Research Project

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August 26, 2021

Contents

1 Past and current research

My general area of research is algebraic geometry and, more precisely, moduli theory. The study of moduli problems led me to form a strong background in commutative algebra, scheme theory and stack theory, which are the languages used in this context.

1.1 Ph.D. thesis and works on ramified G-covers

In my Ph.D. thesis I have introduced the notion of ramified Galois cover under a finite group scheme G (briefly a G-cover) and studied the geometry of the moduli stack G-Cov they form. The study of G-covers essentially splits in two cases, namely the abelian case, where I have provided a toric description of a special irreducible component of G-Cov (see [Ton13]), and the non-abelian one. This last case is much harder than the abelian one, due to the complexity of the representation theory of a non-abelian group. Torsors under the group G are special cases of G-covers and Tannaka's reconstruction asserts that they correspond to particular strong monoidal functor. Following this point of view I have been able to identify a special class of (non-strong) monoidal functors and extend Tannaka's correspondence between them and G-covers and, more generally, to interpret any cover with an action of a group in terms of a functor. This point of view allowed me to prove results about the geometry of G-Cov and give a divisorial condition assuring that the total space of a G-cover of a normal scheme is normal too (see [Ton17]).

In [Ton21], starting from the results in my thesis, I have described the geometry of G-Cov for the simplest non abelian group, namely $G = S_3$. In particular I proved that its main irreducible component (closure of the open substack of torsors) is universally closed, more precisely a stack quotient by GL_2 of a smooth non degenerate projective surface in $\mathbb{P}^7$ which is intersection of five quadrics.

1.2 Sheafification of linear functors

Equivariant problems are known to correspond to problems over the stack $\mathrm{B}\,G$ of G-torsors and, using this strategy, I have generalized the correspondence between G-covers and particular monoidal functors (see Section 1.1) replacing $\mathrm{B}\,G$ by an arbitrary algebraic stack (see [Ton14]): given a collection $\mathcal{C}$ of quasi-coherent sheaves on an algebraic stack $\mathcal{X}$ it is possible to map quasi-coherent sheaves (of algebras) to particular (monoidal) linear functors and conversely. For instance considering $\mathcal{X} = \mathrm{B}\,G$ and by $\mathcal{C}$ the category of locally free sheaves on $\mathrm{B}\,G$ (that is locally free G-representations) one obtains the characterization of G-covers. Quite surprisingly, considering instead by $\mathcal{X}$ a projective scheme and setting $\mathcal{C} = \{\mathcal{O}_{\mathcal{X}}(n)\}_{n\in\mathbb{Z}}$ one recovers the correspondence between quasi-coherent sheaves on $\mathcal{X}$ and graded modules over a coordinate ring of $\mathcal{X}$. More generally this theory works well under the assumption that $\mathcal{C}$ form a generator of the category of quasi-coherent sheaves on the stack $\mathcal{X}$.

In [Ton20] the above theory is used to define and study certain stacks of fiber functors and prove a generalization of classical Tannaka's reconstruction for gerbes to arbitrary stacks with the resolution property (all quasi-coherent sheaves are quotient of a sum of vector bundles).

Another application is in [DTZ20] (see Section 1.9)

1.3 Picard group of the moduli of uniform cyclic covers of curves

I have also studied moduli problems of covers of curves and, in particular, described the Picard group of some of those moduli (see the joint work [PTT15] with Mattia Talpo (University of Pisa, Italy) and Flavia Poma).

1.4 Nori and algebraic fundamental gerbes and their Tannakian interpretation

I have been collaborating with Lei Zhang (Freie University of Berlin, Germany) on the study of the Nori fundamental gerbe and algebraic fundamental gerbes of a stack. Let $\mathcal{X}$ be a geometrically connected and geometrically reduced stack of finite type over a field k. One can associate with $\mathcal{X}$ a profinite gerbe $\Pi_{\mathcal{X}}$ together with a map $\pi\colon \mathcal{X} \longrightarrow \Pi_{\mathcal{X}}$ having the following universal property: all maps from $\mathcal{X}$ to a finite gerbe factor uniquely through π. The gerbe $\Pi_{\mathcal{X}}$ is called the Nori fundamental gerbe of $\mathcal{X}$.

