rangle_k \subseteq I$$

as k-vector spaces. This implies that

$$\{a_0, \dots, a_n\} \subseteq I \cap A$$

hence $I \subseteq (I \cap A)A_k \subseteq I$, so $I = (I \cap A)A_k$. □

Using 4.2 we prove a variant of Theorem 1.1.

Lemma 4.6. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$ and $\mathcal{M}$ be a stack in the étale topology with the following property: for all $\xi, \eta \in \mathcal{M}(T)$ for a scheme T the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(\xi, \eta) \to T$ is a sheaf in the fpqc topology and its diagonal is an immersion. Then the functor*

$$\alpha_{\mathcal{M},\mathcal{X}} \colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

is fully faithful.

Proof. By 3.4 we can assume that $\mathcal{X} = X = \operatorname{Spec} A$ is affine. Let $\xi, \eta \in \mathcal{M}(X)$ and set $H = \underline{\mathrm{Hom}}_{\mathcal{M}}(\xi, \eta) \to X$. By 3.5 we need to show that a commutative diagram

$$\begin{tikzcd} & H \arrow[d] \\ \mathbf{D}_k(X) \arrow[ur] \arrow[r] & X \end{tikzcd}$$

induces a section of $H \to X$. We have Cartesian diagrams of solid arrows

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & \mathbf{D}_k(X) \arrow[r, dashed] \arrow[d] & Z \arrow[d, dashed] \\ H_k \arrow[r] \arrow[d] & \mathbf{D}_k(H) \arrow[r] \arrow[d] & H \arrow[d] \\ X_k \arrow[r] & \mathbf{D}_k(X) \arrow[r] & X \end{tikzcd}$$

Since the diagonal of $H \to X$ is an immersion, the same property holds for the diagonal of $\mathbf{D}_k(X) \to \mathbf{D}_k(H)$. It follows that the section $\mathbf{D}_k(X) \to \mathbf{D}_k(H)$ is an immersion as well. Thanks to 4.2 we find an immersion $Z \to H$ and a Cartesian diagram like in the above diagram. Since Z is a sheaf in the fpqc topology and the map $Z \to X$ become an isomorphism after the fpqc base change $X_k \to X$ it follows that $Z \to X$ is an isomorphism as well. In conclusion the section $X \to Z \to H$ satisfies the requests. □

5. The proper case

The goal of this section is to prove Theorem 1.3, that is the case $\mathcal{M} = \mathrm{QCoh}_f$, which is the stack (not in groupoid) of finitely presented quasi-coherent sheaves over $\mathbb{F}_q$. Using this we will then prove Theorem 1.4.

Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$. An object of $\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$ is a pair $(\mathcal{F},\sigma)$ where $\mathcal{F} \in \mathrm{QCoh}_f(\mathcal{X}_k)$ and $\sigma\colon \phi_k^*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

We start showing how to prove Theorem 1.4 as a consequence of Theorem 1.3.

Lemma 5.1. *Let $\mathcal{M}$ be a quasi-compact category fibered in groupoid. Then the "taking colimit" functor*

$$\mathsf{ind}(\mathrm{QCoh}_f(\mathcal{M})) \to \mathrm{QCoh}(\mathcal{M})$$

is a faithful additive tensor functor. If $\mathcal{M}$ is quasi-separated then it is also fully faithful.

Proof. If $\mathcal{M}$ is an affine scheme the result is standard. In particular, in general, faithfulness follows taking an fpqc covering from an affine scheme.

So assume that $\mathcal{M}$ is a quasi-compact and quasi-separated fibered category. We have to show that if $\mathcal{F} \in \mathrm{QCoh}_f(\mathcal{M}))$ and if $\{\mathcal{G}_i\}_{i\in I} \in \mathsf{ind}(\mathrm{QCoh}_f(\mathcal{M}))$, then

$$\mathrm{Hom}_{\mathsf{ind}(\mathrm{QCoh}_f(\mathcal{M}))}(\mathcal{F},\{\mathcal{G}_i\}_{i\in I}) = \varinjlim_{i\in I} \mathrm{Hom}_{\mathrm{QCoh}_f(\mathcal{M})}(\mathcal{F},\mathcal{G}_i) \longrightarrow \mathrm{Hom}_{\mathrm{QCoh}_f(\mathcal{M})}(\mathcal{F},\varinjlim_{i\in I}\mathcal{G}_i)$$

is an isomorphism. Let $\pi\colon U \to \mathcal{M}$ be a fpqc covering from an affine scheme, set $R = U \times_{\mathcal{X}} U$, which is a quasi-compact algebraic space, let $V \to R$ be an étale atlas from an affine scheme and set $\alpha,\beta\colon V \to U$ the induced maps. If $\mathcal{F},\mathcal{G} \in \mathrm{QCoh}(\mathcal{M})$ then the following sequence is exact

$$\mathrm{Hom}_{\mathcal{M}}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_U(\pi^*\mathcal{F},\pi^*\mathcal{G}) \rightrightarrows \mathrm{Hom}_V(\alpha^*\pi^*\mathcal{F},\alpha^*\pi^*\mathcal{G})$$

Using this we can reduce the problem to the case when $\mathcal{M}$ is an affine scheme. □

