CHAPTER 9

Nonlinear Differential Equations and Stability

9.1 INTRODUCTION

Up to this point most of our discussion has dealt with methods for solving differential equations. Primarily, we have considered linear equations for which there is an elegant and extensive theory. However, some simple first order nonlinear equations were considered in Chapter 2, two special cases of second order nonlinear equations were discussed in the problems following Section 3.1, and the numerical methods of Chapter 8 apply equally well to nonlinear equations.

We must face the fact that it is usually very difficult, if not impossible, to find a solution of a given differential equation in a reasonably convenient and explicit form, especially if the equation is nonlinear. Therefore, it is important to consider what qualitative information can be obtained about the solutions of differential equations, particularly nonlinear ones, without actually solving the equations. The questions considered in this chapter are mainly associated with the idea of "stability" of a solution; this idea will be made more precise in Section 9.4. As an example, in applications such as automatic control theory, an important question is whether small changes in the initial conditions (input)* lead to small changes (stability) or to large changes (instability) in the solution (output). A second type of question arises when a nonlinear equation is approximated by a simpler linear one; for example, as mentioned in Section 1.1, when the nonlinear pendulum equation $d^2\theta/dt^2 + (g/l)\sin\theta = 0$ is replaced by $d^2\theta/dt^2 + (g/l)\theta = 0$ for θ small. If the solution of the linear equation is not a reasonable approximation to the solution of the original nonlinear equation, the linearization is of dubious value. Many of the arguments that we will present follow from geometric

* For purposes of discussion it may be helpful to think of the initial state as the input and the differential equation as an operation that starts once the input is given and gives an output, the solution.

considerations and were developed by the French mathematician H. Poincaré* and the Russian mathematician A. M. Liapounov.†

To illustrate some of the ideas in their simplest form, consider the nonlinear first order equation

$$\frac{dA}{dt} = \epsilon A - \sigma A^2, \tag{1}$$

with the initial condition

$$A(0) = A_0 \geq 0, \tag{2}$$

where ϵ and σ are given positive numbers. While negative values of A_0 are permissible mathematically, in physical applications A_0 will be nonnegative. Equation (1) occurs in many different applications. Let us see how it arises in ecology in an analysis of the growth of a single species having access to limited resources.

Let A be the size of a population at time t. In the simplest model of population growth the rate of growth is assumed to be directly proportional to the size of the population: $dA/dt = \epsilon A$, where the constant $\epsilon > 0$ is known as the *growth rate*. As we know, the solution of this differential equation satisfying the initial condition (2) is

$$A(t) = A_0 e^{\epsilon t}, \tag{3}$$

which gives an exponentially growing population that becomes unbounded as $t \to \infty$. However, the growth of a population must eventually be limited by the availability of resources, which in turn implies that the growth rate should be a function of the population:

$$dA/dt = Af(A). \tag{4}$$

When the population size A is small, there will be sufficient resources so the growth rate will be independent of the resources and hence approximately proportional to the population, $f(A) \approx \epsilon$. However, as the population increases the available resources per individual will decrease and produce an inhibitory effect on the population growth. Thus $f(A)$ should decrease as A increases, so eventually $df(A)/dA < 0$. The simplest function that meets these conditions is $f(A) = \epsilon - \sigma A$ where $\sigma > 0$. This gives Eq. (1). In ecology Eq. (1) is known as the *Verhulst*‡ equation or the *logistic* equation. An equation similar to Eq. (1)

* H. Poincaré (1854–1912) was generally regarded as the outstanding mathematician in the world around 1900. He made many fundamental contributions in several different areas of mathematics. He founded the subject of topological dynamics and was the father of modern topology, which is one of the most active fields in mathematical research. In his work on celestial mechanics he developed his theory of asymptotic expansions, which today is one of the most powerful tools of the applied mathematician.

† A. M. Liapounov's (1857–1918) work on stability and nonlinear differential equations, which is now realized to be of great importance, was not fully appreciated during his lifetime.

‡ P. F. Verhulst (1804–1849) was a Belgian mathematician who in 1839 introduced the solution of Eq. (1) into the theory of population dynamics.

also arises in the study of the transition from laminar to turbulent flow in fluid dynamics; there it is often called the *Landau** equation.

As we have noted, if the nonlinear term σA^2 is neglected in Eq. (1) then the population grows exponentially with time. The situation is entirely different if the nonlinear term is retained. Note that if $A = 0$ or if $A = \epsilon/\sigma$ the right-hand side of Eq. (1) vanishes. Thus we have the two constant solutions $A = \phi_1(t) = 0$ and $A = \phi_2(t) = \epsilon/\sigma$. As we shall see, these two solutions are of particular importance. In the theory of differential equations they are called *critical points* or, since $dA/dt = 0$ at these points, *equilibrium points*. Provided that $A \neq 0$ and $A \neq \epsilon/\sigma$, we can write Eq. (1) in the form

$$\frac{dA}{A(\epsilon - \sigma A)} = dt. \tag{5}$$

Using partial fractions we have

$$\left[\frac{1}{A} + \frac{\sigma}{\epsilon - \sigma A}\right] dA = \epsilon\, dt. \tag{6}$$

Integrating,

$$\ln|A| - \ln|\epsilon - \sigma A| = \epsilon t + K, \tag{7}$$

where K is a constant of integration to be chosen so that the initial condition $A(0) = A_0$ is satisfied. The cases $0 < A_0 < \epsilon/\sigma$, and $A_0 > \epsilon/\sigma$ must be considered separately since the sign of $(\epsilon - \sigma A)$ is different in the intervals $0 < A < \epsilon/\sigma$ and $A > \epsilon/\sigma$. Nevertheless, the final result can be expressed in a single formula. Omitting several steps of algebra, we find that

$$A = \phi(t) = \frac{\epsilon}{\sigma + [(\epsilon - \sigma A_0)/A_0]e^{-\epsilon t}} \tag{8}$$

or $A_0 \neq 0$. While the formula (8) is not valid for $A_0 = 0$, it does contain the constant solution $A = \phi_2(t) = \epsilon/\sigma$ corresponding to $A_0 = \epsilon/\sigma$. Since $\epsilon > 0$, it follows that $\exp(-\epsilon t) \to 0$ as $t \to \infty$, and hence $A \to \epsilon/\sigma$ as $t \to \infty$. Several typical solutions for different values of A_0 as well as the solutions $A = \phi_1(t) = 0$ and $A = \phi_2(t) = \epsilon/\sigma$ are shown in Figure 9.1.

For A_0 sufficiently small (actually $A_0 < \epsilon/2\sigma$, see Problem 14) the curve is S-shaped or sigmoid. While there are numerous experimental data for the growth of protozoa and bacteria that fit a sigmoid curve, there are also populations for which the fit of the growth curve is not so good. In ecology the ratio ϵ/σ is called the *saturation level* or *total carrying capacity*; a population starting below this level cannot surpass it.

It is clear from Figure 9.1 or from Eq. (8) that any solution of Eq. (1) starting at an initial point $A_0 > 0$ ultimately approaches the solution $A = \phi_2(t) = \epsilon/\sigma$ as $t \to \infty$. This limiting behavior as $t \to \infty$ is an example of the type of qualitative

* L. D. Landau (1908–1968) was a Russian physicist who received the Nobel prize in 1962 for his contributions to the understanding of condensed states of matters, particularly liquid helium. He was also the coauthor, with E. M. Lifshitz, of a well-known series of physics textbooks.

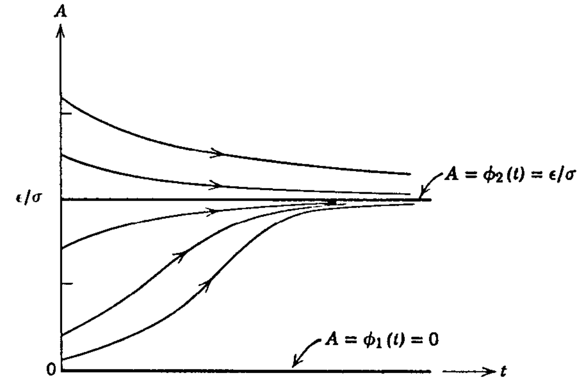

FIGURE 9.1 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$.

information that is often of interest. Here we have obtained it by first solving the differential equation, but it is significant for more difficult problems that the limiting behavior can often be determined without a complete knowledge of the solutions. To show this for the present problem, let us plot dA/dt as a function of A as given by Eq. (1). The graph is a parabola with $dA/dt = 0$ at $A = 0$ and at $A = \epsilon/\sigma$. Furthermore, dA/dt is positive for A small; thus the graph must have the appearance indicated in Figure 9.2. If $0 < A < \epsilon/\sigma$, then $dA/dt > 0$ and A increases toward ϵ/σ. On the other hand, if $A > \epsilon/\sigma$, then $dA/dt < 0$ and A decreases toward ϵ/σ. As indicated, A always approaches the solution $A = \phi_2(t) = \epsilon/\sigma$ provided that the initial value $A_0 > 0$.

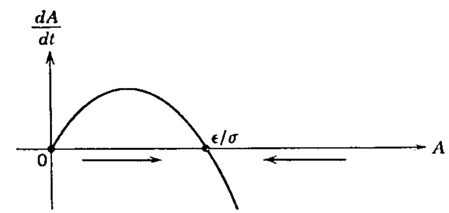

FIGURE 9.2 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$.

Let us examine the solution $A = \phi_1(t) = 0$ in more detail. To emphasize the point we wish to make, suppose that we think of Eq. (1) as representing a physical system, or black box, as depicted in Figure 9.3. If $A_0 = 0$ is the input, then the output is $A = 0$. But suppose that a slight error is made, and a small but nonzero value of A_0 is the input. The question is whether the output remains close to $A = 0$. It is clear from Figure 9.1 that the answer is no. Instead, $A \to \epsilon/\sigma$ as

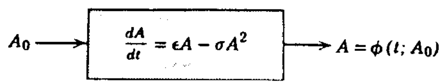

FIGURE 9.3

$t \to \infty$. It is natural to say that the solution $A = \phi_1(t) = 0$ is an *unstable solution* of Eq. (1) or that $A = 0$ is an *unstable critical (equilibrium) point.*

Now consider the solution $A = \phi_2(t) = \epsilon/\sigma$. If the input is $A_0 = \epsilon/\sigma$, then the output of the black box is $A = \epsilon/\sigma$. If a small error is made in the input, the output will approach ϵ/σ asymptotically (in fact, exponentially) as $t \to \infty$. Hence we say that the solution $A = \phi_2(t) = \epsilon/\sigma$ is an *asymptotically stable solution* of Eq. (1) or that $A = \epsilon/\sigma$ is an *asymptotically stable critical (equilibrium) point.*

An ecological interpretation of these two cases is as follows. First, suppose we are trying to perform a bacteria-free experiment in an environment for which the growth of bacteria is given by Eq. (1). After considerable effort the bacteria population is reduced to an extremely low, but nonzero, level. Is it safe to carry out the experiment on the assumption that the bacteria population will stay very low? No; the initial level of bacteria (A_0) is greater than zero and the bacteria will grow toward ϵ/σ as t increases. Second, suppose that we are trying to perform an experiment with a level of bacteria of approximately ϵ/σ but that from time to time contaminants are introduced that kill a few bacteria. Is it possible that this will cause the bacteria population to continue to decrease and eventually become extinct? No; if the population is moved slightly below ϵ/σ it will increase toward ϵ/σ; if it is moved slightly above ϵ/σ it will decrease toward ϵ/σ.

As a second mathematical example again consider Eq. (1) with the initial condition (2), but suppose that $\epsilon < 0$ and $\sigma < 0$. Without computing the explicit solution of Eq. (1), let us consider what information can be obtained from the graph of dA/dt vs A shown in Figure 9.4. Since $dA/dt < 0$ for $0 < A < \epsilon/\sigma$, it is clear that if the initial value A_0 is in the interval $0 < A_0 < \epsilon/\sigma$, then A will approach the solution $A = \phi_1(t) = 0$ as $t \to \infty$. On the other hand, since $dA/dt > 0$ for $A > \epsilon/\sigma$ it follows that if $A_0 > \epsilon/\sigma$, then A will increase without bound as t

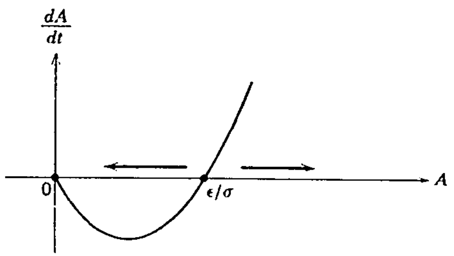

FIGURE 9.4 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon < 0$, $\sigma < 0$.

increases. It also follows from Figure 9.4 that the solution $A = \phi_2(t) = \epsilon/\sigma$ is unstable to small changes in the initial condition from $A_0 = \epsilon/\sigma$; a small positive change causes A to increase without bound, and a small negative change causes A to approach zero with increasing t.

On the other hand, the solution $A = \phi_1(t) = 0$ is asymptotically stable to small changes in the initial condition $A_0 = 0$, in the sense that the system causes a small initial deviation from the zero solution to decay. However, the solution $A = \phi_1(t) = 0$ is not stable to all initial deviations from $A_0 = 0$. If $A_0 > \epsilon/\sigma$, then rather than dying out, A will grow without bound. A sketch of the solutions of Eq. (1) with $\epsilon < 0$ and $\sigma < 0$ for different values of A_0 is shown in Figure 9.5.

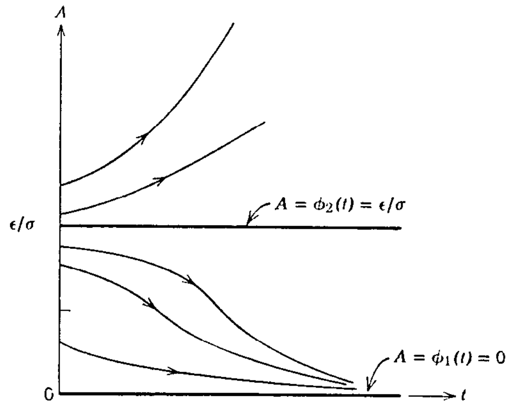

FIGURE 9.5 $dA/dt = \epsilon A - \sigma A^2$, $\epsilon < 0$, $\sigma < 0$.