Following ideas from [BV14] and [EH10], in [TZ19a] we introduced the notion of algebraic fundamental gerbe of $\mathcal{X}$ and prove that $\Pi_{\mathcal{X}}$ is its profinite quotient, generalizing what is known for smooth varieties. In other words, we introduced a Tannakian category $\mathrm{Strat}_{\infty}(\mathcal{X})$, whose corresponding gerbe is the algebraic fundamental gerbe of $\mathcal{X}$ and showed that the category $\mathrm{Rep}\,\Pi_{\mathcal{X}}$ of representations of $\Pi_{\mathcal{X}}$ can be identified with the subcategory of $\mathrm{Strat}_{\infty}(\mathcal{X})$ of essentially finite objects. Being essentially finite is an algebraic condition and, in [TZ19b], we find an equivalent but more geometric description by looking at objects of $\mathrm{Strat}_{\infty}(\mathcal{X})$ who are trivialized by some covering.

In particular this yields a geometric criterion for the essential finiteness of vector bundles over a pseudo-proper stack of finite type. This criterion has made possible to solve the problem of finding a Galois closure of a tower of torsors, which is the subject of [ABE+19], joint work with Marco Antei (University of Costa Rica), Indranil Biswas (Tata Institute of Fundamental Research, Bombay, India), Michel Emsalem (Université des Sciences et des Technologies de Lille, France) and Lei Zhang.

In [TZ20] we also apply the machinery to generalize a characterization of essentially finite vector bundles on normal proper varieties to stacks, which yields an alternative proof in the classical case.

1.5 Moduli of formal torsors

I have been collaborating with Takehiko Yasuda (Osaka University, Japan) about the construction of a moduli stack of Galois covers of a formal power series ring. This moduli is needed in a project about motivic integration and the wild McKay correspondence. Given a finite group G, in [TY20a] we consider the fiber category given by $\Delta_G(B) = \mathrm{B}\,G(B[[t]]_t)$, so that the k-rational points of Δ_G are the G-torsors of $k((t))$. When G is a semidirect product of a p-group (where $\mathrm{char}\, k = p > 0$) and a cyclic tame group, we prove that Δ_G is a direct limit of separated Deligne-Mumford stacks whose transition maps are composition of closed embeddings and universal homeomorphisms. In particular Δ_G admits a "ind coarse moduli space" (defined as one can expect).

In the subsequent paper [TY21] we consider the case of a general group G. In order to study the stack Δ_G we introduce the category of P-schemes, a category in which finitely presented and universally bijective morphisms become isomorphisms, and the notion of P-moduli spaces. Starting from the results of [TY20a], we show that Δ_G admits a P-moduli space $\overline{\Delta}_G$, so that it is possible to integrate (in a motivic sense) locally constructible functions $f\colon \Delta_G \to \frac{1}{l}\mathbb{Z}$ over $\overline{\Delta}_G$. We furthermore prove that some special functions on Δ_G are locally constructible. All this machinery made possible to state a wild McKay correspondence, expressed in terms of an equality between integrals, which later has been proved in [Yas19].

1.6 Arithmetic Nori local fundamental group scheme

In the paper [RTZ21], building up from results in [TZ19a], we introduce and study the Nori local fundamental group scheme of a reduced scheme (or stack in general) X over a perfect field k of positive characteristic p, denoted by $\pi^{\mathrm{L}}(X/k)$. This group "parametrizes" torsors of X under finite and connected group schemes, also called local, and it is pro-local, i.e. a projective limit of local groups.