Lemma 5.2. *Let $\mathcal{X}$ and $\mathcal{M}$ as follows:*

(1) $\mathcal{X}$ is a category fibered in groupoids over $\mathbb{F}_q$ for which

$$\alpha_{\mathrm{QCoh}_f,\mathcal{X}}\colon \mathrm{QCoh}_f(\mathcal{X}) \to \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence;

(2) $\mathcal{M}$ is a stack in groupoids over $\mathbb{F}_q$ for the fpqc (resp. fppf, étale) topology with quasi-affine diagonal and admitting a representable fpqc (resp. fppf, étale) covering from an affine scheme;

(3) for any map $\mathbf{D}_k(\mathcal{X}) \xrightarrow{\psi} \mathcal{M}$ and any quasi-coherent sheaf of algebras $\mathscr{A}$ over $\mathcal{M}$, the pullback $\psi^*\mathscr{A}$ is the colimit of a ring object in $\mathsf{ind}(\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})))$.

Then the functor

$$\alpha_{\mathcal{M},\mathcal{X}}\colon \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence.

Proof. By 4.6 we already know that $\alpha_{\mathcal{M},\mathcal{X}}$ is fully faithful.

By hypothesis there is an fpqc (resp. fppf, étale) covering $f\colon \tilde{W} = \operatorname{Spec} A \to \mathcal{M}$. Since $\mathcal{M}$ has quasi-affine diagonal, it follows that f is quasi-affine. Therefore, if $W \coloneqq \operatorname{Spec} f_*\mathcal{O}_{\tilde{W}}$ then $\tilde{W} \to W$ is a quasi-compact open immersion [Sta18, 01SM]. Now consider a functor $\psi\colon \mathbf{D}_k(\mathcal{X}) \to \mathcal{M}$ and the Cartesian diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"] & W \arrow[d, "f"] \\ \mathbf{D}_k(\mathcal{X}) \arrow[r, "\psi"] & \mathcal{M} \end{tikzcd}$$

Then by hypothesis $\psi^* f_*\mathcal{O}_{\tilde{W}} \simeq g_*\mathcal{O}_B$ is the colimit of a ring object in $\mathsf{ind}(\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})))$. We have a commutative diagram

$$\begin{tikzcd} \mathsf{ind}(\mathrm{QCoh}_f(\mathcal{X})) \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathcal{X}) \arrow[d] \\ \mathsf{ind}(\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))) \arrow[r] & \mathrm{QCoh}(\mathbf{D}_k(\mathcal{X})) \end{tikzcd}$$

Since the first vertical map is an equivalence by hypothesis, there is a ring object in $\mathsf{ind}(\mathrm{QCoh}_f(\mathcal{X}))$ inducing a quasi-coherent sheaf of algebra on $\mathcal{X}$ and therefore an affine map $V \to \mathcal{X}$ with a Cartesian diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"] & V \arrow[d] \\ \mathbf{D}_k(\mathcal{X}) \arrow[r] & \mathcal{X} \end{tikzcd}$$

In particular $B \simeq \mathbf{D}_k(V)$. Consider the Cartesian diagrams on the left in

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] & \tilde{W} \arrow[d] & & \tilde{B} \arrow[r] \arrow[d] & \tilde{V} \arrow[d] \\ \mathbf{D}_k(V) \arrow[r] \arrow[d] & W \arrow[d] & \Longrightarrow & \mathbf{D}_k(V) \arrow[r] \arrow[d] & V \arrow[d] \\ \mathbf{D}_k(\mathcal{X}) \arrow[r] & \mathcal{M} & & \mathbf{D}_k(\mathcal{X}) \arrow[r] & \mathcal{X} \end{tikzcd}$$

Applying 4.2 to the immersion $\tilde{B} \to \mathbf{D}_k(V)$ we obtain the Cartesian diagrams on the right for some immersion $\tilde{V} \to V$. Since $\tilde{V}_k \to \mathcal{X}_k$ is a base change of $\tilde{W} \to \mathcal{M}$ which is an fpqc (resp. fppf, étale) covering, it follows that $\tilde{V} \to \mathcal{X}$ is an fpqc (resp. fppf, étale) covering as well. From 3.9 with $B = \tilde{W}$ and 3.11 we obtain the desired map $\mathcal{X} \to \mathcal{M}$. □

Remark 5.3. (1) If $\mathcal{M}$ is a quasi-compact and quasi-separated algebraic stack, then each quasi-coherent sheaf on $\mathcal{M}$ is the union of its quasi-coherent subsheaves of finite type (see the main result of [Rydh16]).

(2) If $\mathcal{M}$ is an affine gerbe over a field, then each quasi-coherent sheaf is a filtered direct limit of vector bundles (which are the only quasi-coherent sheaf of finite type).

Proof of Theorem 1.4 as a consequence of Theorem 1.3. We apply 5.2. Condition (1) is satisfied thanks to Theorem 1.3.

We have to prove that conditions (2) and (3) of 5.2 are satisfied in the cases mentioned in Theorem 1.4.