We can interpret these results in the context of our black box of Figure 9.3. Usually in applications where a mechanism is to deliver a certain output when a certain input (or something very close to that input) is provided, we are interested only in stability with respect to small disturbances. It is, of course, conceivable that large disturbances can occur. For example, suppose an automatic control is set to keep a flap on an airplane wing at a certain inclination. In the normal motion of the plane the changing aerodynamic forces on the flap will cause it to move from its set position, but then the automatic control (the black box) will come into action to damp out the small deviation and return the flap to its set position. (Mathematically, for Eq. (1) with $\epsilon < 0$ and $\sigma < 0$ the set position is $A = 0$.) However, if the airplane is caught in a high gust of wind the flap may be deflected so much that the automatic control cannot bring it back to the set position (this would correspond to a deviation greater than ϵ/σ). Presumably the pilot would then take control!

In concluding this introductory section let us consider the question of neglecting the nonlinear term σA^2 in Eq. (1). In the first case ($\epsilon > 0$, $\sigma > 0$) the linear equation does give the correct behavior of A for a small time interval if A is initially small. However, in studying the solution for all time it *is not* permissible to neglect the term σA^2. The reason is that even if A is initially very small, so that the term σA^2 is initially small compared to the term ϵA, A grows exponentially with t, and the term σA^2 eventually becomes important. In the second case ($\epsilon < 0$, $\sigma < 0$), it *is* permissible to neglect the quadratic term σA^2 provided that A is small. The solution of the linear equation, $A_0 \exp(\epsilon t)$, approaches zero for any value of A_0. The same result is obtained from the nonlinear equation provided that $A_0 < \epsilon/\sigma$, and this is what is meant by "small" in this particular example. Thus, in the first case, the solution of the linearized problem is not a good approximation (at least not for all time) to that of the nonlinear problem; in the second case it is, provided only that $A_0 < \epsilon/\sigma$.

In the following sections we will develop the ideas suggested here, and others, in greater depth. While many books dealing with nonlinear differential equations have appeared in recent years, almost all are at an advanced level and assume considerable mathematical sophistication. However, several texts, listed in the references at the end of this chapter, have chapters dealing with theory or applications that should be readable after completing the present chapter.

PROBLEMS

In Problems 1 through 7 determine the limiting behavior of the solution of each equation as $t \to \infty$ for different initial conditions $A(0) = A_0$ by plotting dA/dt vs A.

1. $dA/dt = \epsilon A - \sigma A^2, \quad \epsilon > 0, \quad \sigma > 0, \quad -\infty < A_0 < \infty$
2. $dA/dt = \epsilon A - \sigma A^2, \quad \epsilon < 0, \quad \sigma < 0, \quad -\infty < A_0 < \infty$
3. $dA/dt = \epsilon A + \sigma A^2, \quad \epsilon > 0, \quad \sigma > 0, \quad A_0 \geq 0$
4. $dA/dt = \epsilon A + \sigma A^2, \quad \epsilon > 0, \quad \sigma > 0, \quad -\infty < A_0 < \infty$
5. $dA/dt = A(A - 1)(A - 2), \quad A_0 \geq 0$
6. $dA/dt = A(1 - A)(A - 2), \quad A_0 \geq 0$
7. $dA/dt = A^2(A^2 - 1), \quad -\infty < A_0 < \infty$
8. The population equation

$$dA/dt = \epsilon A^{1/2}(1 - A/K), \qquad \epsilon > 0, \qquad K > 0$$

appears to give results in good agreement with some experimental data. Determine the behavior of A as $t \to \infty$ for different values of $A(0) = A_0 \geq 0$.

9. Suppose that the number of trout $N(t)$ in a pond with a limited food supply is governed by the logistic equation $dN/dt = \alpha N(N_E - N)$ where α and N_E are positive constants.

(a) Show that $N = N_E$ is a stable critical point and that $N = 0$ is an unstable critical point.

(b) How is the logistic equation modified if a fisherman removes trout at a rate proportional to the number of trout in the pond? What happens to the stable critical point? Make a reasonable assumption about the constant of proportionality.

(c) How is the logistic equation modified if a fisherman removes trout at a uniform rate per unit time? Determine the number of trout in the pond for large time as a function of the initial number N_0. Make a reasonable assumption about the rate at which trout are removed.

10. In a simple model of economic growth the differential equation

$$dk/dt = sf(k) - gk$$

is obtained. Here k is the ratio of the supply of capital to the supply of labor, $f(k)$ is the productivity function (output per unit labor), g is the growth rate of the supply of labor, and s is a constant with $0 < s < 1$. It is known that $f'(k) > 0$, $f''(k) < 0$, and $f(k)$ is bounded. Show that if $0 < g/s < f'(0)$, then there exists a unique stable equilibrium solution (critical point) k given by the solution of the equation $sf(k) - gk = 0$.

11. Show that all solutions of the linear equation $dA/dt = -\epsilon A$ with $\epsilon > 0$ approach the solution $A = \phi(t) = 0$ as $t \to \infty$ independent of the choice of the initial condition $A(0) = A_0$, $-\infty < A_0 < \infty$. The solution $A = \phi(t) = 0$ is said to be *globally asymptotically stable* to emphasize that this solution is approached as $t \to \infty$ regardless of the initial state. This may be contrasted with the situation in Figure 9.4, where the solution $A = 0$ is approached as $t \to \infty$ only if $A_0 < \epsilon/\sigma$.

12. Derive Eq. (8) by choosing K in Eq. (7) so that the initial condition $A(0) = A_0$ is satisfied.

13. Determine the solution of the equation

$$dA/dt = \epsilon A - \sigma A^2, \qquad \epsilon < 0, \qquad \sigma < 0$$

with $A(0) = A_0$, $0 \leq A_0 < \infty$, and verify that the sketch shown in Figure 9.5 is correct.

14. Show that the graph of the solution of the logistic equation $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$ is concave up if $0 < A < \epsilon/2\sigma$, concave down if $\epsilon/2\sigma < A < \epsilon/\sigma$, and concave up if $A > \epsilon/\sigma$.
Hint: Compute d^2A/dt^2 by differentiating the logistic equation and then eliminate dA/dt to express d^2A/dt^2 as a function of A.

15. **Linear Stability.** Consider the logistic equation $dA/dt = \epsilon A - \sigma A^2$, $\epsilon > 0$, $\sigma > 0$. We have seen that the constant solutions (critical points) $A = 0$ and $A = \epsilon/\sigma$ play a crucial role in the analysis of this differential equation. In this problem we will discuss an analytic method, rather than the geometric arguments of the text, for analyzing the stability of these solutions.

Let A_E be any constant solution (critical point) of the logistic equation. Suppose that this solution is very slightly perturbed, what happens? We write $A(t) = A_E + u(t)$ where $u(0)$ is very small and ask what happens to u as $t \to \infty$.

(a) Derive the differential equation satisfied by $u(t)$.

(b) If $u(0)$ is very small then, at least initially, $u^2 \ll u$. Neglecting quadratic terms in u, derive a linear first order differential equation for u. This is known as *linearization.*

(c) If $u(t) \to \infty$ as $t \to \infty$ the constant solution is said to be linearly unstable; on the other hand, if $u(t) \to 0$ as $t \to \infty$ the constant solution is said to be linearly stable (actually asymptotically stable). Show that the constant solution $A_E = 0$ is linearly unstable and that the constant solution $A_E = \epsilon/\sigma$ is linearly stable. Note that the stability characteristics may be modified by the inclusion of the nonlinear term involving u^2 which has been neglected. That is why the stability is referred to as *linear stability* or *linear instability*.

16. Following the outline of Problem 15 analyze the linear stability of the constant solutions of the equation $dA/dt = \epsilon A - \sigma A^2$, $\epsilon < 0$, $\sigma < 0$.

9.2 SOLUTIONS OF AUTONOMOUS SYSTEMS

In the remainder of this chapter we will be concerned with systems of two simultaneous differential equations of the form

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y). \tag{1}$$

We assume that the functions F and G are continuous and have continuous partial derivatives in some domain D in the xy plane. Then, by Theorem 7.1, if (x_0, y_0) is a point in this domain, there exists a unique solution $x = \phi(t)$, $y = \psi(t)$ of the system (1) satisfying the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0. \tag{2}$$

The solution is defined in some interval $\alpha < t < \beta$ which contains the point t_0.

Notice that the independent variable t does not appear explicitly in Eqs. (1). Such a system is said to be *autonomous*. Physically, an autonomous system is one in which the parameters of the system are not time-dependent. Autonomous systems occur frequently in practice; for example, the motion of an undamped pendulum of length l is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{3}$$

Letting $x = \theta$ and $y = d\theta/dt$, we can rewrite Eq. (3) as a nonlinear autonomous system of two equations:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\left(\frac{g}{l}\right)\sin x. \tag{4}$$

The geometric reasoning introduced in the previous section can be effectively extended to systems of two autonomous equations. In order to initiate this discussion and to motivate some of the more mathematical details that will appear in the following sections we consider the ecological problem of two competing species.

Two Competing Species. Suppose that we have two similar species competing for a limited food supply; for example, two species of fish in a pond that do not prey on each other but do compete for the available food.* Let x and y be the populations of the two species at time t. The discussion of the previous section

* Another example is that of two yeasts competing in an anaerobic (absence of oxygen) medium. Note further that yeasts produce alcohol, which in turn reduces their growth rate; this is consistent with Eqs. (5), (6), and (7).

suggests that in the absence of species y the growth of species x is governed by an equation of the form

$$dx/dt = x(\epsilon_1 - \sigma_1 x), \tag{5}$$

and that in the absence of species x the growth of species y is governed by an equation of the form

$$dy/dt = y(\epsilon_2 - \sigma_2 y). \tag{6}$$

However, when both species are present each will impinge on the available food supply for the other. In effect, they reduce the growth rates and saturation populations of each other. The simplest expression for reducing the growth rate of species x due to the presence of species y is to replace the growth rate factor $\epsilon_1 - \sigma_1 x$ of Eq. (5) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$ where α_1 is a measure of the degree to which species y interferes with species x. Similarly, in Eq. (6) we replace $\epsilon_2 - \sigma_2 y$ by $\epsilon_2 - \sigma_2 y - \alpha_2 x$. Thus we have the system of equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \tag{7a}$$

$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x). \tag{7b}$$

The actual values of the positive constants $\epsilon_1, \sigma_1, \alpha_1, \epsilon_2, \sigma_2, \alpha_2$ depend on the physical problem under consideration.

By analogy with our discussion of the logistic equation in Section 9.1, we determine the constant solutions of Eqs. (7). This is done by setting the right-hand sides of Eqs. (7) equal to zero

$$\begin{aligned} x(\epsilon_1 - \sigma_1 x - \alpha_1 y) &= 0, \\ y(\epsilon_2 - \sigma_2 y - \alpha_2 x) &= 0. \end{aligned} \tag{8}$$

The solutions corresponding to either $x = 0$ or $y = 0$ are readily seen to be $x = 0, y = 0$; $x = 0, y = \epsilon_2/\sigma_2$; $x = \epsilon_1/\sigma_1, y = 0$. In addition, there is a constant solution corresponding to the intersection of the lines $\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$ and $\epsilon_2 - \sigma_2 y - \alpha_2 x = 0$ if these lines intersect. There are no other constant solutions of Eqs. (7). Geometrically these solutions can be represented as points in the xy plane; they are called critical points or equilibrium points. Moreover, in the same xy plane it is very helpful to visualize a solution of the system (7) as a point (x, y) moving as a function of time. At time $t = 0$ the initial populations of the two species provide an initial point (x_0, y_0) in the plane; then we follow the motion of the point (x, y) representing the populations of the two species at time t as it traces a curve in the plane. We can obtain considerable information about the behavior of solutions of Eqs. (7) without actually solving the problem.

First, from Eq. (7a) we observe that x increases or decreases as $\epsilon_1 - \sigma_1 x - \alpha_1 y > 0$ or $\epsilon_1 - \sigma_1 x - \alpha_1 y < 0$; similarly from Eq. (7b) we see that y increases or decreases as $\epsilon_2 - \sigma_2 y - \alpha_2 x > 0$ or $\epsilon_2 - \sigma_2 y - \alpha_2 x < 0$. This situation is depicted geometrically in Figure 9.6. In order to see what is happening to the two populations simultaneously we must superimpose the diagrams of Figure 9.6.

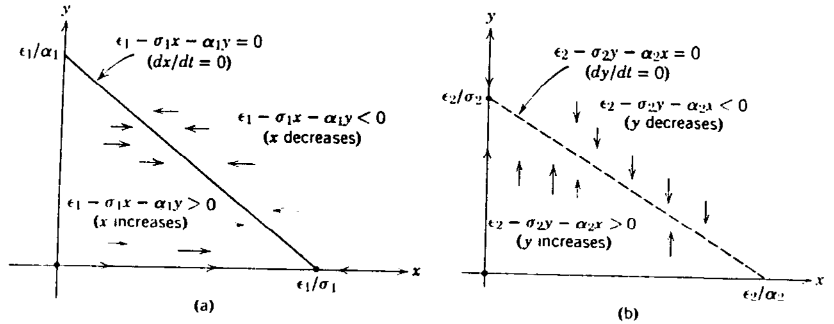

FIGURE 9.6

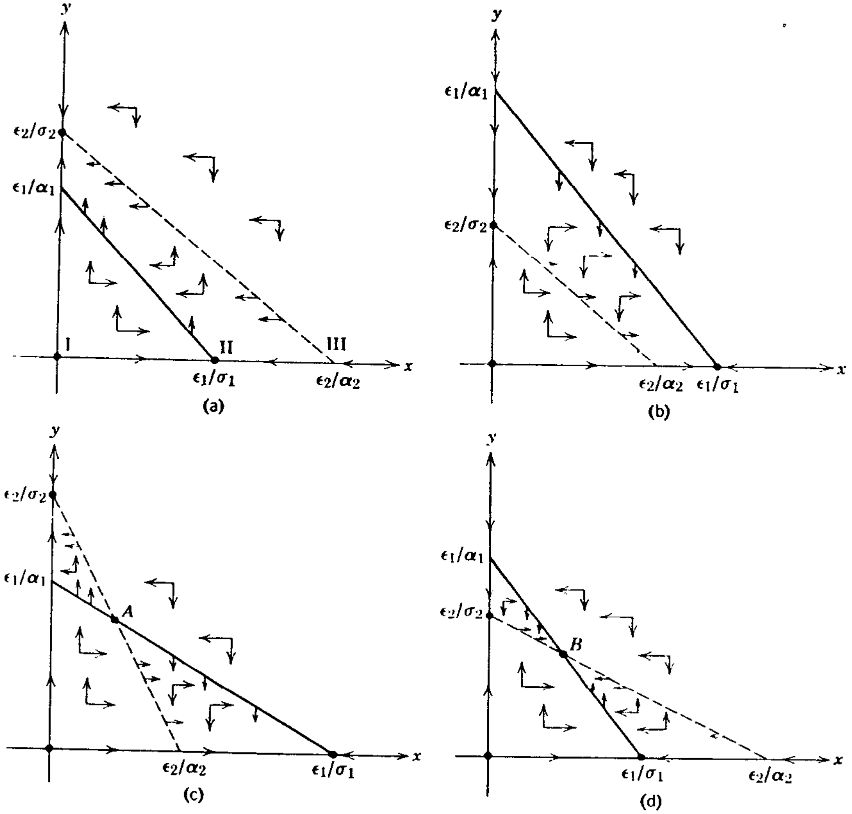

FIGURE 9.7

There are four possibilities as shown in Figure 9.7. The critical points are indicated by heavy dots. We will examine only cases (a) and (d) in more detail; cases (b) and (c) are similar and are left as problems. Also, for convenience we will assume that the initial populations x_0 and y_0 are each nonzero.