We then focus on a specific case: if K/k is a field extension then $\pi^{\mathrm{L}}(K/k) = \pi^{\mathrm{L}}((\operatorname{Spec} K)/k)$ is a pro-local group scheme over k. It turns out that there is a parallel between this group and the absolute Galois group, where purely inseparable extensions corresponds to separable ones. For example, given K/k we show that the map

$$\begin{aligned} \{\text{p.i. extensions of } K\} &\longrightarrow \{\text{subgroups of } \pi^{L}(K)\} \\ F/K &\longmapsto \pi^{L}(F) \subseteq \pi^{L}(K) \end{aligned}$$

is a order-reverse-embedding. We mention that this map is not surjective, so we cannot talk about a correspondence in this case. In [RTZ21] we state two conjectures that we discuss in Sections 2.4.2 and 2.4.3.

1.7 Berthelot's conjecture for isocrystals and Künneth formula

In the paper [DTZ21] with Lei Zhang and Valentina Di Proietto (University of Exeter, United Kingdom) we prove the Berthelot's conjecture for isocrystals. This conjecture predicts that under a proper and smooth morphism of schemes in characteristic p, the higher direct images of an overconvergent F-isocrystal are overconvergent F-isocrystals. In this paper we prove that this is true for crystals up to isogeny. As an application we prove a Künneth formula for the crystalline fundamental group.

1.8 Cox rings for algebraic stacks

In the paper [HMT20] with Andreas Hochenegger (Politecnico di Milano, Italy) and Elena Martinengo (University of Torino, Italy) we extended the classical definition of Cox rings from algebraic varieties to algebraic stacks. Given an algebraic stack X and the graded module

$$S = \bigoplus_{\mathcal{L} \in \operatorname{Pic}(X)} \mathrm{H}^0(X, \mathcal{L})$$

we show that all possible multiplications on S compatible with tensor products of invertible sheaves are parametrized by $\operatorname{Ext}^1(\operatorname{Pic}(X), \mathrm{H}^0(\mathcal{O}_X)^*)$. Thus S is a ring, called a Cox ring of X, and if the previous Ext^1 is zero (e.g. $\operatorname{Pic}(X)$ is free or $\mathrm{H}^0(\mathcal{O}_X^*) = k^*$ for an algebraically closed field k) this structure is unique.

When X is Noetherian, normal and excellent it is possible to replace line bundles with rank 1 reflexive sheaves, parametrize possible multiplications by a similar Ext^1 group and therefore define a/the Cox ring of reflexive sheaves.

The existence of the Cox ring for stacks makes possible the definition of Mori dream stacks (see [HM15]), which are a generalization of Mori dream spaces introduced by Y. Hu and S. Keel in [HK00].

1.9 Frobenius fixed objects of moduli

In the paper [DTZ20] with Valentina Di Proietto and Lei Zhang we work on a generalization of a lemma of Drinfeld.

Let X be a category fibered in groupoids over a finite field $\mathbb{F}_q$ and k be an algebraically closed field containing $\mathbb{F}_q$. Denote by $\phi_k \colon X_k \to X_k$ the arithmetic Frobenius of X_k/k and suppose that $\mathcal{M}$ is a stack over $\mathbb{F}_q$ (not necessarily in groupoids). Then there is a natural functor $\alpha_{\mathcal{M},} \colon \mathcal{M}(X) \longrightarrow \mathcal{M}(D_k(X))$, where $\mathcal{M}(D_k(X))$ is the category of ϕ_k-invariant maps $X_k \to \mathcal{M}$. A version of Drinfeld's lemma (see [Laf97, Lemma 4]) states that if X is a projective scheme and $\mathcal{M}$ is the stack of quasi-coherent sheaves of finite presentation, then $\alpha_{\mathcal{M},X}$ is an equivalence.

We extend this result in several directions. For proper algebraic stacks or affine gerbes X, we prove Drinfeld's lemma and deduce that $\alpha_{\mathcal{M},\mathcal{X}}$ is an equivalence for very general algebraic stacks $\mathcal{M}$. Here we use [Ton14]: where Drinfeld used sheafification of graded modules for projective spaces, we use sheafification of certain linear functors.