In cases (2) and (3), taking into account 5.1 and 5.3, the functor

$$\mathsf{ind}(\mathrm{QCoh}_f(\mathcal{M})) \to \mathrm{QCoh}(\mathcal{M})$$

is an equivalence, so quasi-coherent sheaf of algebras on $\mathcal{M}$ are colimit of ring objects in $\mathsf{ind}(\mathrm{QCoh}_f(\mathcal{M}))$.

In case (1), if $\mathscr{A}$ is a quasi-coherent sheaf of algebras on $\mathcal{Y}$, by 5.3 we can write $\mathscr{A}$ as the filtered union of its quasi-coherent subsheaves $\mathscr{A}_i$ of finite type. Thus we have an object $\mathscr{A}_* = \{\mathscr{A}_i\}_i \in \mathsf{ind}(\mathrm{QCoh}(\mathcal{M}))$ whose limit is $\mathscr{A}$.

Since the image of $\mathscr{A}_i \otimes \mathscr{A}_j \to \mathscr{A}$ is of finite type, we obtain a map $\mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_*$ inducing the multiplication on $\mathscr{A}$. Since the image of $\mathcal{O}_{\mathcal{X}} \to \mathscr{A}$ is of finite type, we also have a unit map $\mathcal{O}_{\mathcal{X}} \to \mathscr{A}_*$.

This data define a ring structure on $\mathscr{A}_*$. Indeed this can be check on an affine atlas, using the following property: if R is a ring, $M \in \mathsf{Mod}(R)$ is finitely generated, $N_* \in \mathsf{ind}(\mathsf{Mod}(M))$ and $\alpha\colon M \to N_*$ is a map whose colimit is zero then $\alpha = 0$.

Notice that quasi-coherent sheaves of finite type over $\mathbf{D}_k(\mathcal{X})$ are finitely presented by fpqc descent because $\mathcal{X}_k$ is covered by a locally Noetherian scheme. We can therefore conclude that $\psi^*\mathscr{A}_*$ is a ring object in $\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$ as required. □

We now come back to the proof of Theorem 1.3. We first show some cases when the hypothesis of Theorem 1.3 are satisfied.

Remark 5.4.
1. Let $\mathcal{X}$ be a quasi-separated algebraic stack over $\mathbb{F}_q$ such that $\mathcal{X}$ and $\mathcal{X}_k$ are Noetherian and $\psi\colon \mathcal{X}_k \to \operatorname{Spec} k$ pushes coherent sheaves to finite dimensional vector spaces. Then $\mathcal{X}$ satisfies the hypothesis of 1.3 by 5.3.
2. If $\mathcal{X}$ satisfies the property in (1) and $f\colon \mathcal{Y} \to \mathcal{X}$ is a proper map of algebraic stacks, then $\mathcal{Y}$ also satisfies the property in (1). Indeed in this case f_* maps coherent sheaves into coherent sheaves. See [Fal03].
3. If $\mathcal{X}$ satisfies the property in (1) and $f\colon \mathcal{Y} \to \mathcal{X}$ is a relative (fppf) gerbe banded by a sheaf of groups $\mathcal{G} \to \mathcal{X}$ which is representable by flat algebraic spaces of finite type over $\mathcal{X}$, then $\mathcal{Y}$ also satisfies the property in (1).

   We claim more generally that: $\mathcal{Y}$ and $\mathcal{Y}_k$ are algebraic stacks; $f\colon \mathcal{Y} \to \mathcal{X}$ is smooth and quasi-separated, so that $\mathcal{Y}$ and $\mathcal{Y}_k$ are Noetherian and quasi-separated; f_* maps coherent sheaves to coherent sheaves.

   Taking into account [Sta18, 06DC] we see that all claims above are fppf local on $\mathcal{X}$. Thus we can assume $\mathcal{X} = \operatorname{Spec} A$, for some Noetherian ring A and $\mathcal{Y} = \mathrm{B}_A\,\mathcal{G}$, for some flat algebraic group space G of finite type over A. Again [Sta18, 06DC] tells us that $\mathcal{Y}$ is algebraic. Moreover $\mathrm{B}_A\,\mathcal{G} \to \operatorname{Spec} A$ is smooth: it is flat with smooth geometric fibers. The diagonal of $\mathcal{Y} \to \operatorname{Spec} A$ is fppf locally of the form $\mathcal{G} \to \operatorname{Spec} A$, which is quasi-compact by hypothesis.

Finally, a quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{B}_A\,\mathcal{G}$ is equivalent to a A-module M with an action of $\mathcal{G}$ and $f_*\mathcal{F}$ is just the submodule of $\mathcal{G}-$invariants $M^{\mathcal{G}} \subseteq M$. Thus if $\mathcal{F}$ is coherent, that is of finite type, then M is finitely generated, and so is $M^{\mathcal{G}}$.

We will actually work in a slightly more general situation than the one in 1.3.