Consider case (a). If the initial populations are in region I then both x and y will increase; if the point moves into region II then species y will continue to increase but species x will start to decrease. Similarly, if the initial point is in region III, then both x and y will decrease; if the point moves into region II then x will continue to decrease while y now starts to increase. This suggests, for populations initially reasonably close to $(0, \epsilon_2/\sigma_2)$ that the point (x, y) representing the populations at time t approaches the critical point $(0, \epsilon_2/\sigma_2)$ as $t \to \infty$. This is shown in Figure 9.8a for several different initial states. This situation corresponds to the extinction of population x with population y reaching an equilibrium state of size ϵ_2/σ_2.

One might ask whether the point $(0, \epsilon_1/\alpha_1)$ is also a possible limiting state, since superficially populations that start near this point may seem to approach it as $t \to \infty$. The answer is no. In region I the point (x, y) moves away from the y axis while moving upward, and in region II, while moving toward the y axis the point (x, y) still moves upward. Moreover, note that $(0, \epsilon_1/\alpha_1)$ is not a critical point; that is $x = 0, y = \epsilon_1/\alpha_1$ is not a solution of Eqs. (7). The other critical points in Figure 9.7a are $(0, 0)$ and $(\epsilon_1/\sigma_1, 0)$. However, an inspection of Figure 9.8a readily shows that a solution (x, y) starting from nonzero values (x_0, y_0) cannot approach either of these points as $t \to \infty$.

Consider case (d). An examination of Figure 9.7d suggests that the population point (x, y) will move toward the intersection of the two straight dividing lines as t increases. This is shown schematically in Figure 9.8b for several different initial states. In this case both species can coexist with equilibrium populations given by the coordinates of the critical point B.

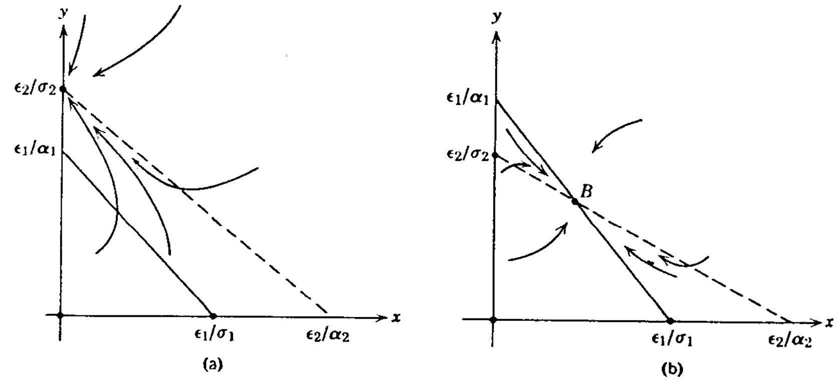

FIGURE 9.8

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(12)

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(15)

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(16)

$s\Big)$,

where C and s are arbitrary constants, is the general solution of Eqs. (15). Eliminating t by dividing the equation for y by $\sqrt{g/l}$ and then squaring and adding the equations for x and y shows that the trajectories of the system (15) lie on the ellipses

$$x^2 + \frac{y^2}{g/l} = C^2.$$

Several trajectories are sketched in Figure 9.10. The direction of motion with increasing t can be determined from the differential equations (15). For points in the first quadrant ($x > 0, y > 0$), Eqs. (15) imply that $dx/dt > 0$ and $dy/dt < 0$; therefore the motion is clockwise. The value of C in Eqs. (16) singles out one particular ellipse, and the value of s simply represents a displacement along the ellipse. For example, the solution $x = \sqrt{l/g}\sin\sqrt{g/l}\,t$, $y = \cos\sqrt{g/l}\,t$, which satisfies the initial conditions $x = 0$, $y = 1$ at $t = 0$, describes the trajectory $(g/l)x^2 + y^2 = 1$. Similarly, the solution $x = \sqrt{l/g}\sin(\sqrt{g/l}\,t + \pi/2)$, $y = \cos(\sqrt{g/l}\,t + \pi/2)$, which satisfies the initial conditions $x = \sqrt{l/g}$, $y = 0$ at $t = 0$, also lies on the same trajectory. Clearly, infinitely many solutions follow the same trajectory.

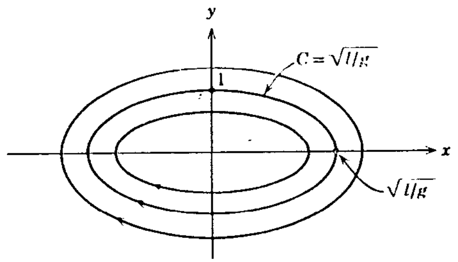


FIGURE 9.10

For this example, it is clear from Eqs. (16) that every solution of the linearized pendulum equations (15) is periodic with period $2\pi/\sqrt{g/l}$. More generally, a solution $x = \phi(t)$, $y = \psi(t)$ of Eqs. (1) is said to be *periodic* with period T if it exists for all t, and if $\phi(t + T) = \phi(t)$, $\psi(t + T) = \psi(t)$ for all t. The trajectory of a periodic solution is a closed curve, as illustrated by the elliptical trajectories in Figure 9.10.

In determining the trajectories of the system (1), it is often convenient to eliminate the parameter t from the solution $x = \phi(t)$, $y = \psi(t)$, obtaining a relation between x and y, as was just illustrated for Eqs. (16). Alternatively, in a region where $F(x, y) \neq 0$, we have from Eqs. (1)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x, y)}{F(x, y)}, \tag{17}$$

which is a first order differential equation* for $y = f(x)$. The one parameter family of solutions of Eq. (17) is the set of trajectories of the system (1). Equation (17) is particularly convenient for determining the slope at a point on a trajectory. Near a point where $F(x, y) = 0$, but $G(x, y) \neq 0$ one can consider in place of Eq. (17) the equation $dx/dy = F(x, y)/G(x, y)$.

However, at a point (x_0, y_0) at which both F and G vanish we cannot solve for dy/dx or dx/dy. Thus we again see that such a point, a *critical point*, has a special significance. If the point (x_0, y_0) is a critical point of the system (1), then $x = x_0$, $y = y_0$ is a solution of the system (1). Indeed, it follows from the existence and uniqueness theorem that the only solution of the system (1) passing through the point (x_0, y_0) is the constant solution itself. The trajectory of this solution is, of course, simply the single point (x_0, y_0). The particle at (x_0, y_0) is then often said to be at rest or in equilibrium. A trajectory, represented by the solution $x = \phi(t)$, $y = \psi(t)$, $t \geq \alpha$, is said to approach the critical point (x_0, y_0) as $t \to \infty$ if $\phi(t) \to x_0$ and $\psi(t) \to y_0$ as $t \to \infty$.

We have already seen the important role that critical points played in our preliminary analysis of the problem of two competing species. The nature of a critical point of an autonomous system can be further illustrated by an appeal to mechanics. The equation

$$m\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right) \tag{18}$$

can be interpreted as describing the motion of a particle (position x at time t) of a mass m acted on by a force that depends on the position x and velocity dx/dt. If we let $y = dx/dt$, then Eq. (18) can be rewritten as

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \frac{1}{m}f(x, y). \tag{19}$$

At a critical point of this system, the velocity $y = dx/dt$ and acceleration dy/dt vanish simultaneously. Thus, when the mass is at a critical point it is at rest (zero velocity) with zero force acting on it; in mechanics such a state (point) is called an *equilibrium state* (*point*).

A question of interest is whether a particle slightly displaced from an equilibrium point moves back toward the equilibrium point, away from it, or perhaps just moves in the neighborhood. Imagine the possibilities of a small displacement of a pendulum bob from: (a) a vertical downward position with air resistance, (b) a vertical upward position with or without air resistance, and (c) a vertical downward position without air resistance (see Figure 9.11). In case (a) we say that the critical (equilibrium) point is asymptotically stable, in case (b) it is unstable, and in case (c) it is stable but not asymptotically stable.

* The condition $F(x, y) \neq 0$ is needed so that we can divide by $F(x, y)$ to form Eq. (17). Moreover, the condition $F(x, y) \neq 0$ implies that $dx/dt \neq 0$; as a consequence we are assured that the solution $x = \phi(t)$ can be solved for t as a function of x and, in turn, y becomes a function of x.

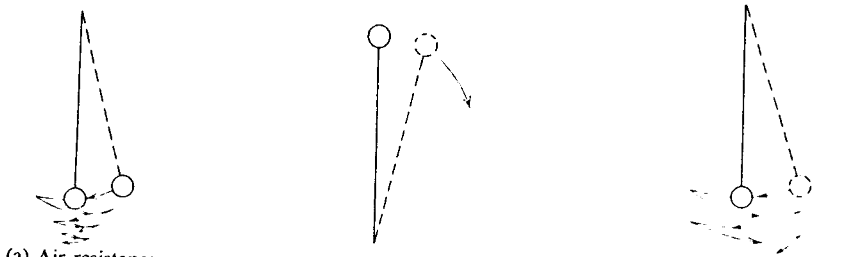


FIGURE 9.11

For the problem of competing species, the pattern of trajectories in Figure 9.8a suggest that the critical point $(0, \epsilon_2/\sigma_2)$ is an asymptotically stable critical point. If the populations are slightly disturbed from these values they will return to $(0, \epsilon_2/\sigma_2)$. The critical points $(0, 0)$ and $(\epsilon_1/\sigma_1, 0)$ are unstable. Similarly, when coexistence is possible (Figure 9.8b) it would appear that the coexistence critical point is asymptotically stable, and that the critical points $(0, 0)$, $(\epsilon_1/\sigma_1, 0)$, and $(0, \epsilon_2/\sigma_2)$ are unstable. The intuitive understanding of stability is clear for these specific physical examples; precise mathematical definitions will be given in Section 9.4.

The significance of critical points in the analysis of the autonomous system (1) is due to the fact that they are constant solutions of the system, and that the qualitative behavior of all trajectories in the phase plane is determined to a very large extent by the location of the critical points and the behavior of trajectories near them.

The following statements can be made about the trajectories of autonomous systems. The proofs follow from the existence and uniqueness theorem and are outlined in the problems.

1. Through any point (x_0, y_0) in the phase plane there is at most one trajectory of the system (1) (see Problem 8).

2. A particle starting at a point that is not a critical point cannot reach a critical point in a finite time. Thus if (x_0, y_0) is a critical point, and a trajectory corresponding to a solution approaches (x_0, y_0), then necessarily $t \to \infty$ (see Problem 9).

3. A trajectory passing through at least one point that is not a critical point cannot cross itself unless it is a closed curve. In this case the trajectory corresponds to a periodic solution of the system (1) (see Problem 10).

The substance of these three results is the following. If a particle (solution) starts at a point that is not a critical point, then it moves on the same trajectory no matter at what time it starts, it can never come back to its initial point unless the motion is periodic, it can never cross another trajectory, and it can only "reach" a critical point in the limit as $t \to \infty$. This suggests that such a particle (solution) either approaches a critical point (x_0, y_0), moves on a closed trajectory or approaches a closed trajectory as $t \to \infty$, or else goes off to infinity. Thus for autonomous systems a study of the critical points and periodic solutions is of fundamental importance. These facts will be developed in greater detail in the next two sections.

In conclusion, we note that many results for autonomous systems can be generalized to nonautonomous systems though the analysis becomes more complicated. Formally, a nonautonomous system of n first order equations for $x_1(t), x_2(t), \ldots, x_n(t)$ can be written as an autonomous system of $n + 1$ equations by letting $t = x_{n+1}$ wherever t appears explicitly and by appending the equation $dx_{n+1}/dt = 1$. For $n = 2$, this would require an analysis of the trajectories in three-dimensional space, which is considerably more difficult than in two dimensions.

PROBLEMS

1. Sketch the trajectory corresponding to the solution satisfying the specified initial conditions, and indicate the direction of motion for increasing t.

(a) $dx/dt = -x, \quad dy/dt = -2y; \quad x(0) = 4, \quad y(0) = 2$

(b) $dx/dt = -x, \quad dy/dt = 2y;$
$x(0) = 4, \quad y(0) = 2 \quad$ and $\quad x(0) = 4, \quad y(0) = 0$

(c) $dx/dt = -y, \quad dy/dt = x;$
$x(0) = 4, \quad y(0) = 0 \quad$ and $\quad x(0) = 0, \quad y(0) = 4$

2. Determine the critical points for each of the following systems.

(a) $dx/dt = 2x - 3y, \quad dy/dt = 2x - 2y$

(b) $dx/dt = x - xy, \quad dy/dt = y + 2xy$

(c) $dx/dt = x - x^2 - xy, \quad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$

(d) $dx/dt = y, \quad dy/dt = -(g/l)x - (c/ml)y; \quad g, l, c, m > 0$

(e) $dx/dt = y, \quad dy/dt = -(g/l)\sin x - (c/lm)y; \quad g, l, c, m > 0$

(f) The van der Pol equation: $dx/dt = y, \quad dy/dt = \mu(1 - x^2)y - x, \quad \mu > 0$

3. Consider the linear system

$$dx/dt = ax + by, \qquad dy/dt = cx + dy,$$

where a, b, c, and d are real constants.

(a) Show that if $ad - bc \neq 0$, then the only critical point is $(0, 0)$.

(b) Show that if $ad - bc = 0$, then in addition to the critical point $(0,0)$ there is a line through the origin for which every point is a critical point of the system. Thus in the latter case the critical point $(0, 0)$ is not "isolated" from the other critical points of the system.

4. Sketch the trajectories of the following system either by solving the system and eliminating t to obtain a one-parameter family of curves, or by integrating the differential equation [Eq. (17)], which gives the slope of the tangent of a trajectory. Indicate the direction of motion with increasing t.