For arbitrary $\mathcal{X}$, we show that $\alpha_{\mathcal{M},X}$ is an equivalence when $\mathcal{M}$ is the stack of immersions, the stack of quasi-compact separated étale morphisms or any quasi-separated Deligne-Mumford stack with separated diagonal.

2 Future research

2.1 Homotopy exact sequence for isocrystals

The original motivation of [DTZ21] was to prove the homotopy exact sequence for isocrystals. If $f \colon X \longrightarrow S$ is a proper, smooth and geometrically connected map between smooth varieties over a field k, $x \in X(k)$, $s = f(x) \in S(k)$ and X_s is the fiber over s one can ask if the sequence

$$\pi(X_s, x) \longrightarrow \pi(X, x) \longrightarrow \pi(S, s) \longrightarrow 0$$

is exact. Here π denotes a fundamental group attached to a variety over k. The sequence is known to be exact if π is the étale fundamental group or π is the stratified fundamental group. We want to prove that the same is true for the isocrystal fundamental group. The Künneth formula proved in [DTZ21] represents the special case $X = X_s \times S$.

2.2 Picard group of compactification of moduli of uniform cyclic covers of curves

In [PTT15] we have computed the Picard group of the stacks of "uniform" cyclic covers of curves. Those spaces admit a natural compactification using admissible covers. We have recently understood that the information on the Picard group of the "open" part should be enough to describe the Picard group of the compactification. We know that the boundary divisors are independent and the last ingredient we need (and we are looking for) is one explicit relation among the given generators. We plan to find this relation by using test curves and, perhaps, Grothendieck-Riemann-Roch. This is a work in progress with Mattia Talpo and Nicola Pagani (University of Liverpool, United Kingdom).

2.3 Cox rings for algebraic stacks

2.3.1 Classical results, new setting

In [HMT20] a classical construction (Cox rings) is carried to a new setting (algebraic stacks) (see Section 1.8). We plan to extend known results for varieties to stacks, with an eye towards positive characteristic. For example we plan to prove that the Cox ring is still factorially graded.

2.3.2 Homogeneous sheafification for modules over the Cox ring

As a consequence of [Ton14], we construct a functor from graded modules over the Cox ring to the category of quasi-coherent sheaves, which is analogous to the sheafification of graded modules for projective spaces and toric varieties (see Section 1.2). We plan to investigate this relation further and exploit it for the study of vector bundles. For example [MS04], in the toric case, connects local cohomology of modules with sheaf cohomology and find applications in the study of syzygies of certain line bundles (see [HSS06]). Our goal is to provide similar results for toric stacks and Mori dream stacks.

2.3.3 Complete exceptional collections from Mori dream spaces

In [Kaw06] Kawamata shows that projective toric varieties with at most quotient singularities can be "modified" into smooth Deligne-Mumford stacks whose bounded derived category has a complete exceptional collection. The proof proceed by applying Minimal Model Program (MMP) to the initial toric variety. This process inevitably leads to consider the general situation concerning Deligne-Mumford stacks even if one only need results for smooth varieties. We plan to generalize this result from toric varieties to Mori Dream Spaces along the lines of Kawamata's proof. Mori Dream Spaces are indeed varieties defined to behave nicely with respect to MMP. Moreover we expect stacks and, in particular, Mori Dream stacks to naturally appear in the process.

2.3.4 Cox rings for logarithmic schemes

We plan to apply the theory developed in [HMT20] (see Section 1.8) in order to obtain a notion of Cox ring for a general logarithmic scheme. The main idea is to use [TV18], which allows to attach to any logarithmic scheme an infinite root stack. Even though this stack is not algebraic, it has an "fpqc presentation" by affine schemes, thus satisfies the condition required in [HMT20] for the existence of a Cox ring. We expect this theory to agree with a recent construction of the Cox ring for log pairs (see [BM21]).