Situation 5.5. Let $\mathcal{X}$ be a quasi-compact category fibered in groupoids over $\mathbb{F}_q$. Let $\mathcal{C}$ be a full subcategory of $\mathrm{QCoh}_f(\mathcal{X})$ which generates it, that is, such that all quasi-coherent sheaves are quotient of a sum of objects in $\mathcal{C}$. Assume that for all $\mathcal{F} \in \mathrm{QCoh}_f(\mathcal{X}_k)$ the k-vector space $\mathrm{Hom}_{\mathcal{X}_k}(\mathcal{E} \otimes k, \mathcal{F})$ has finite dimension for all $\mathcal{E} \in \mathcal{C}$, where $\mathcal{E} \otimes k$ is the pullback of $\mathcal{E}$ along $\pi\colon \mathcal{X}_k \to \mathcal{X}$. For simplicity we also assume that $\mathcal{C}$ is stable under finite direct sums.

In the hypothesis of 1.3 we can take $\mathcal{C} = \mathrm{QCoh}_f(\mathcal{X})$ and the above requirements are satisfied. We prove

Theorem 5.6. *If $\mathcal{X}$ is as in Situation 5.5 then*

$$\mathrm{QCoh}_f(\mathcal{X}) \to \mathrm{QCoh}(\mathbf{D}_k(\mathcal{X}))$$

is an equivalence of categories. In particular Theorem 1.3 holds.

The main result we are going to use is:

Proposition 5.7. *[Ton20, Theorem B] Let $\mathcal{X}$ be a quasi-compact category fibered in groupoid over a ring R, A be an R-algebra and denote by $\mathrm{L}_R\mathcal{C}, A)$ the category of contravariant R-linear functors $\mathcal{C} \to \mathsf{Mod}(A)$. Then the functor*

$$\Gamma_*\colon\ \mathrm{QCoh}(\mathcal{X} \times_R A) \to \mathrm{L}_R(\mathcal{C}, A),\ \ \mathcal{F} \longmapsto \Gamma_{\mathcal{F}} = \mathrm{Hom}_{\mathcal{X}\times_R A}(- \otimes_R A, \mathcal{F})$$

is an equivalence onto the full subcategory of functors $\Omega\colon \mathcal{C} \to \mathsf{Mod}(A)$ which are exact on right exact sequences with objects in $\mathcal{C}$.

Proof of Theorem 5.6. Using 5.7 we want to define a functor $\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})) \to \mathrm{QCoh}_f(\mathcal{X})$. Consider an object in $\mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$, which we think of as a pair $(\mathcal{F}, \sigma)$ where $\mathcal{F} \in \mathrm{QCoh}_f(\mathcal{X}_k)$ and $\sigma\colon \phi_k^*\mathcal{F} \to \mathcal{F}$ is an isomorphism. With such a pair we can associate the $\mathbb{F}_q$-linear functor

$$\Gamma_{\mathcal{F}} = \mathrm{Hom}_{\mathcal{X}_k}(- \otimes_{\mathbb{F}_q} k, \mathcal{F})\colon \mathcal{C} \to \mathsf{Mod}(k)$$

and a natural isomorphism $\Gamma_{\phi_k^*\mathcal{F}} \to \Gamma_{\mathcal{F}}$. Notice that

$$\Gamma_{\phi_k^*\mathcal{F}}(\mathcal{E}) = \phi_k^*(\Gamma_{\mathcal{F}}(\mathcal{E})) \text{ for } \mathcal{E} \in \mathcal{C}$$

Since by hypothesis $\Gamma_{\mathcal{F}}$ has value in $\mathsf{Vect}(k)$ we can conclude that $(\mathcal{F}, \sigma)$ defines a functor $\Gamma_{(\mathcal{F},\sigma)}\colon \mathcal{C} \to \mathsf{Vect}(\mathbf{D}_k(\mathbb{F}_q))$. Via the canonical equivalence of 1.2

$$\mathsf{Vect}(\mathbb{F}_q) \to \mathsf{Vect}(\mathbf{D}_k(\mathbb{F}_q))$$

we obtain a functor

$$\Omega\colon\ \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})) \to \mathrm{L}_R(\mathcal{C}, \mathbb{F}_q)$$

together with an isomorphism $\Omega_{\mathcal{F}} \otimes_{\mathbb{F}_q} k \to \Gamma_{\mathcal{F}}$ compatible with the ϕ_k equivariant maps. Notice moreover that by construction the composition

$$\mathrm{QCoh}_f(\mathcal{X}) \to \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})) \to \mathrm{L}_R(\mathcal{C}, \mathbb{F}_q)$$

is just Γ_* over $\mathbb{F}_q$. Using 5.7 the following lemma concludes the proof of 5.6. ☐

Lemma 5.8. *The functor $\Omega\colon \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X})) \to \mathrm{L}_R(\mathcal{C}, \mathbb{F}_q)$ is fully faithful and its image is made by functors which are exact on right exact sequences.*

Proof. Let $\mathcal{F} = (\mathcal{F}, \sigma), \mathcal{G} = (\mathcal{G}, \delta) \in \mathrm{QCoh}_f(\mathbf{D}_k(\mathcal{X}))$. The second claim is clear: $\Omega_{\mathcal{F}}$ is exact because $\Gamma_{\mathcal{F}}\colon \mathcal{C} \to \mathsf{Vect}(k)$ is so and $\Omega_{\mathcal{F}} \otimes_{\mathbb{F}_q} k \simeq \Omega_{\mathcal{F}}$.