(a) $dx/dt = -x, \quad dy/dt = -2y$

(b) $dx/dt = -x, \quad dy/dt = 2y$

(c) $dx/dt = x, \quad dy/dt = -2y$

(d) $dx/dt = -x, \quad dy/dt = 3x + 2y$

5. Given that $x = \phi(t)$, $y = \psi(t)$ is a solution of the autonomous system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y)$$

for $\alpha < t < \beta$, show that $x = \Phi(t) = \phi(t - s)$, $y = \Psi(t) = \psi(t - s)$ is a solution for $\alpha + s < t < \beta + s$.

6. Show by direct integration that even though the right-hand side of the system

$$\frac{dx}{dt} = \frac{x}{1 + t}, \qquad \frac{dy}{dt} = \frac{y}{1 + t}$$

depends on t, the paths followed by particles emitted at (x_0, y_0) at $t = s$ are the same regardless of the value of s. Why is this so?
Hint: Consider Eq. (17) for this case.

7. Consider the system

$$dx/dt = F(x, y, t), \qquad dy/dt = G(x, y, t).$$

Show that if the functions $F/(F^2 + G^2)^{1/2}$ and $G/(F^2 + G^2)^{1/2}$ are independent of t, then the solutions corresponding to the initial conditions $x(s) = x_0$, $y(s) = y_0$ give the same trajectories regardless of the value of s.
Hint: Use Eq. (17).

8. Prove that for the system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y)$$

there is at most one trajectory passing through a given point (x_0, y_0).
Hint: Let C_0 be the trajectory generated by the solution $x = \phi_0(t)$, $y = \psi_0(t)$, with $\phi_0(t_0) = x_0$, $\psi_0(t_0) = y_0$, and let C_1 be the trajectory generated by the solution $x = \phi_1(t)$, $y = \psi_1(t)$, with $\phi_1(t_1) = x_0$, $\psi_1(t_1) = y_0$. Use the fact that the system is autonomous and the existence and uniqueness theorem to show that C_0 and C_1 are the same.

9. Prove that if a trajectory starts at a noncritical point of the system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y)$$

then it cannot reach a critical point (x_0, y_0) in a finite length of time.
Hint: Assume the contrary; that is, assume that the solution $x = \phi(t)$, $y = \psi(t)$ satisfies $\phi(a) = x_0$, $\psi(a) = y_0$. Then use the fact that $x = x_0$, $y = y_0$ is a solution of the given system satisfying the initial condition $x = x_0$, $y = y_0$, at $t = a$.

10. Assuming that the trajectory corresponding to a solution $x = \phi(t)$, $y = \psi(t)$, $-\infty < t < \infty$, of an autonomous system is closed, show that the solution is periodic.
Hint: Since the trajectory is closed there exists at least one point (x_0, y_0) such that $\phi(t_0) = x_0$, $\psi(t_0) = y_0$ and a number $T > 0$ such that $\phi(t_0 + T) = x_0$, $\psi(t_0 + T) = y_0$. Show that $x = \Phi(t) = \phi(t + T)$ and $y = \Psi(t) = \psi(t + T)$ is a solution and then use the existence and uniqueness theorem to show that $\Phi(t) = \phi(t)$ and $\Psi(t) = \psi(t)$ for all t.*

Problems 11 through 13 deal with two competing species as described by Eqs. (7) of the text.

11. For the cases in which coexistence is possible (Figures 9.7c and 9.7d), determine the coordinates of this critical point.

* If it is originally known only that the functions ϕ and ψ are defined on an interval containing t_0 and $t_0 + T$, the proof is more difficult. However, it can be shown that the interval of definition must be $-\infty < t < \infty$ and that the solution is periodic with period T.

12. Draw several typical trajectories for the case shown in Figure 9.7b and determine what happens as $t \to \infty$.

13. Draw several typical trajectories for the case shown in Figure 9.7c and determine what happens as $t \to \infty$.

9.3 THE PHASE PLANE; LINEAR SYSTEMS

In this section we will continue our discussion of the autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y). \tag{1}$$

In particular we will be concerned with the behavior of the trajectories of the system (1) in the neighborhood of a critical point (x_0, y_0) and the significance of this behavior. It is convenient to choose the critical point to be at the origin of the phase plane: $x_0 = 0$, $y_0 = 0$. This involves no loss of generality, since if $x_0 \neq 0$, $y_0 \neq 0$, it is always possible to make the substitution $x = x_0 + u$, $y = y_0 + v$ in Eqs. (1), so that u and v will satisfy an autonomous system of equations with a critical point at the origin.

We will further assume that the origin is an *isolated critical point* of the system (1); that is, we assume that there is some circle about the critical point, inside which there are no other critical points. Finally, we assume that in the neighborhood of (0, 0), the functions F and G have the form

$$F(x, y) = ax + by + F_1(x, y),$$
$$G(x, y) = cx + dy + G_1(x, y), \tag{2}$$

where $ad - bc \neq 0$. Note that there are no constant terms in Eqs. (2) since $F(0, 0) = 0$ and $G(0, 0) = 0$. Also, we require that the functions F_1 and G_1 be continuous, have continuous first partial derivatives, and be small in the sense* that

$$\frac{F_1(x, y)}{r} \to 0, \qquad \frac{G_1(x, y)}{r} \to 0 \quad \text{as} \quad r \to 0, \tag{3}$$

where $r = (x^2 + y^2)^{1/2}$. Such a system is often referred to as an *almost linear system* in the neighborhood of the critical point (0, 0). Essentially, Eqs. (2) and (3) say that for (x, y) near (0, 0) the functions F and G are well approximated by the linear functions $ax + by$ and $cx + dy$, respectively.

The conditions on F and G are satisfied by many functions of two variables. For example, any polynomial in x and y with zero constant term is an almost linear function in the neighborhood of (0, 0).

* Actually, if the functions F and G are continuous and have continuous first partial derivatives, as we assumed in Section 9.2, then it can be shown that F and G have the form (2) in the neighborhood of the critical point (0, 0) and that condition (3) is satisfied. The reason for the restriction $ad - bc \neq 0$ will be made clear after Eq. (8) of this section; also see Problem 3 of Section 9.2.

As another example, consider the autonomous system for the undamped motion of a pendulum given in Eqs. (4) of Section 9.2: $dx/dt = y$, $dy/dt = -(g/l)\sin x$. The point (0, 0) is a critical point of this system. In order to compare these equations with those of Eqs. (2), we write them in the form

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = -\frac{g}{l}x - \frac{g}{l}(\sin x - x). \tag{4}$$

Thus $a = 0$, $b = 1$, $c = -g/l$, $d = 0$, $F_1(x, y) = 0$, and

$$G_1(x, y) = -\frac{g}{l}(\sin x - x) = -\frac{g}{l}\left(-\frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right). \tag{5}$$

From Eq. (5), $\sin x - x$ is similar to $-x^3/3! = -(r^3\cos^3\theta)/3!$ for x very small, and hence $(\sin x - x)/r \to 0$ as $r \to 0$. Consequently, the autonomous system (4) is almost linear in the neighborhood of the critical point (0, 0).

A consequence of the assumption that $F_1(x, y)$ and $G_1(x, y)$ are small compared to the linear terms $ax + by$ and $cx + dy$ near the origin is that in many cases (but not all) the trajectories of the linear autonomous system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy, \tag{6}$$

are good approximations to those of the almost linear system in the neighborhood of the critical point (0, 0). Thus we first consider the linear system (6), and then in Section 9.4 we will relate our conclusions to the almost linear system.

The system of equations (6) can be readily solved by elimination (Section 7.2) or by the matrix methods of Chapter 7. For our purposes it is not necessary to introduce the vector-matrix notation of that chapter. Of course, for the student who has read Chapter 7 it may be more convenient to rephrase some of the following results in vector-matrix notation.

We look for solutions of Eqs. (6) of the form $x = A\exp(rt)$, $y = B\exp(rt)$. Substituting for x and y in Eqs. (6), we obtain

$$(a - r)A + bB = 0$$
$$cA + (d - r)B = 0. \tag{7}$$

In order for these two linear homogeneous equations to have a nontrivial solution, it is necessary for the determinant of the coefficients to vanish. Thus r must be a root of the characteristic equation

$$r^2 - (a + d)r + (ad - bc) = 0. \tag{8}$$

Equation (8) is also obtained if one eliminates either x or y in system (6) and then looks for a solution proportional to $\exp(rt)$. The student who has read Section 7.4 will note that the roots r_1 and r_2 of Eq. (8) are the eigenvalues of the coefficient matrix of the system (6). When r is a root of Eq. (8), then Eqs. (7) are not independent; hence either one of them can be used to calculate the relation between A and B. For example, for $r = r_1$, we have from the first of Eqs. (7) that $B = -(a - r_1)A/b$, assuming that $b \neq 0$.

Notice that $r = 0$ cannot be a root of Eq. (8). If it were, then necessarily $ad - bc = 0$. As a consequence, the system of equations $ax + by = 0$, $cx + dy = 0$ would have nontrivial solutions since the determinant of the coefficients would be zero. These nontrivial solutions would be determined only up to a multiplicative constant and would lie on a line passing through the origin. Thus other critical points would be located arbitrarily close to the origin, violating the assumption that the origin is an isolated critical point.

A number of different cases must be considered depending on whether the roots of Eq. (8) are both positive, both negative, one positive and one negative, complex with positive real parts, complex with negative real parts, or pure imaginary. We will consider each case in turn, illustrating the behavior of the trajectories by a specific example. We emphasize that the general situation for each case is similar to that for the corresponding example, since a suitable change of variables can be made to transform a given system into a system having the form of the example for that case. Once the problem is solved in the new coordinate system the solution can be transformed back to the original coordinate system and constitutes a solution of the original system. For example, a circle in the transformed coordinate system may be an ellipse in the original coordinate system with its major axis not necessarily parallel to one of the coordinate axes. Also for each case we will show a more "general" picture of a typical set of trajectories. It is important that the student become familiar with the type of behavior that the trajectories have for each case, because these are the basic building blocks of the qualitative theory of differential equations.

Case 1. Real Unequal Roots of the Same Sign. The general solution of Eqs. (6) is

$$x = A_1e^{r_1t} + A_2e^{r_2t}, \qquad y = B_1e^{r_1t} + B_2e^{r_2t}, \tag{9}$$

where only one each of the constants A_1, B_1 and A_2, B_2, respectively, are independent. Suppose first that r_1 and r_2 are negative; then both x and y approach zero and the point (x, y) approaches the critical point (0, 0) as $t \to \infty$ independent of the choice of the constants A_1, A_2, B_1, and B_2. Every solution of the system (6) approaches the solution $x = 0$, $y = 0$ exponentially as $t \to \infty$.

To illustrate this case consider the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = -2y, \tag{10}$$

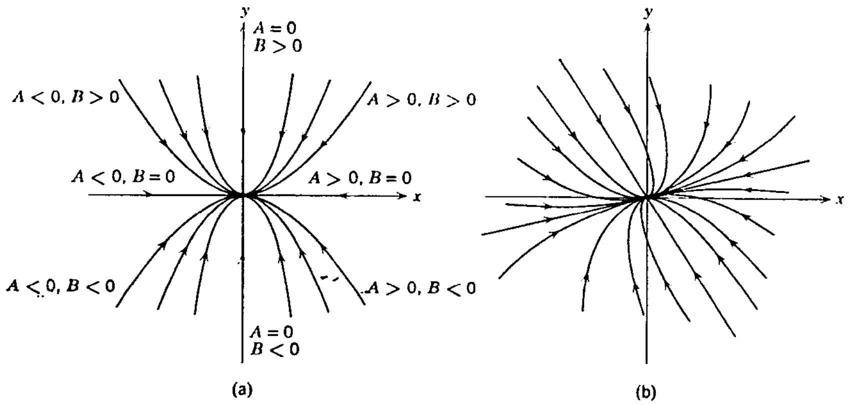


FIGURE 9.12 Improper node, $r_1 \neq r_2$.

which has the general solution

$$x = Ae^{-t}, \qquad y = Be^{-2t}, \tag{11}$$

where A and B are arbitrary. A few trajectories for this example are sketched in Figure 9.12a. If $A > 0$, $B = 0$, then $y = 0$ and $x \to 0$ through positive values as $t \to \infty$. The other halves of the coordinate axes correspond to the cases $A < 0$, $B = 0$; $A = 0$, $B > 0$; and $A = 0$, $B < 0$. If $A \neq 0$, $B \neq 0$, then $y = (B/A^2)x^2$, and we obtain the portions of the parabolas in the first, second, third, and fourth quadrants, as indicated in Figure 9.12a.

This type of critical point is called a *node*, or sometimes an *improper node*, to distinguish it from another type of node to be mentioned later. The distinguishing feature is that all trajectories except one pair approach the critical point tangent to the same line, while the exceptional pair of trajectories approaches tangent to a different line. In the example all trajectories approach the origin tangent to the x axis except one pair, which approaches along the y axis. A more general sketch of trajectories near an improper node is shown in Figure 9.12b. If the roots are real, unequal, and positive the situation is similar except that the direction of motion on the trajectories is away from the critical point (0, 0). In this case every solution, except the solution $x = 0$, $y = 0$, even a solution started very near (0, 0), recedes from the origin as $t \to \infty$.

Case 2. Real Roots of Opposite Sign. The general solution of Eqs. (6) is still given by Eqs. (9), but with $r_1 > 0$ and $r_2 < 0$. It is now possible for the direction of motion to be toward the critical point on some trajectories and away from the critical point on other trajectories. Typical of this case is the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = 2y, \tag{12}$$

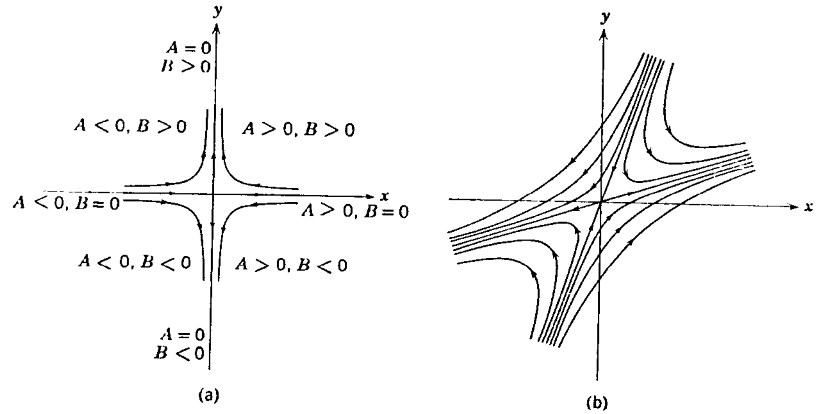


FIGURE 9.13 Saddle point, $r_1r_2 < 0$.

which has the general solution

$$x = Ae^{-t}, \qquad y = Be^{2t}, \tag{13}$$

where A and B are arbitrary. For any initial point not on the x axis, $x \to 0$, $y \to \pm\infty$, depending on the sign of B, as $t \to \infty$; while for an initial point on the x axis, y stays zero and $x \to 0$ as $t \to \infty$. The trajectories are sketched in Figure 9.13a where use has been made of the fact that for $A \neq 0$ and $B \neq 0$, the solution lies on a portion of the curve $y = BA^2/x^2$. In this case the critical point is called a *saddle point*.