2.4 Infinitesimal group schemes

2.4.1 Infinitesimal motivic McKay correspondence

Together with Takehiko Yasuda we are looking at possible ways to extend motivic McKay correspondence in positive characteristic p beyond the wild case (see Section 1.5), to general finite group schemes. The first step we are considering is the case of local (infinitesimal) group schemes. In [TY20b] we considered the simplest case, that is $G = \alpha_p$, the kernel of the relative Frobenius $\mathbb{G}_a \to \mathbb{G}_a$.

2.4.2 Infinitesimal Neukirch-Uchida conjecture

In [RTZ21, Conjecture I] we state a conjecture about the local Nori fundamental group scheme (see Section 1.6), which is analog to the classical Neukirch-Uchida conjecture. Our counjecture states that, if k is a perfect field of positive characteristic p and $K, F/k$ are finitely generated field extensions which are not algebraic, then

$$\pi^{\mathrm{L}}(K/k) \simeq \pi^{\mathrm{L}}(F/k) \implies K \simeq_k F$$

Classical Neukirch-Uchida theorem states the same result for $\mathbb{Q}$ and the absolute Galois group in place of k and $\pi^{\mathrm{L}}(-/k)$.

In support of this conjecture is the fact that the representation theory of $\pi^{\mathrm{L}}(K/k)$ seems to contain a lot of information about K/k: the group of characters of $\pi^{\mathrm{L}}(K/k)$ (that is its one dimensional representations) is isomorphic to $(K^{1/p^\infty})^*/K^*$, where K^{1/p^∞} is the perfect closure of K, and we expect higher dimensional representations to encode the additive structure. This is a work in progress with Matthieu Romagny and Lei Zhang.

2.4.3 Infinitesimal generic Abhyankar's conjecture

I am collaborating with Lei Zhang and Shusuke Otabe (University of Tokyo, Japan) on the following problem: given a perfect field k of positive characteristic p and a local (i.e. finite and connected) group scheme G over k does there exist a G-torsor over $k(t)$ which is minimal, that is not induced by a torsor under a proper subgroup of G? Equivalently, are all local groups of k a quotient of the Nori local fundamental group scheme $\pi^{\mathrm{L}}(k(t)/k)$ (see Section 1.6). Classical Abhyankar's conjecture, in its generic version, which can also be thought of as the inverse Galois problem for $k(t)$, asserts the same result over an algebraically closed field if we replace local group schemes by finite groups.

We state this conjecture in [RTZ21, Conjecture II]. A stronger version, which deals with quotients of $\pi^{\mathrm{L}}(X/k)$ for a rational affine curve X, has been proposed by Otabe in [Ota18] and here proved for solvable groups. In [OTZ21] we proved it for simple groups. The conjecture is still open and we are actively working on it.

3 References

[ABE+19] Marco Antei, Indranil Biswas, Michel Emsalem, Fabio Tonini, and Lei Zhang. Nori fundamental gerbe of essentially finite covers and Galois closure of towers of torsors. *Selecta Mathematica*, 25(2):18, jun 2019.

[BM21] Lukas Braun and Joaquín Moraga. Iteration of Cox rings of klt singularities. *arXiv:2103.13524*, mar 2021.

[BV14] Niels Borne and Angelo Vistoli. The Nori fundamental gerbe of a fibered category. *Journal of Algebraic Geometry*, 24(2):311–353, sep 2014.

[DTZ20] Valentina Di Proietto, Fabio Tonini, and Lei Zhang. Frobenius fixed objects of moduli. *arXiv:2012.14075*, page 21, dec 2020.

[DTZ21] Valentina Di Proietto, Fabio Tonini, and Lei Zhang. A crystalline incarnation of Berthelot's conjecture and Künneth formula for isocrystals. *Accepted in Journal of Algebraic Geometry*, dec 2021.

[EH10] Hélène Esnault and Amit Hogadi. On the algebraic fundamental group of smooth varieties in characteristic p>0. *Transactions of the American Mathematical Society*, 364(5):2429–2442, may 2010.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Mathematical Journal*, 48(1):331–348, jan 2000.

[HM15] Andreas Hochenegger and Elena Martinengo. Mori dream stacks. *Mathematische Zeitschrift*, 280(3-4):1185–1202, aug 2015.