By 5.7 the functor

$$\Gamma_*\colon\ \mathrm{QCoh}(\mathcal{X}_k) \to \mathrm{L}_{\mathbb{F}_q}(\mathcal{C}, k)$$

is fully faithful. This easily implies that the functor in the statement is faithful.

Let $\beta\colon \Omega_{\mathcal{F}} \to \Omega_{\mathcal{G}}$ be a morphism. Via the equivalence $\mathsf{Vect}(\mathbb{F}_q) \to \mathsf{Vect}(\mathbf{D}_k(\mathbb{F}_q))$ the map β is a natural morphism $\beta\colon \Gamma_{\mathcal{F}} \to \Gamma_{\mathcal{G}}$ such that

$$\begin{tikzcd}
\Gamma_{\phi_k^*\mathcal{F}} = \phi_k^*\Gamma_{\mathcal{F}} \arrow[r, "\Gamma_\sigma"] \arrow[d, "\phi_k^*\beta"] & \Gamma_{\mathcal{F}} \arrow[d, "\beta"] \\
\Gamma_{\phi_k^*\mathcal{G}} = \phi_k^*\Gamma_{\mathcal{G}} \arrow[r, "\Gamma_\delta"] & \Gamma_{\mathcal{G}}
\end{tikzcd}$$

By 5.7 $\beta = \Gamma_\zeta$ for a unique $\zeta\colon \mathcal{F} \to \mathcal{G}$. Moreover, ζ makes the following diagram commutative as required:

$$\begin{tikzcd}
\phi_k^*\mathcal{F} \arrow[r, "\sigma"] \arrow[d, "\phi_k^*\zeta"] & \mathcal{F} \arrow[d, "\zeta"] \\
\phi_k^*\mathcal{G} \arrow[r, "\delta"] & \mathcal{G}
\end{tikzcd}$$

This concludes the proof. ☐

Theorem 5.9. *Affine gerbes over $\mathbb{F}_q$ are trivial.*

Proof. We apply 1.4 with $\mathcal{X} = \operatorname{Spec} \mathbb{F}_q$ and $\mathcal{M}$ an affine gerbe. Choose an algebraically closed field k such that there is an object $\xi \in \mathcal{M}(k)$. It could be that ξ and $\phi_k^*\xi$ are not isomorphic, but they surely become so enlarging the field k. Thus we get an object of $\mathcal{M}(\mathbf{D}_k(\mathbb{F}_q)) \simeq \mathcal{M}(\mathbb{F}_q)$. ☐

6. The etalé case

In this section we consider the case when $\mathcal{M} = \mathrm{Et}$ is the stack over $\mathbb{F}_q$ of representable, étale, quasi-compact and quasi-separated morphisms from algebraic spaces. Let $\mathcal{M} = \mathrm{Et}_s$ denote the substack of Et of étale maps which are separated. The goal is to prove Theorem 1.5. We use the abbreviation *qcqs* for quasi-compact and quasi-separated, and *qcs* for quasi-compact and separated.

If $\mathcal{X}$ is a category fibered in groupoids then $\mathrm{Et}(\mathcal{X})$ (resp. $\mathrm{Et}_s(\mathcal{X})$) is the category of maps $\mathcal{V} \to \mathcal{X}$ which are representable, étale and qcqs (resp. qcs). Arrows are any morphisms between such objects: Et and Et_s are not fibered in groupoids. Since locally quasi-finite and separated morphisms of algebraic spaces are schematically representable (see

it follows that if $V \to X$ is an object of $\mathrm{Et}_s(X)$ and X is a scheme then also V is a scheme.

We denote by FP the fibered category over $\mathbb{Z}$ of finitely presented (in particular qcqs) morphisms from algebraic spaces: if $\mathcal{X}$ is a category fibered in gropoids then $\mathrm{FP}(\mathcal{X}) = \mathrm{Hom}(\mathcal{X}, \mathrm{FP})$ is the category of representable morphisms $\mathcal{Y} \to \mathcal{X}$ which are finitely presented.

Proposition 6.1. *Let $A = \varinjlim_i A_i$ be a filtered direct limit of rings. Then the natural functor*

$$\varinjlim_i \mathrm{FP}(A_i) \longrightarrow \mathrm{FP}(A)$$

is an equivalence and it preserves all the following properties of morphisms: schematic, affine, separated, smooth, étale.

Proof. The first claim is [Sta18, 076K]. For the other ones we have instead: schematic by [Sta18, 01ZM]; affine by [Sta18, 01ZN]; separated by [Sta18, 0851]; smooth by [Sta18, 0CN2]; étale by [Sta18, 07SL]. □

Proof of Theorem 1.5. By 3.4 we can assume that $\mathcal{X} = X = \mathrm{Spec}\, A$ is affine. An object of $\mathrm{Et}_s(\mathbf{D}_k(A))$ is a pair (V, σ) where $V \to \mathrm{Spec}\,(A_k)$ is an etale and qcs map of schemes and $\sigma \colon V \to V$ is an isomorphism making the following diagram commutative

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,(A_k) & \xrightarrow{\phi_k} & \mathrm{Spec}\,(A_k) \end{array}$$

Let $\{A_i\}_i$ be the set of all $\mathbb{F}_q$-subalgebras of A which are of finite type over $\mathbb{F}_q$, so that A is a filtered direct limit of the A_i. We have that

$$\varinjlim_i \mathrm{Et}_s(A_i) \to \mathrm{Et}_s(A) \qquad \text{and} \qquad \varinjlim_i \mathrm{Et}_s((A_i)_k) \to \mathrm{Et}_s(A_k)$$

are equivalences thanks to 6.1. Therefore the functor

$$\varinjlim_i \mathrm{Et}_s(\mathbf{D}_k(A_i)) \to \mathrm{Et}_s(\mathbf{D}_k(A))$$

is an equivalence. In particular we can assume that A is of finite type over $\mathbb{F}_q$.