It is important to note that only the trajectories on the x axis approach the critical point; all others approach infinity as $t \to \infty$ because of the presence of $y = B\exp(2t)$, $B \neq 0$. The situation is always similar for a saddle point. To see this, note that the general solution of the system (6) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} k_{11} \\ k_{21} \end{pmatrix}e^{r_1t} + B\begin{pmatrix} k_{12} \\ k_{22} \end{pmatrix}e^{r_2t},$$

where A and B are arbitrary constants, k_{11}, k_{21}, k_{12}, and k_{22} are known, and r_1 and r_2 are real but of opposite sign ($r_1 < 0$, $r_2 > 0$). Hence only the trajectories corresponding to solutions for which $B = 0$ will approach the critical point. Taking $A > 0$ gives one trajectory, $A < 0$ gives a second. A more general sketch of trajectories near a saddle point is shown in Figure 9.13b.

Case 3. Equal Roots. The general solution of Eqs. (6) is of the form

$$x = (A_1 + A_2t)e^{rt}, \qquad y = (B_1 + B_2t)e^{rt}, \tag{14}$$

where only two of the four constants A_1, A_2, B_1, and B_2 are independent. Regardless of the values of A_1, A_2, B_1, and B_2 it is clear that if $r < 0$, the direction of motion on all trajectories is toward the critical point and, if $r > 0$, it is away from the critical point. We will consider the case $r < 0$.

Depending on whether the term te^{rt} is present, the trajectories for this case are of two quite different types. The simpler case is when this term is not present; that is, when $A_2 = B_2 = 0$. Illustrative of this situation is the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = -y, \tag{15}$$

which has the general solution

$$x = Ae^{-t}, \qquad y = Be^{-t}, \tag{16}$$

where A and B are arbitrary. For $A \neq 0$, $B \neq 0$ the trajectories are the straight lines $y = (B/A)x$. The trajectories are sketched in Figure 9.14. Every trajectory has a different slope as it approaches (or recedes from, if r is greater than zero) the origin, and the critical point is called a *proper node*.

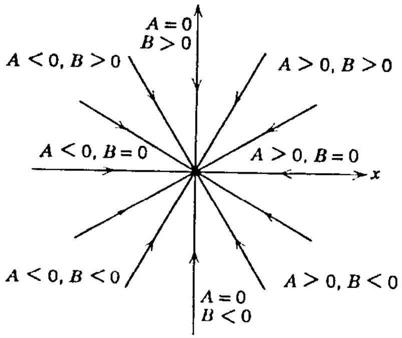


FIGURE 9.14 Proper node, $r_1 = r_2$.

Typical of the situation when the term te^{rt} is present is the system

$$\frac{dx}{dt} = -2x, \qquad \frac{dy}{dt} = x - 2y, \tag{17}$$

which has the general solution

$$x = Ae^{-2t}, \qquad y = Be^{-2t} + Ate^{-2t}, \tag{18}$$

where A and B are arbitrary. A detailed sketch of the trajectories represented by the solutions (18) is fairly complicated since, if t is eliminated between the equations for x and y, an expression is obtained for y that involves both x and $\ln x$. Assuming that $A > 0$, it then follows from the first of Eqs. (18) that x is positive and $e^{-2t} = x/A$ so $t = -(\frac{1}{2}) \ln (x/A)$. Substituting this expression for t in the second of Eqs. (18) yields

$$y = \frac{Bx}{A} - \frac{x}{2} \ln \frac{x}{A}. \tag{19}$$

For assigned values of $A > 0$ and B, the trajectories can be obtained from I (19). For $A < 0$ a similar equation can be derived.

It is clear from Eq. (18) that all of the trajectories approach the origin $t \to \infty$. The slope at any point can be obtained from Eq. (19) for $A > 0$ or fr a similar equation for $A < 0$, or by recalling that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2Be^{-2t} - 2Ate^{-2t} + Ae^{-2t}}{-2Ae^{-2t}} = \frac{(-2B + A) - 2At}{-2A}. \qquad ($$

For $t \to \infty$, $dy/dx \to \infty$; thus all of the trajectories enter the origin along the axis. A qualitative picture of the trajectories is shown in Figure 9.15a. Aga the critical point is called an *improper node*. A more general sketch is shown Figure 9.15b.

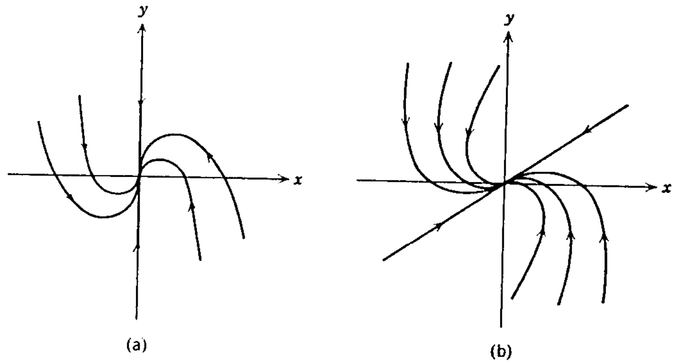


FIGURE 9.15 Improper node, $r_1 = r_2$.

Case 4. Complex Roots. In this case the general solution of Eqs. (6) is

$$x = e^{\lambda t}(A_1 \cos \mu t + A_2 \sin \mu t), \qquad y = e^{\lambda t}(B_1 \cos \mu t + B_2 \sin \mu t), \qquad (2$$

where the roots of Eq. (8) are $r = \lambda \pm i\mu$, and only two of the constants A A_2, B_1, and B_2 are independent. If $\lambda < 0$ the motion on every trajectory is towa the critical point and, if $\lambda > 0$, it is away from the critical point.

Consider the specific system

$$\frac{dx}{dt} = -x + 2y, \qquad \frac{dy}{dt} = -2x - y, \qquad (2.$$

which has the general solution

$$x = e^{-t}(A \cos 2t + B \sin 2t), \qquad y = e^{-t}(B \cos 2t - A \sin 2t), \qquad (2.$$

where A and B are arbitrary. The trajectories associated with the solution (2 can best be visualized by introducing polar coordinates. If we let

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and

$$R = (A^2 + B^2)^{1/2}, \qquad R\cos\alpha = A, \qquad R\sin\alpha = B, \tag{24}$$

then Eqs. (23) take the form

$$r\cos\theta = Re^{-t}\cos(2t - \alpha), \qquad r\sin\theta = -Re^{-t}\sin(2t - \alpha). \tag{25}$$

Thus

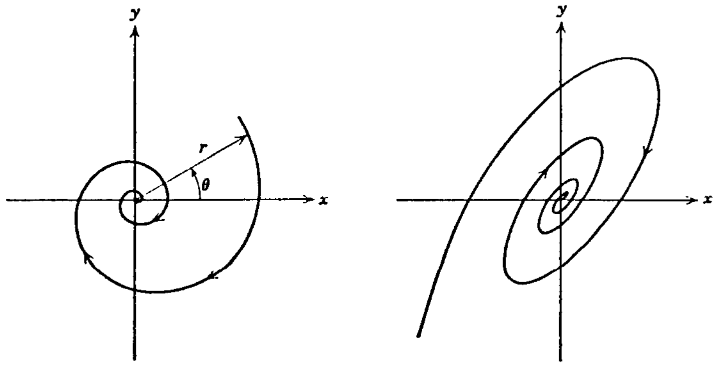
$$r = Re^{-t}, \qquad \theta = -(2t - \alpha), \tag{26}$$

Finally, eliminating t gives

$$r = Re^{(\theta-\alpha)/2}, \tag{27}$$

which represents a family of spirals, one of which is sketched in Figure 9.16a. Since $\lambda = -1 < 0$, the direction of motion is toward the critical point (0, 0); since θ decreases with increasing t, the motion is clockwise. Whether the motion is clockwise or counterclockwise can also readily be obtained directly from the original differential equations (22). It is convenient to choose a point on one of the coordinate axes and to determine the direction of motion there. For example, if we choose a point on the positive x axis, then it follows from the second of Eqs. (22) that $dy/dt < 0$. Hence the point (x, y) on the trajectory is moving downward when it crosses the positive x axis, so the direction of motion on the trajectory is clockwise.

The critical point is called a *spiral point*. A more general sketch of trajectories near a spiral point is shown in Figure 9.16b. If λ were positive the spirals would be similar, but the direction of motion would be away from (0, 0).



(a) (b)

FIGURE 9.16 Spiral point, $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$.

Case 5. Pure Imaginary Roots. While this is a special case of the previou one given by Eqs. (21) with $\lambda = 0$, it must be treated separately, since the tra jectories are no longer spirals. In this case the motion is periodic in time and th trajectories are closed curves. The direction of motion around the closed curve can be either clockwise or counterclockwise, depending on the equations. Typica of this case are the linearized undamped pendulum equations discussed as a example in Section 9.2. The trajectories, as shown in Figure 9.10 of the previou section, are ellipses centered at the origin. The origin is called a *center*. A typica case in which the major axes of the ellipses are inclined at an angle to the coordinat axes is shown in Figure 9.17. The trajectories neither approach nor recede fron the critical point.

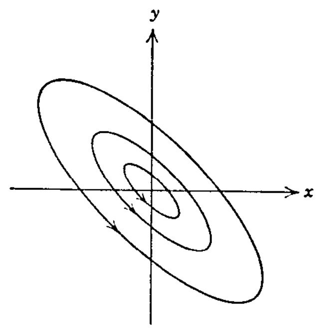


FIGURE 9.17 Center, $r_1 = i\mu$, $r_2 = -i\mu$.

In each of the cases we have discussed, the trajectories of the system exhibite one of the following three types of behavior.

1. All of the trajectories approach the critical point as $t \to \infty$. This will b the case if the roots of the characteristic equation (8) are real and negative or com plex with negative real parts.

2. The trajectories neither approach the critical point nor tend to infinity a $t \to \infty$. This will be the case if the roots of the characteristic equation (8) ar pure imaginary.

3. At least one (possibly all) of the trajectories tends to infinity as $t \to \infty$ This will be the case if at least one of the roots of the characteristic equation (8 is positive or if the roots have positive real parts.

These three possibilities illustrate the concepts of asymptotic stability, stability, and instability, respectively, of the critical point at the origin of the system (6). The precise definitions of these terms will be given in Section 9.4, but their basic meaning should be clear from the geometrical discussion above and the discussion following Eq. (19) of Section 9.2. Our analysis of the trajectories of the linear system (6) is summarized in Table 9.1. Also see Problems 16 and 17 and Figure 9.20, which accompanies those problems.

TABLE 9.1

$dx/dt = ax + by$ $r^2 - (a + d)r + (ad - bc) = 0$
$dy/dt = cx + dy$ $ad - bc \neq 0$

Roots of the Characteristic Equation	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Improper node	Unstable
$r_1 < r_2 < 0$		Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$		Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$	Spiral point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable
$r_1 = i\mu,\ r_2 = -i\mu$	Center	Stable

It is clear from our discussion and Table 9.1 that the roots r_1, r_2 of Eq. (8) determine the type of critical point and its stability characteristics. In turn, the values of r_1, r_2 depend on the coefficients a, b, c, d in the system of differential equations (6). When such a system (6) arises in some applied field, the coefficients a, b, c, and d usually result from the measurements of certain physical quantities. Such measurements are often subject to small uncertainties, so it is of interest to investigate whether small changes (perturbations) in the coefficients can affect the stability or instability of a critical point and/or significantly alter the pattern of trajectories.

It is possible to show from Eq. (8) that *small* perturbations in some or all of the coefficients a, b, c, and d are reflected in *small* perturbations in the roots r_1 and r_2. The most sensitive situation occurs for $r_1 = i\mu$ and $r_2 = -i\mu$; that is, when the critical point is a center and the trajectories are closed curves. If a slight change is made in the coefficients, then the roots r_1 and r_2 will take on new values, $r_1' = \lambda' + i\mu'$ and $r_2' = \lambda' - i\mu'$, where λ' is small in magnitude and $\mu' \cong \mu$ (see Figure 9.18). If $\lambda' \neq 0$, which almost always occurs, then the trajectories of the

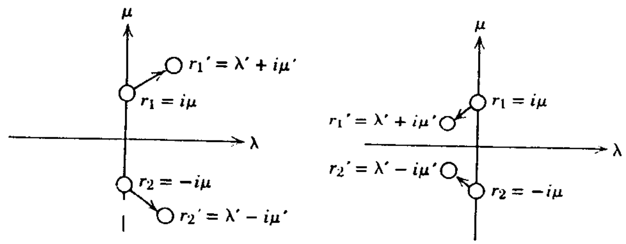


FIGURE 9.18 Schematic perturbation of $r_1 = i\mu$, $r_2 = -i\mu$.

perturbed system are spirals, rather than closed curves. The system is asymptotically stable if $\lambda' < 0$, but unstable if $\lambda' > 0$. Thus in the case of a center, small perturbations in the coefficients may well change a stable system into an unstable one, and in any case may be expected to alter radically the pattern of trajectories in the phase plane (see Problem 14).

Another slightly less sensitive case occurs if the roots r_1 and r_2 are equal; in this case the critical point is a node. Small perturbations in the coefficients will normally cause the two equal roots to separate (bifurcate). If the separated roots are real, then the critical point of the perturbed system remains a node, but if the separated roots are complex conjugates, then the critical point becomes a spiral point. These two possibilities are shown schematically in Figure 9.19. In this case the stability or instability of the system is not affected by small perturbations in the coefficients, but the trajectories may be altered considerably (see Problem 15).

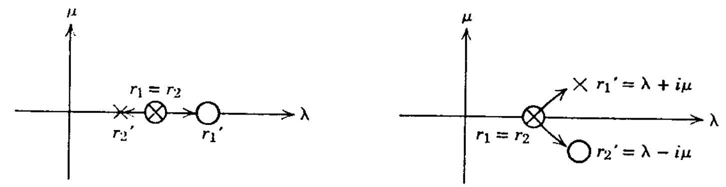


FIGURE 9.19 Schematic perturbation of $r_1 = r_2$.