[HMT20] Andreas Hochenegger, Elena Martinengo, and Fabio Tonini. Cox ring of an algebraic stack. *arXiv:2004.01445*, apr 2020.

[HSS06] Milena Hering, Hal Schenck, and Gregory G. Smith. Syzygies, multigraded regularity and toric varieties. *Compositio Mathematica*, 142(6):1499–1506, nov 2006.

[Kaw06] Yujiro Kawamata. Derived categories of toric varieties. *Michigan Mathematical Journal*, 54(3):517–535, nov 2006.

[Laf97] Laurent Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. *Astérisque*, (243):331, 1997.

[MS04] Diane Maclagan and Gregory G. Smith. Multigraded castelnuovo-mumford regularity. *Journal fur die Reine und Angewandte Mathematik*, (571):179–212, 2004.

[Ota18] Shusuke Otabe. On a purely inseparable analogue of the Abhyankar conjecture for affine curves. *Compositio Mathematica*, 154(8):1633–1658, aug 2018.

[OTZ21] Shusuke Otabe, Fabio Tonini, and Lei Zhang. A generalized Abhyankar's conjecture for simple Lie algebras in characteristic $p > 5$. *Accepted for publication in Mathematische Annalen*, mar 2021.

[PTT15] Flavia Poma, Mattia Talpo, and Fabio Tonini. Stacks of uniform cyclic covers of curves and their Picard groups. *Algebraic Geometry*, 2(1):91–122, mar 2015.

[RTZ21] Matthieu Romagny, Fabio Tonini, and Lei Zhang. The arithmetic local Nori fundamental group. *Accepted in Transactions of the American Mathematical Society*, nov 2021.

[Ton13] Fabio Tonini. Stacks of Ramified Covers Under Diagonalizable Group Schemes. *International Mathematics Research Notices*, 2014(8):2165–2244, jan 2013.

[Ton14] Fabio Tonini. Sheafification of linear functors. *arXiv:1409.4073*, sep 2014.

[Ton17] Fabio Tonini. Ramified Galois covers via monoidal functors. *Transformation Groups*, 22(3):845–868, sep 2017.

[Ton20] Fabio Tonini. Stacks of fiber functors and Tannaka's reconstruction. *arXiv:2010.12445*, oct 2020.

[Ton21] Fabio Tonini. Stack of S_3-covers. *arXiv:2104.08387*, apr 2021.

[TV18] Mattia Talpo and Angelo Vistoli. Infinite root stacks and quasi-coherent sheaves on logarithmic schemes. *Proceedings of the London Mathematical Society*, 116(5):1187–1243, may 2018.

[TY20a] Fabio Tonini and Takehiko Yasuda. Moduli of formal torsors. *Journal of Algebraic Geometry*, 29(4):753–801, jun 2020.

[TY20b] Fabio Tonini and Takehiko Yasuda. Notes on the motivic McKay correspondence for the group scheme α_p. *Singularities - Kagoshima 2017*, pages 215–233, jun 2020.

[TY21] Fabio Tonini and Takehiko Yasuda. Moduli of formal torsors II. *Accepted in Annales de l'Institut Fourier*, sep 2021.

[TZ19a] Fabio Tonini and Lei Zhang. Algebraic and Nori fundamental gerbes. *Journal of the Institute of Mathematics of Jussieu*, 18(4):855–897, jul 2019.

[TZ19b] Fabio Tonini and Lei Zhang. F-divided sheaves trivialized by dominant maps are essentially finite. *Transactions of the American Mathematical Society*, 371(8):5529–5549, aug 2019.

[TZ20] Fabio Tonini and Lei Zhang. Essentially Finite Vector Bundles on Normal Pseudo-proper Algebraic Stacks. *Documenta Mathematica*, 25:159–169, feb 2020.

[Yas19] Takehiko Yasuda. Motivic integration over wild Deligne-Mumford stacks. *arXiv:1908.02932*, aug 2019.