Let $\overline{X}$ be any compactification of $X = \mathrm{Spec}\, A$, so that X is an open subset of $\overline{X}$. Composing along $X \to \overline{X}$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Et}_s(X) & \longrightarrow & \mathrm{Et}_s(\mathbf{D}_k(X)) \\ \downarrow & & \downarrow \\ \mathrm{Et}_s(\overline{X}) & \longrightarrow & \mathrm{Et}_s(\mathbf{D}_k(\overline{X})) \end{array}$$

The vertical functor are fully faithful. It is easy to see that the result for $\overline{X}$ implies the result for X. Thus we can assume that X is a projective scheme over $\mathbb{F}_q$. Since the small étale site does not change under a nilpotent closed immersion, we can moreover assume that X is reduced. In conclusion we can assume that X is a projective variety over $\mathbb{F}_q$.

By [Sta18, 02LR] and [Sta18, 03GR] if Z is a reduced scheme of finite type over some field (hence Nagata) and $V \to Z$ is a map of schemes étale and separated then the normalization $N(V) \to Z$ of Z in V is finite and $V \to N(V)$ is an open immersion.

Let Fin denote the stack of finite morphisms. By the functoriality of normalizations [Sta18, 035J] we get a functor

$$\mathrm{Et}_s(X) \longrightarrow \mathrm{Fin}(X)$$

Again by functoriality [Sta18, 035J], if $(V \to X_k, \sigma) \in \mathrm{Et}_s(\mathbf{D}_k(X))$, then $\sigma\colon V \to V$ extends to an *isomorphism* $N(\sigma)\colon N(V) \to N(V)$ over $\phi_k\colon X_k \to X_k$. Thus we get a morphism $\mathrm{Et}_s(\mathbf{D}_k(X)) \to \mathrm{Fin}(\mathbf{D}_k(X))$. We claim that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Et}_s(X) \arrow[r, "\alpha_{\mathrm{Et}_s,X}"] \arrow[d] & \mathrm{Et}_s(\mathbf{D}_k(X)) \arrow[d] \\
\mathrm{Fin}(X) \arrow[r] & \mathrm{Fin}(\mathbf{D}_k(X))
\end{tikzcd}$$

We are going to use that the bottom functor is an equivalence thanks to 1.3. Given $U \to X$ in $\mathrm{Et}_s(X)$ we can construct the following Cartesian diagrams:

$$\begin{tikzcd}
U_k \arrow[r] \arrow[d] & N(U_k) \arrow[r, "v"] \arrow[d] & N(U)_k \arrow[r] \arrow[d] & X_k \arrow[d] \\
W \arrow[r] & N \arrow[r, "u"] & N(U) \arrow[r] & X
\end{tikzcd}$$

where the top arrows should be thought of morphisms over $\mathbf{D}_k(X)$. The finite map $N \to X$ and the morphism u are obtained using that $\alpha_{\mathrm{Fin},X}$ is an equivalence. The map $W \to N$ is the open immersion obtained from 4.2. Since $W \to N(U)$ pullback to an open immersion, it is an open immersion. Applying again 4.2 we can conclude that $W = U \to N(U)$ is the given open immersion.

By the universal property of normalization we can conclude that $N \to N(U)$ is an isomorphism.

We now show that $\alpha_{\mathrm{Et}_s,X}$ is an equivalence.

For the fully faithfulness, given $U \to X$ and $W \to X$ in $\mathrm{Et}_s(X)$ and a morphism $\mathbf{D}_k(U) \to \mathbf{D}_k(W)$, this map extends to a morphism $\mathbf{D}_k(N(U)) \to \mathbf{D}_k(N(W))$ induced by a map $a\colon N(U) \to N(W)$. We have $a(U) \subseteq W$ because this relation holds after pulling back to $N(W)_k$.

For the essential surjectivity, starting with $(V \to X_k, \sigma) \in \mathrm{Et}_s(\mathbf{D}_k(X))$ we get $(N(V) \to X_k, N(\sigma)) \in \mathrm{Fin}(\mathbf{D}_k(X))$ which descents to $(N \to X) \in \mathrm{Fin}(X)$. Then we get the descent $U \subseteq N$ of the open embedding $V \subseteq N(V) = N_k$ by 4.2. ☐

Proposition 6.2. *Let $\mathcal{X}$ be a category fibered in groupoids over $\mathbb{F}_q$. Then the functor*

$$\mathrm{Et}(\mathcal{X}) \longrightarrow \mathrm{Et}(\mathbf{D}_k(\mathcal{X}))$$

is fully faithful.