In all other cases the stability or instability of the system is not changed, nor is the type of critical point altered, by sufficiently small perturbations in the coefficients of the system. For example, if r_1 and r_2 are real, negative, and unequal, then a *small* change in the coefficients will not change the sign of r_1 and r_2 nor allow them to coalesce. Thus the critical point remains an asymptotically stable improper node.

In Section 9.4 we will discuss how the results of this section may be used to investigate almost linear systems.

PROBLEMS

In each of Problems 1 through 8 classify the critical point (0, 0), and determine whether it is stable, asymptotically stable, or unstable.

1. $dx/dt = 3x - 2y$, $dy/dt = 2x - 2y$
2. $dx/dt = 5x - y$, $dy/dt = 3x + y$
3. $dx/dt = 2x - y$, $dy/dt = 3x - 2y$
4. $dx/dt = x - 4y$, $dy/dt = 4x - 7y$
5. $dx/dt = x - 5y$, $dy/dt = x - 3y$
6. $dx/dt = 2x - 5y$, $dy/dt = x - 2y$
7. $dx/dt = 3x - 2y$, $dy/dt = 4x - y$
8. $dx/dt = -x - y$, $dy/dt = -\frac{1}{4}y$

In each of Problems 9 through 12 determine the critical point (x_0, y_0), and then classify its type and examine its stability by making the transformation $x = x_0 + u$, $y = y_0 + v$.

9. $dx/dt = x + y - 2$, $dy/dt = x - y$

10. $dx/dt = -2x + y - 2$, $dy/dt = x - 2y + 1$

11. $dx/dt = -x - y - 1$, $dy/dt = 2x - y + 5$

12. $dx/dt = \alpha - \beta y$, $dy/dt = -\gamma + \delta x$, $\alpha, \beta, \gamma, \delta > 0$

13. The equation of motion of a spring-mass system with damping (see Section 3.7) is

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0,$$

where m, c, and k are positive. Write this second order equation as a system of two first order equations for $x = u$, $y = du/dt$. Show that $x = 0$, $y = 0$ is a critical point, and analyze the nature and stability of the critical point as a function of the parameters m, c, and k. A similar analysis can be applied to the electric circuit equation (see Section 3.8)

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

14. In this problem we show how small changes in the coefficients of a system of linear equations can affect a critical point that is a center. Consider the system

$$dx/dt = 0x + y, \qquad dy/dt = -x + 0y.$$

Show that the roots of the characteristic equation are $\pm i$ so that (0, 0) is a center. Now consider the system

$$dx/dt = \epsilon x + y, \qquad dy/dt = -x + \epsilon y,$$

where $|\epsilon|$ is arbitrarily small. Show that the characteristic equation has roots $\epsilon \pm i$. Thus no matter how small $|\epsilon| \neq 0$ is, the center is shifted to a spiral point. If $\epsilon < 0$ then the spiral point is asymptotically stable; if $\epsilon > 0$ the spiral point is unstable.

15. In this problem we show how small changes in the coefficients of a system of linear equations can affect the nature of a critical point when the roots of the characteristic equation are equal. Consider the system

$$dx/dt = -x + y, \qquad dy/dt = 0x - y.$$

Show that the roots of the characteristic equation are $r_1 = -1, r_2 = -1$ so that the critical point (0, 0) is an asymptotically stable node. Now consider the system

$$dx/dt = -x + y, \qquad dy/dt = -\epsilon x - y,$$

where $|\epsilon|$ is arbitrarily small. Show that the characteristic equation has roots $-1 \pm i\sqrt{\epsilon}$. Thus if $\epsilon > 0$ then the asymptotically stable node is shifted to an asymptotically stable spiral point. If $\epsilon < 0$ then the roots are $-1 \pm \sqrt{|\epsilon|}$, and the critical point remains an asymptotically stable node.

16. Consider the linear autonomous system

$$dx/dt = ax + by, \qquad dy/dt = cx + dy,$$

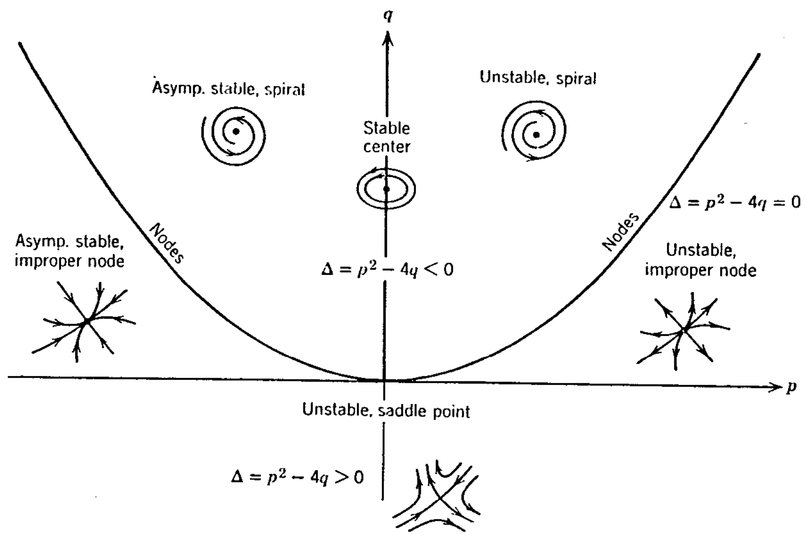


FIGURE 9.20 Stability diagram.

where a, b, c, and d are real constants. Let $p = a + d$, $q = ad - bc$, and $\Delta = p^2 - 4q$. Show that the critical point (0, 0) is a

(a) node if $q > 0$ and $\Delta \geq 0$; (b) saddle point if $q < 0$;
(c) spiral point if $p \neq 0$ and $\Delta < 0$; (d) center if $p = 0$ and $q > 0$.

Hint: These conclusions can be obtained by studying the roots r_1 and r_2 of the characteristic equation. It may also be helpful to show, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

17. Continuing Problem 16, show that the critical point (0, 0) is
(a) asymptotically stable if $q > 0$ and $p < 0$;
(b) stable if $q > 0$ and $p = 0$;
(c) unstable if $q < 0$ or $p > 0$.

Notice that the results (a), (b), and (c) together with the fact that $q \neq 0$ show that the critical point is asymptotically stable if, and only if, $q > 0$ and $p < 0$

The results of Problems 16 and 17 are conveniently summarized in Figure 9.20.

9.4 STABILITY; ALMOST LINEAR SYSTEMS

In the previous sections of this chapter, we have on several occasions referred to the concepts of stability, asymptotic stability, and instability of a solution of the autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y). \tag{1}$$

In this section we will finally give a precise mathematical meaning to these concepts, discuss an important theorem dealing with the stability of an almost linear system, and explore its consequences by considering an illustrative example.

A critical point $x = x_0, y = y_0$ (an equilibrium solution $x = x_0, y = y_0$) of the autonomous system (1) is said to be a *stable critical point* if, given any $\epsilon > 0$, it is possible to find a δ such that every solution $x = \phi(t), y = \psi(t)$ of the system (1), which at $t = 0$ satisfies

$$\{[\phi(0) - x_0]^2 + [\psi(0) - y_0]^2\}^{1/2} < \delta, \tag{2}$$

exists and satisfies

$$\{[\phi(t) - x_0]^2 + [\psi(t) - y_0]^2\}^{1/2} < \epsilon \tag{3}$$

for all $t \geq 0$. This is illustrated geometrically in Figures 9.21a and 9.21b. These mathematical statements say that all solutions that start "sufficiently close" to (x_0, y_0) stay "close" to (x_0, y_0). Note that in Figure 9.21a the trajectory is within the circle $(x - x_0)^2 + (y - y_0)^2 = \delta^2$ at $t = 0$ and, while it soon passes outside of this circle, it remains within the circle $(x - x_0)^2 + (y - y_0)^2 = \epsilon^2$ for all $t \geq 0$. However, the trajectory of the solution does not have to approach the critical point (x_0, y_0), as is illustrated in Figure 9.21b.

A critical point (x_0, y_0) is said to be *asymptotically stable* if it is stable and if there exists a δ_0, $0 < \delta_0 < \delta$, such that if a solution $x = \phi(t), y = \psi(t)$ satisfies

$$\{[\phi(0) - x_0]^2 + [\psi(0) - y_0]^2\}^{1/2} < \delta_0, \tag{4}$$

then

$$\lim_{t\to\infty} \phi(t) = x_0, \qquad \lim_{t\to\infty} \psi(t) = y_0. \tag{5}$$

Trajectories that start "sufficiently close" to (x_0, y_0) must not only stay "close" but must eventually approach (x_0, y_0) as t approaches infinity. This is the case for the trajectory in Figure 9.21a but not for the one in Figure 9.21b. Note that asymptotic stability is a stronger requirement than stability, since a critical point must be stable before we can even talk about whether it is asymptotically stable. On the other hand, the limit condition (5), which is an essential feature of asymptotic stability, does not by itself imply even ordinary stability. Indeed, examples can be constructed in which all of the trajectories approach (x_0, y_0) as $t \to \infty$, but for which (x_0, y_0) is not a stable critical point. Geometrically, all that is needed

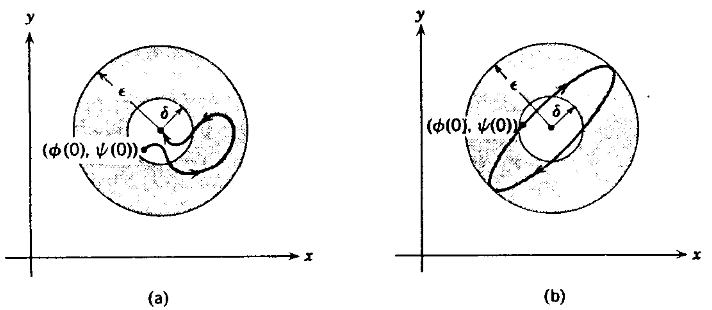


FIGURE 9.21

is a family of trajectories having members that start arbitrarily close to (x_0, y_0), then recede an arbitrarily large distance before eventually approaching (x_0, y_0) as t approaches infinity.* A critical point that is not stable is said to be *unstable*.

For the linear system

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy. \tag{6}$$

with $ad - bc \neq 0$, the type and stability of the critical point $(0, 0)$ as a function of the roots $r_1 \neq 0$ and $r_2 \neq 0$ of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{7}$$

were listed in Table 9.1 of the preceding section. The stability characteristics are summarized in the following theorem.

Theorem 9.1. *The critical point* $(0, 0)$ *of the linear system* (6) *is*

(i) *asymptotically stable if the roots* r_1, r_2 *of the characteristic equation* (7) *are real and negative or have negative real parts;*
(ii) *stable, but not asymptotically stable, if* r_1 *and* r_2 *are pure imaginary;*
(iii) *unstable if* r_1 *and* r_2 *are real and either is positive, or if they have positive real parts.*

This theorem was established by intuitive arguments in the preceding section. For a rigorous proof, it is necessary to show how to compute δ for a given ϵ so that Eq. (3) is satisfied and, for asymptotic stability, how to compute δ_0 so that Eq. (5) is true. We will not take up this type of detailed analysis.

Notice that if a critical point of the linear system (6) is asymptotically stable, then not only do trajectories that start close to the critical point approach the critical point, but, in fact, since every solution is a linear combination of $e^{r_1 t}$ and $e^{r_2 t}$, *every* trajectory approaches the critical point. In this case the critical point is said to be *globally asymptotically stable*. This property of linear systems is not, in general, true for nonlinear systems. This was illustrated by the example $dA/dt = \epsilon A - \sigma A^2$ with $\epsilon < 0$ and $\sigma < 0$, whose solutions are sketched in Figure 9.5 in Section 9.1. The critical point $A = 0$ is asymptotically stable but not globally asymptotically stable.

Often an important practical problem in considering an asymptotically stable critical point of a nonlinear system is to estimate the set of initial conditions for which the critical point is asymptotically stable. This set of initial points is called the *region of asymptotic stability* for the critical point. Alternatively, we may wish to determine whether the critical point is asymptotically stable for a given set of initial conditions. Again, for the example sketched in Figure 9.5, the critical point $A = 0$ is asymptotically stable for all initial conditions $A(0)$ such that $A(0) < \epsilon/\sigma$.

* Such examples are fairly complicated and not often encountered in practice (see Cesari, page 96).

We now want to relate the results for the linear system (6) to the nonlinear system

$$\frac{dx}{dt} = ax + by + F_1(x, y),$$
$$\frac{dy}{dt} = cx + dy + G_1(x, y), \qquad (8)$$

mentioned at the beginning of Section 9.3. We assume that $(0, 0)$ is a critical point of the system (8) and that $ad - bc \neq 0$. Also we assume that F_1 and G_1 have continuous first partial derivatives and are small near the origin in the sense that $F_1(x, y)/r \to 0$ and $G_1(x, y)/r \to 0$ as $r \to 0$, where $r = (x^2 + y^2)^{1/2}$. Recall that such a system is said to be almost linear in the neighborhood of the origin. In our discussion we will usually not mention the phrase "near the origin," since it will be clear that we are talking about the neighborhood of the critical point $(0, 0)$.

As an example, the system

$$\frac{dx}{dt} = x - x^2 - xy,$$
$$\frac{dy}{dt} = \tfrac{1}{2}y - \tfrac{1}{4}y^2 - \tfrac{3}{4}xy, \qquad (9)$$

satisfies the stated conditions. Here $a = 1$, $b = 0$, $c = 0$, $d = \frac{1}{2}$, $F_1(x, y) = -x^2 - xy$, and $G_1(x, y) = -\frac{1}{4}y^2 - \frac{3}{4}xy$. To show that $F_1(x, y)/r \to 0$ as $r \to 0$, let $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{F_1(x, y)}{r} = \frac{-r^2\cos^2\theta - r^2\sin\theta\cos\theta}{r} = -r(\cos^2\theta + \cos\theta\sin\theta) \to 0 \qquad (10)$$

as $r \to 0$. The argument that $G_1(x, y)/r \to 0$ as $r \to 0$ is similar.

The type and stability of a critical point of the almost linear system (8) are closely related in most cases to the type and stability of the critical point of the corresponding linear system (6).

Theorem 9.2. *Let r_1 and r_2 be the roots of the characteristic equation* (7) *of the linear system* (6) *corresponding to the almost linear system* (8). *Then the type and stability of the critical point* $(0, 0)$ *of the linear system* (6) *and the almost linear system* (8) *are as shown in the table on page* 413.

At this stage, the proof of this theorem is too difficult to give, and we will accept the results of Theorem 9.2 without proof. The results for asymptotic stability and for instability follow as a consequence of a result discussed in Section 9.6, and a proof is sketched in Problems 5 to 7 of Section 9.6. Essentially, Theorem 9.2 says that for x and y near zero the nonlinear terms $F_1(x, y)$ and $G_1(x, y)$ are small and do not affect the stability and type of critical point as determined by the linear terms except in two sensitive cases: r_1 and r_2 pure imaginary, and r_1 and r_2 real and equal. Recall that at the end of Section 9.3 we stated that small perturbations in the coefficients of the linear system (6), and hence in the roots r_1 and r_2 can alter the type and stability of the critical point only in these two sensitive cases. When r_1 and r_2 are pure imaginary, a small perturbation can change the stable center into either an asymptotically stable or an unstable spiral point or even leave it as a center. When $r_1 = r_2$ a small perturbation does not affect the stability of the critical point, but may change the node into a spiral point. It is reasonable to expect that the small nonlinear terms in Eqs. (8) might have a similarly substantial effect, at least in these two sensitive cases. This is so, but the main significance of Theorem 9.2 is that in *all other cases* the small nonlinear terms do not alter the type or stability of the critical point. Thus, except in the two sensitive cases, the type and stability of the critical point of the nonlinear system (8) can be determined from a study of the much simpler linear system (6).

r_1, r_2	Linear System		Almost Linear System	
	Type	*Stability*	*Type*	*Stability*
$r_1 > r_2 > 0$	*IN*	*Unstable*	*IN*	*Unstable*
$r_1 < r_2 < 0$	*IN*	*Asymptotically stable*	*IN*	*Asymptotically stable*
$r_2 < 0 < r_1$	*SP*	*Unstable*	*SP*	*Unstable*
$r_1 = r_2 > 0$	*PN or IN*	*Unstable*	*PN, IN, or SpP*	*Unstable*
$r_1 = r_2 < 0$	*PN or IN*	*Asymptotically stable*	*PN, IN or SpP*	*Asymptotically stable*
$r_1 = r_2 = \lambda \pm i\mu$				
$\lambda > 0$	*SpP*	*Unstable*	*SpP*	*Unstable*
$\lambda < 0$	*SpP*	*Asymptotically stable*	*SpP*	*Asymptotically stable*
$r_1 = i\mu, r_2 = -i\mu$	*C*	*Stable*	*C or SpP*	*Indeterminate*

IN = *Improper node;* *PN* = *Proper node;* *SP* = *Saddle point;* *SpP* = *Spiral point;* *C* = *Center*

Even if the critical point is of the same type as that of the linear system, the trajectories of the almost linear system may be considerably different in appearance from those of the corresponding linear system. This is illustrated in Problems 12 and 13. However, it can be shown that the slope at which trajectories "enter" or "leave" the critical point is given correctly by the linear equations.

We will illustrate some of these ideas by considering the motion of a damped pendulum and several problems in ecology. The pendulum problem is discussed here; the ecological problems are discussed in Section 9.5.

Damped Pendulum. We consider the motion of a damped pendulum for which the damping is proportional to the speed (see Figure 9.22). It is not difficult to

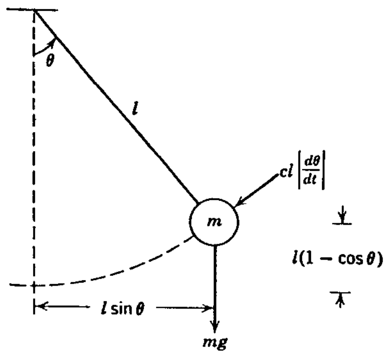


FIGURE 9.22 Pendulum.

show, by reasoning similar to that used in the derivation of the spring-mass equation (Section 3.7), that the governing equation is

$$ml^2\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mgl\sin\theta = 0, \tag{11}$$

where the damping constant $c > 0$. Letting $x = \theta$ and $y = d\theta/dt$ gives the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{g}{l}\sin x - \frac{c}{ml}y. \tag{12}$$

The point $x = 0$, $y = 0$ is a critical point of the system (12). Because of the damping mechanism, we expect any small motion about $\theta = 0$ to decay in amplitude. Thus intuitively the equilibrium point (0, 0) should be asymptotically stable. To show this we rewrite the system (12) as

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -\frac{g}{l}x - \frac{c}{ml}y - \frac{g}{l}(\sin x - x).\end{aligned} \tag{13}$$

It was shown in Section 9.3 [see Eq. (5)] that $(\sin x - x)/r \to 0$ as $r \to 0$, so the system (13) is an almost linear system; hence Theorem 9.2 is applicable. The roots of the characteristic equation associated with the corresponding linear system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{g}{l}x - \frac{c}{ml}y \tag{14}$$

are

$$r_1, r_2 = \frac{-c/ml \pm \sqrt{(c/ml)^2 - 4g/l}}{2}. \tag{15}$$

1. If $(c/ml)^2 - 4g/l > 0$ the roots are real, unequal, and negative. The critical point (0, 0) is an asymptotically stable improper node of the linear system (14) and of the almost linear system (13).

2. If $(c/ml)^2 - 4g/l = 0$ the roots are real, equal, and negative. The critical point (0, 0) is an asymptotically stable node of the linear system (14). It may be either an asymptotically stable node or an asymptotically stable spiral point of the almost linear system (13).

3. If $(c/ml)^2 - 4g/l < 0$ the roots are complex with negative real parts. The critical point (0, 0) is an asymptotically stable spiral point of the linear system (14) and of the almost linear system (13).

In addition to the critical point (0, 0), the almost linear system (12) has the critical points $x = n\pi$, $y = 0$, $n = \pm 1, \pm 2, \pm 3, \ldots$ corresponding to $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots, d\theta/dt = 0$. Again we expect (from Figure 9.22) that the points corresponding to $\theta = \pm 2\pi, \pm 4\pi, \ldots$ are asymptotically stable spiral points, and the points corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$ are unstable saddle points. Consider the critical point $x = \pi$, $y = 0$. To examine the stability of this point, let

$$x = \pi + u, \qquad y = 0 + v. \tag{16}$$

Substituting for x and y in Eqs. (12), and making use of the fact that $\sin(\pi + u) = -\sin u$, we obtain

$$\begin{aligned}\frac{du}{dt} &= v,\\ \frac{dv}{dt} &= -\frac{c}{ml}v + \frac{g}{l}\sin u.\end{aligned} \tag{17}$$

We are interested in studying the critical point $u = v = 0$ of the system (17). We can rewrite the second of Eqs. (17) as

$$\frac{dv}{dt} = \frac{g}{l}u - \frac{c}{ml}v + \frac{g}{l}(\sin u - u). \tag{18}$$

It is clear that the first of Eqs. (17) and Eq. (18) are the same as the system (13), except that $-g/l$ is replaced by g/l. Thus the system is almost linear, and the roots of the characteristic equation of the corresponding linear system are given by

$$r_1, r_2 = \frac{-c/ml \pm \sqrt{(c/ml)^2 + 4g/l}}{2}. \tag{19}$$

One root (r_1) is positive and the other (r_2) is negative. Therefore, the critical point $x = \pi$, $y = 0$ is an unstable saddle point of both the linear system and the almost linear system, as expected.

Let us consider the case $(c/ml)^2 - 4g/l < 0$ in more detail. We will try to make a schematic sketch of the trajectories in the xy (phase) plane. The critical points are at $y = 0$, $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.

We have seen that the critical point (0, 0) is an asymptotically stable spiral point. This point corresponds to the pendulum hanging vertically downward with zero velocity. It is not difficult to show that the critical points $(2n\pi, 0)$, $n = \pm 1, \pm 2, \ldots$, which also correspond to the pendulum hanging vertically downward with zero velocity, are also asymptotically stable spiral points. (Let $x = 2n\pi + u, y = 0 + v$ and proceed as above.) The direction of motion on the spirals near (0, 0) can be obtained from Eqs. (12). Consider the point at which a spiral intersects the positive y axis ($x = 0, y > 0$). At such a point it follows from Eqs. (12) that $dx/dt > 0$. Thus the point (x, y) on the trajectory is moving to the right so the direction of motion on the spirals is clockwise. The situation is the same for the spirals near the critical points $(2n\pi, 0)$, $n = \pm 1, \pm 2, \ldots$. The spirals are sketched in Figure 9.23.

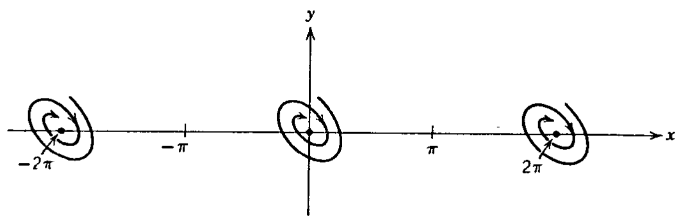


FIGURE 9.23 Asymptotically stable spiral points for the damped pendulum.

We have shown above that the critical point $(\pi, 0)$ is an unstable saddle point. This point corresponds to the pendulum in a vertically upward position with zero velocity. Again, we can establish that the critical points $((2n + 1)\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$, are unstable saddle points. Thus in the phase plane the asymptotically stable spiral points are separated by unstable saddle points. Recall from Section 9.3 that only one pair of trajectories "enters" a saddle point. In order to determine the direction of the entering trajectories at the saddle point $(\pi, 0)$, we consider the linear system corresponding to Eqs. (17):

$$\frac{du}{dt} = v,$$
$$\frac{dv}{dt} = \frac{g}{l}u - \frac{c}{ml}v. \qquad (20)$$

On making use of Eq. (19) and the first of Eqs. (20), we can write the general solution of Eq. (20) in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = C_1\begin{pmatrix} 1 \\ r_1 \end{pmatrix}e^{r_1 t} + C_2\begin{pmatrix} 1 \\ r_2 \end{pmatrix}e^{r_2 t},$$

where C_1 and C_2 are arbitrary constants. Since $r_1 > 0$ and $r_2 < 0$, it follows that the solution that approaches zero as $t \to \infty$ corresponds to $C_1 = 0$. For this solution $v/u = r_2$ so the slope of the entering trajectories is negative; one lies in the second quadrant ($C_2 < 0$) and the other lies in the fourth quadrant ($C_2 > 0$). For $C_2 = 0$ we obtain the pair of trajectories "exiting" from the saddle point. These trajectories have slope $r_1 > 0$; one lies in the first quadrant ($C_1 > 0$) and the other lies in the third quadrant ($C_1 < 0$). The situation is the same for the other unstable saddle points $((2n + 1)\pi, 0)$, $n = \pm 1, \pm 2, \ldots$. A possible picture of the trajectories in the neighborhood of the unstable saddle points is shown in Figure 9.24.

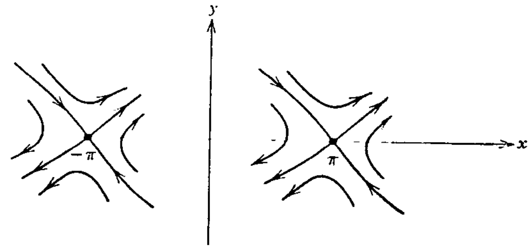


FIGURE 9.24 Unstable saddle points for the damped pendulum.

If we put Figures 9.23 and 9.24 together, we obtain the schematic picture of the trajectories in the phase plane shown in Figure 9.25. Note that the trajectories entering the saddle points separate the phase plane into sectors. Such a trajectory is called a *separatrix*. The initial conditions on θ and $d\theta/dt$ determine the position of an initial point (x, y) in the phase plane. The subsequent motion of the pendulum is represented by the trajectory passing through the initial point as it spirals towards the asymptotically stable point of that sector. Note that it is mathematically possible (but physically unrealizable) to choose initial conditions on a separatrix

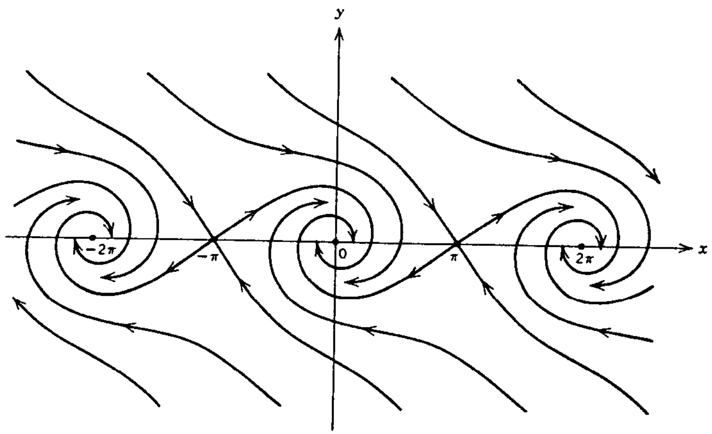


FIGURE 9.25 Phase plane for the damped pendulum.

so that the resulting motion leads to a balanced pendulum in a vertically upward position of unstable equilibrium.

PROBLEMS

In each of Problems 1 through 10, verify that (0, 0) is a critical point, show that the system is almost linear, and discuss the type and stability of the critical point (0, 0).

1. $dx/dt = x - y + xy$, $dy/dt = 3x - 2y - xy$
2. $dx/dt = x + x^2 + y^2$, $dy/dt = y - xy$
3. $dx/dt = -2x - y - x(x^2 + y^2)$, $dy/dt = x - y + y(x^2 + y^2)$
4. $dx/dt = y + x(1 - x^2 - y^2)$, $dy/dt = -x + y(1 - x^2 - y^2)$
5. $dx/dt = 2x + y + xy^3$, $dy/dt = x - 2y - xy$
6. $dx/dt = x + 2x^2 - y^2$, $dy/dt = x - 2y + x^3$
7. $dx/dt = y$, $dy/dt = -x + \mu y(1 - x^2)$, $\mu > 0$
8. $dx/dt = 1 + y - e^{-x}$, $dy/dt = y - \sin x$
9. $dx/dt = (1 + x)\sin y$, $dy/dt = 1 - x - \cos y$
10. $dx/dt = e^{-x+y} - \cos x$, $dy/dt = \sin(x - 3y)$

11. Determine all real critical points of each of the following systems of equations and discuss their type and stability.

(a) $dx/dt = x + y^2$, $dy/dt = x + y$
(b) $dx/dt = 1 - xy$, $dy/dt = x - y^3$
(c) $dx/dt = x - x^2 - xy$, $dy/dt = 3y - xy - 2y^2$
(d) $dx/dt = 1 - y$, $dy/dt = x^2 - y^2$

12. Consider the autonomous system

$$dx/dt = y, \qquad dy/dt = x + 2x^3.$$

(a) Show that the critical point (0, 0) is a saddle point.

(b) Sketch the trajectories for the corresponding linear system by integrating the equation for dy/dx. Show from the parametric form of the solution that the only trajectory on which $x \to 0$, $y \to 0$ as $t \to \infty$ is $y = -x$.