Proof. According to 3.4 we may assume that $\mathcal{X} = X = \operatorname{Spec} A$.

Let $Y \to X$ be an object of $\mathrm{Et}(X)$, that is an étale and quasi-compact morphism of algebraic spaces and $U \to Y$ be a map from an algebraic space. If $(U \to X) \in \mathrm{Et}_s(X)$, then $(U \to Y) \in \mathrm{Et}_s(Y)$. Indeed $U \to Y$ is quasi-compact because Y is quasi-separated and it is separated by [Sta18, 03KR].

If $U \to Y$ is an étale atlas from a qcs scheme and $R = U \times_Y U$, then R is qcs because Y is quasi-separated and $R \to U \times_X U$ is a monomorphism. In particular $R \rightrightarrows U$ defines a groupoid in $\mathrm{Et}_s(X)$ but also in $\mathrm{Et}_s(Y)$.

We come back to the problem of fully faithfulness. Faithfulness follows from 3.5.

Suppose that $Z \longrightarrow X$ is another object in $\mathrm{Et}(X)$, and suppose that $\lambda\colon Y_k \longrightarrow Z_k$ define a morphism $\mathbf{D}_k(Y) \to \mathbf{D}_k(Z)$. Let $R \rightrightarrows U \longrightarrow Y$ be an étale presentation by qcs étale schemes. A map $Y \longrightarrow Z$ is the same as a map $U \longrightarrow Z$ which equalizes $R \rightrightarrows U$. By 1.5 we may assume that Y is a qcs étale scheme over X.

Now suppose that $R \rightrightarrows U \longrightarrow Z$ is an étale presentation by qcs étale schemes. Then the pullback

$$R'_k \rightrightarrows U'_k \longrightarrow Y_k$$

of $R_k \rightrightarrows U_k \longrightarrow Z_k$ along λ is an étale presentation of Y_k. Moreover this presentation belongs to $\mathrm{Et}_s(Y_k)$. More precisely it defines a groupoid in $\mathrm{Et}_s(\mathbf{D}_k(Y))$. By 1.5 it descend to an étale presentation

$$R' \rightrightarrows U' \longrightarrow Y$$

in $\mathrm{Et}_s(Y)$. On the other hand, since $Y \to X$ is separated, it follows that the presentation actually belongs to $\mathrm{Et}_s(X)$.

By construction the original morphism $\lambda\colon Y_k \to Z_k$ defines a morphism of the groupoid presentations in $\mathrm{Et}_s(\mathbf{D}_k(X))$. By 1.5 this morphism induces a morphism of the corresponding groupoid presentations in $\mathrm{Et}_s(X)$. By étale descent we get a map $Y \longrightarrow Z$ inducing λ. □

7. The non proper case

The main of goal of this section is to prove Theorem 1.6.

Proof of Theorem 1.6. Let $\mathcal{M}$ be a Deligne Mumford stack over $\mathbb{F}_q$ with qcs diagonal. In particular, by 4.6, we know that the map $\alpha_{\mathcal{M},\mathcal{V}}$ is fully faithful for all categories fibered in groupoids $\mathcal{V}$ over $\mathbb{F}_q$. So we are interested in the essential surjectivity.

By 3.4 we can assume that $\mathcal{X} = X = \operatorname{Spec} A$ is affine. Let $\mathbf{D}_k(X) \to \mathcal{M}$ be a map. Since $\mathcal{M}$ is a union of quasi-compact open substacks, we see that $\mathbf{D}_k(X) \to \mathcal{M}$ factors through one of this opens. In other words we can assume that $\mathcal{M}$ is quasi-compact, that is there exists an étale map $W \to \mathcal{M}$ where W is an affine scheme. The condition on the diagonal of $\mathcal{M}$ assures that $W \to \mathcal{M}$ is an étale and qcs map. Putting together 1.5, 3.9 with $B = W$ and 3.11 we obtain the desired map $X \to \mathcal{M}$. □

Proof of Theorem 1.1. By 3.4 we can assume that $\mathcal{X} = X = \operatorname{Spec} A$ is affine. Let $\xi, \eta \in \mathcal{M}(X)$, set $I = \underline{\mathrm{Iso}}_{\mathcal{M}}(\xi,\eta) \to X$ and let $\mathbf{D}_k(X) \to I$ be a map. Since I is a quasi-separated algebraic space, by 1.6 we find a factorization $\mathbf{D}_k(X) \to X \to I$. Since $\mathbf{D}_k(X) \to I \to X$ is the canonical map, again 1.6 tell us that $X \to I \to X$ is the identity, as required. □

8. Counterexamples

The following examples shows some cases when the main Theorems of this paper cannot be extended.

Example 8.1. Let $X = \mathbb{G}_m$ and $\mathcal{M} = \mathrm{B}\,\mathbb{G}_m$. We show that the functor

$$\mathcal{M}(X) \to \mathcal{M}(\mathbf{D}_k(X))$$

which is fully faithful thanks to 1.1, is not essentially surjective. In particular Theorem 1.6 cannot be extended to algebraic stacks.

By definition it is easy to write down the exact sequence

$$\mathrm{H}^0(\mathcal{O}^*_{X_k}) \to \mathrm{H}^0(\mathcal{O}^*_{X_k}) \to \mathrm{Pic}(\mathbf{D}_k(X)) \to \mathrm{Pic}(X_k)$$

where the first map is $\omega \mapsto \omega\phi_k(\omega)^{-1}$ and X may be arbitrary. If $X = \mathbb{G}_m = \mathrm{Spec}\,(\mathbb{F}_q[x]_x)$ then $\mathrm{H}^0(\mathcal{O}^*_{X_k}) = \{\lambda x^t \mid \lambda \in k^*,\ t \in \mathbb{Z}\}$ and $\phi_k(\lambda x^t) = \lambda^q x^t$. It is therefore clear that

$$0 = \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{D}_k(X)) \simeq \mathbb{Z}$$

is not an isomorphism.

Example 8.2. We show an example of a finite, flat and finitely presented map $Z \to \mathbf{D}_k(\mathbb{G}_m)$ which does not come from $\mathbb{G}_m$. In particular Theorem 1.5 cannot be extended to the case of finite covers.

It is enough to consider a non trivial invertible sheaf $\mathcal{L}$ on $\mathbf{D}_k(\mathbb{G}_m)$ as in the above example and set

$$Z = \mathrm{Spec}\,(\mathcal{O}_{\mathbf{D}_k(\mathbb{G}_m)} \oplus \mathcal{L}) \to \mathbf{D}_k(\mathbb{G}_m),\ \ \mathcal{L}^2 = 0$$

Example 8.3. Let $\mathcal{X} = \mathrm{Spec}\,(\mathbb{F}_q)$. The map $\mathrm{Spec}\, k \to \mathbf{D}_k(\mathbb{F}_q)$ is a $\underline{\mathbb{Z}}$-torsor, therefore a map which is schematically representable, étale, separated but not quasi-compact. On the other hand this map does not come from a map over $\mathrm{Spec}\,\mathbb{F}_q$, so that Theorem 1.5 cannot be extended in this direction.

Indeed suppose to have Cartesian diagrams as follows

$$\begin{tikzcd} \mathrm{Spec}\, k \times \underline{\mathbb{Z}} \arrow[r] \arrow[d, "v"] & \mathrm{Spec}\, k \arrow[r, "u"] \arrow[d] & V \arrow[d] \\ \mathrm{Spec}\, k \arrow[r] & \mathbf{D}_k(\mathbb{F}_q) \arrow[r] & \mathrm{Spec}\,\mathbb{F}_q \end{tikzcd}$$

The map u is surjective, therefore V must be a point, hence affine. It follows that the map v is quasi-compact, which is not true.

Example 8.4. Let $X = \mathrm{Spec}\,\mathbb{F}_q$ and $\mathcal{M} = \mathrm{B}\,\underline{\mathbb{Z}}$. This is a quasi-compact Deligne-Mumford stack with separated diagonal, but it is not quasi-separated.

We claim that the map $\mathbf{D}_k(\mathbb{F}_q) \to \mathcal{M}$ does not factors through a map $\operatorname{Spec} \mathbb{F}_q \to \mathcal{M}$, which implies that the functor

$$\mathcal{M}(\mathbb{F}_q) \to \mathcal{M}(\mathbf{D}_k(\mathbb{F}_q))$$

which is fully faithful thanks to 1.1, it is not essentially surjective. In particular Theorems 1.4 and 1.6 cannot be extended in this direction.

We cannot have a factorization $\mathbf{D}_k(\mathbb{F}_q) \to \operatorname{Spec} \mathbb{F}_q \to \mathcal{M}$ because otherwise we would have Cartesian diagrams

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{F}_q \arrow[d] \\ \mathbf{D}_k(\mathbb{F}_q) \arrow[r] & \operatorname{Spec} \mathbb{F}_q \arrow[r] & \mathrm{B}\,\underline{\mathbb{Z}} \end{tikzcd}$$

contraddicting what we saw in 8.3.

Example 8.5. Let $X = \mathbb{F}_q$ and consider $\mathcal{M} = \mathbf{D}_k(\mathbb{F}_q)$. Then $\mathcal{M}$ is a quasi-compact algebraic space, but it is not quasi-compact. On the other hand the map

$$\mathcal{M}(X) \to \mathcal{M}(\mathbf{D}_k(X))$$

is not essentially surjective, because the map $\mathrm{id}\colon \mathbf{D}_k(X) \to \mathcal{M}$ does not factor through a map $X \to \mathcal{M}$. In particular Theorems 1.4 and 1.6 cannot be extended in this direction.

If, by contraddiction, we have a factorization $\mathbf{D}_k(\mathbb{F}_q) \to \operatorname{Spec} \mathbb{F}_q \to \mathbf{D}_k(\mathbb{F}_q)$, then we would also have a factorization $\mathbf{D}_k(\mathbb{F}_q) \to \operatorname{Spec} \mathbb{F}_q \to \mathrm{B}\,\underline{\mathbb{Z}}$, which is false as we saw in 8.4.

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