(c) Determine the trajectories for the nonlinear system by integrating the equation for dy/dx. Sketch the trajectories for the nonlinear system that correspond to $y = -x$ and $y = x$ for the linear system.

13. Consider the autonomous system

$$dx/dt = x, \qquad dy/dt = -2y + x^3.$$

(a) Show that the critical point (0, 0) is a saddle point.

(b) Sketch the trajectories for the corresponding linear system [Problem 4(c) of Section 9.2], and show that the trajectory for which $x \to 0$, $y \to 0$ as $t \to \infty$ is given by $x = 0$.

(c) Determine the trajectories for the nonlinear system for $x \neq 0$ by integrating the equation for dy/dx. Show that the trajectory corresponding to $x = 0$ for the linear system is unaltered, but that the one corresponding to $y = 0$ is $y = x^5/5$. Sketch several of the trajectories for the nonlinear system.

14. Theorem 9.2 provides no information about the stability of a critical point of an almost linear system if that point is a center of the corresponding linear system. That this must be the case is illustrated by the following two systems

$$\text{(i)}\ \begin{cases} dx/dt = y + x(x^2 + y^2) \\ dy/dt = -x + y(x^2 + y^2) \end{cases} \qquad \text{(ii)}\ \begin{cases} dx/dt = y - x(x^2 + y^2) \\ dy/dt = -x - y(x^2 + y^2). \end{cases}$$

(a) Show that (0, 0) is a critical point of each system and, furthermore, is a center of the corresponding linear system.

(b) Show that each system is almost linear.

(c) Let $r^2 = x^2 + y^2$, and note that $x\,dx/dt + y\,dy/dt = r\,dr/dt$. For system (ii) show that $dr/dt < 0$ and that $r \to 0$ as $t \to \infty$, and hence the critical point is asymptotically stable. For system (i) show that the solution of the initial value problem for r with $r = r_0$ at $t = 0$ becomes unbounded as $t \to 1/2r_0^2$, and hence the critical point is unstable.

15. The equation of motion of an undamped pendulum is $d^2\theta/dt^2 + (g/l)\sin\theta = 0$. Let $x = \theta$, $y = d\theta/dt$, and $k^2 = g/l$ to obtain the system of equations

$$dx/dt = y, \qquad dy/dt = -k^2 \sin x.$$

(a) Show that the critical points are $(\pm n\pi, 0)$, $n = 0, 1, 2, \ldots$, and that the system is almost linear in the neighborhood of each critical point.

(b) Show that the critical point (0, 0) is a (stable) center of the corresponding linear system. Using Theorem 9.2 what can be said about the almost linear problem? The situation is similar at the critical points $(\pm 2n\pi, 0)$, $n = 1, 2, 3, \ldots$. What is the physical interpretation of these critical points?

(c) Show that the critical point $(\pi, 0)$ is an (unstable) saddle point. The situation is similar at the critical points $[\pm(2n - 1)\pi, 0]$, $n = 1, 2, 3, \ldots$. What is the physical interpretation of these critical points?

16. In this problem we give some of the details of the analysis for sketching the trajectories of the undamped pendulum of Problem 15. Show by eliminating t that the equation of the trajectories can be written as

$$\tfrac{1}{2}y^2 + k^2(1 - \cos x) = E.$$

To discover the significance of the constant E, observe that $\frac{1}{2}y^2 = \frac{1}{2}(dx/dt)^2 = \frac{1}{2}(d\theta/dt)^2$ is proportional to the kinetic energy of the pendulum. Also, $k^2(1 - \cos x) = \int_0^x k^2 \sin x\,dx$ is proportional to the potential energy of the pendulum due to the gravity force. Thus the constant E is the "energy" of the motion. It is constant along a trajectory (during the course of the motion), and is determined by the initial values of x and y.

In sketching the trajectories it is only necessary to consider the interval $-\pi \le x \le \pi$ since the equation is periodic in x with period 2π. For $E = 2k^2$ show that $y = \pm 2k\cos x/2$ and sketch these trajectories. Observe that these trajectories enter or leave the unstable saddle points at $(\pm\pi, 0)$. Determine the direction of motion on each trajectory by using the differential equations given in Problem 15.

It can be shown that the trajectories are closed curves for $E < 2k^2$ and are not closed curves for $E > 2k^2$. We will not pursue these details here, but a schematic sketch of the trajectories for an undamped pendulum is given in Figure 9.26. The trajectories for $E < 2k^2$ correspond to periodic motions about a center; the trajectories for $E > 2k^2$ correspond to whirling motions.

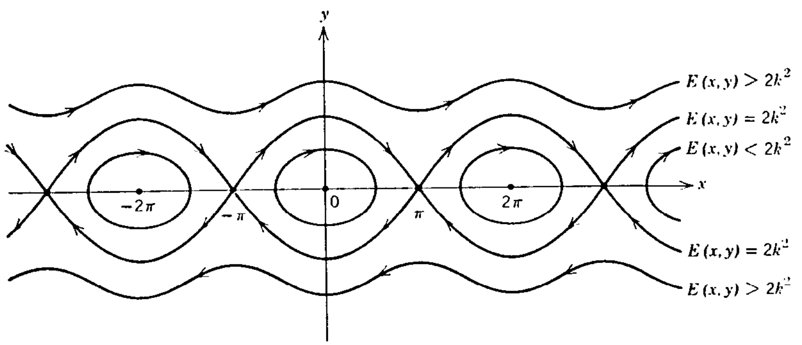


FIGURE 9.26 Phase plane for the undamped pendulum.

*17. In this problem we derive a formula for the natural period of an undamped nonlinear pendulum [$c = 0$ in Eq. (11)]. Suppose that the bob is pulled through a positive angle α and then released with zero velocity. Assume that the relation θ as a function of t can be solved for t as a function of θ so that $d\theta/dt$ can be considered a function of θ. Derive the following sequence of equations:

$$\tfrac{1}{2}ml^2 \frac{d}{d\theta}\left[\left(\frac{d\theta}{dt}\right)^2\right] = -mgl\sin\theta,$$

$$\tfrac{1}{2}m\left(l\frac{d\theta}{dt}\right)^2 = mgl(\cos\theta - \cos\alpha),$$

$$dt = -\sqrt{\frac{l}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Why was the negative square root chosen in the last equation?

If T is the natural period of oscillation, derive the formula

$$\frac{T}{4} = -\sqrt{\frac{l}{2g}}\int_\alpha^0 \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

By making the change of variables $\cos\theta = 1 - 2\sin^2\theta/2$, $\cos\alpha = 1 - 2\sin^2\alpha/2$ followed by $\sin\theta/2 = k\sin\phi$ with $k = \sin\alpha/2$, show that

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = 4\sqrt{\frac{l}{g}}\,F(k, \pi/2).$$

The function F is called the *elliptic integral of the first kind*. Note that the period depends on the ratio l/g and also the initial displacement α through $k = \sin\alpha/2$. The corresponding period for the linearized pendulum is $2\pi(l/g)^{1/2}$ and is independent of the initial displacement. To obtain this special result from the general formula we must consider the limiting case of small α (small angular displacement) in which case k is small. In the limit $k \to 0$ the above formula gives $T = 4(l/g)^{1/2}F(0, \pi/2) = 2\pi(l/g)^{1/2}$.

18. A generalization of the damped pendulum equation discussed in the text, or a damped spring-mass system, is the Liénard equation

$$\frac{d^2x}{dt^2} + c(x)\frac{dx}{dt} + g(x) = 0.$$

If $c(x)$ is a constant and $g(x) = kx$ then this equation has the form of the linear pendulum equation [replace $\sin\theta$ with θ in Eq. (11)]; otherwise the damping force $c(x)\,dx/dt$ and restoring force $g(x)$ are nonlinear. Assume that c is continuously differentiable, g is twice continuously differentiable and $g(0) = 0$.

(a) Write the Liénard equation as a system of two first order equations by introducing the variable $y = dx/dt$.

(b) Show that (0, 0) is a critical point and that the system is almost linear in the neighborhood of (0, 0).

(c) Show that if $c(0) > 0$ and $g'(0) > 0$ then the critical point is asymptotically stable, and that if $c(0) < 0$ or $g'(0) < 0$ then the critical point is unstable.

Hint: Use Taylor series to approximate f and g in the neighborhood of $x = 0$.

19. Consider the equations for two competing species derived in Section 9.2:

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \qquad dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

Suppose that $\epsilon_1/\sigma_1 < \epsilon_2/\alpha_2$ and $\epsilon_2/\sigma_2 < \epsilon_1/\alpha_1$.

(a) Find the critical point (x_E, y_E) for which both species can coexist.

(b) By making the change of variables $x = x_E + u$, $y = y_E + v$ transform the system of equations to one with a critical point at $u = 0$, $v = 0$. Observe that the system is almost linear.

(c) Classify the critical point as to type and stability.

20. Carry out the calculations of Problem 19 for the case $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$.

9.5 COMPETING SPECIES AND PREDATOR-PREY PROBLEMS

In this section we consider two problems in ecology: competing species and predator-prey.

Competing Species. In Section 9.2 we showed that a model for the competition between two species with population densities x and y leads to the differential equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \tag{1a}$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x), \tag{1b}$$

where the parameters $\epsilon_1, \sigma_1, \ldots, \alpha_2$ are positive. As we saw then, we can analyze these equations by dividing the phase plane into regions according to the sign of dx/dt and dy/dt and then drawing typical trajectories. Let us now see how we can use the theory of almost linear systems to obtain a more precise understanding of what happens.

We start by considering the following specific example:

$$dx/dt = x(1 - x - y), \tag{2a}$$
$$dy/dt = y(\tfrac{1}{2} - \tfrac{1}{4}y - \tfrac{3}{4}x). \tag{2b}$$

For convenience we think of x and y as the population densities of two species of bacteria competing with each other for the same supply of food. We ask whether there are equilibrium states that might be reached, or whether a periodic growth and decay will be observed, and how such possibilities depend on the initial state of the two cultures.

The critical points of the system (2) are the solutions of the nonlinear algebraic equations

$$x(1 - x - y) = 0,$$
$$y(\tfrac{1}{2} - \tfrac{1}{4}y - \tfrac{3}{4}x) = 0. \tag{3}$$

Clearly one solution is $x = y = 0$; a second solution is $x = 1$, $y = 0$; a third is $x = 0$, $y = 2$. Finally, if $x \neq 0$, $y \neq 0$, we obtain from Eqs. (3) the system

$$x + y = 1,$$
$$3x + y = 2,$$

which has the solution $x = \frac{1}{2}$, $y = \frac{1}{2}$. These four points in the xy plane are the only critical points of the system (2). We will consider each separately.

x = 0, y = 0. This corresponds to a state in which both bacteria die as a result of their competition. For this case the corresponding linear system, from Eqs. (2), is

$$dx/dt = x, \quad dy/dt = \tfrac{1}{2}y, \tag{4}$$

and the roots of the characteristic equation are 1 and $\frac{1}{2}$. Thus the origin is an unstable improper node: the solution $x = 0$, $y = 0$ of the interaction problem will not occur in practice.

x = 1, y = 0. Clearly this corresponds to a state in which bacteria x survives the competition but bacteria y does not. To examine this critical point, let $x = 1 + u$, $y = 0 + v$. Substituting for x and y in Eqs. (2) and simplifying, we obtain

$$du/dt = -u - v - u^2 - uv,$$
$$dv/dt = -\tfrac{1}{4}v - \tfrac{1}{4}v^2 - \tfrac{3}{4}uv. \tag{5}$$

It is not difficult to show that the system (5) is an almost linear system. The corresponding linear system is $du/dt = -u - v$, $dv/dt = -v/4$ and the roots of the characteristic equation are $-\frac{1}{4}$ and -1. The general solution is

$$u = -\tfrac{4}{3}Ae^{-t/4} + Be^{-t}, \qquad v = Ae^{-t/4}.$$

where A and B are arbitrary. Thus $x = 1$, $y = 0$ is an asymptotically stable improper node. If the initial values of x and y are sufficiently close to $x = 1$, $y = 0$ the interaction will lead finally to that state.

For this critical point we will indicate how the trajectories of the linear system behave in the neighborhood of $x = 1$, $y = 0$. It is clear that $u \to 0$, $v \to 0$ as $t \to \infty$ so that all of the trajectories enter the critical point $(1, 0)$ as $t \to \infty$. For $A = 0$ we have $x = 1 + u = 1 + Be^{-t}$ and $y = v = 0$ so one pair ($B < 0$ and $B > 0$) of trajectories enters along the x axis. For $A \neq 0$ we can compute the slope at any point on a trajectory by noting that

$$\frac{dy}{dx} = \frac{dv/dt}{d(1+u)/dt} = \frac{-\frac{1}{4}Ae^{-t/4}}{\frac{1}{3}Ae^{-t/4} - Be^{-t}} = \frac{-\frac{1}{4}A}{\frac{1}{3}A - Be^{-3t/4}}.$$

In particular as we approach ($t \to \infty$) the critical point along any trajectory with $A \neq 0$ we see that $dy/dx \to -\frac{3}{4}$. Thus all the trajectories except one pair enter the critical point along a line with slope $-\frac{3}{4}$.

x = 0, y = 2. The analysis is exactly similar to that for the critical point $x = 1$, $y = 0$. The critical point $x = 0$, $y = 2$ is also an asymptotically stable improper node. In this case bacteria y survives, but bacteria x does not.

x = 1/2, y = 1/2. This critical point corresponds to a mixed equilibrium state or coexistence; a standoff, so to speak, in the competition between the two bacteria cultures. To examine the nature of this critical point we let $x = \frac{1}{2} + u$, $y = \frac{1}{2} + v$. Substituting for x and y in Eqs. (2) we obtain

$$du/dt = -\tfrac{1}{2}u - \tfrac{1}{2}v - u^2 - uv,$$
$$dv/dt = -\tfrac{3}{8}u - \tfrac{1}{8}v - \tfrac{1}{4}v^2 - \tfrac{3}{4}uv. \tag{6}$$

The system (6) is an almost linear system, and the roots of the characteristic equation of the corresponding linear system are $(-5 \pm \sqrt{57})/16$. Since these roots are real and of opposite sign, the critical point $(\frac{1}{2}, \frac{1}{2})$ is an unstable saddle point. One pair of trajectories enters the critical point; the others recede from it. It can be shown by considering the general solution of the corresponding linear system that the slope of the pair of entering trajectories as $(x, y) \to (\frac{1}{2}, \frac{1}{2})$ is $(\sqrt{57} - 3)/8 \simeq 0.57$ (see Problem 1).

A schematic sketch of what the trajectories might look like in the neighborhood of each critical point is shown in Figure 9.27a. With a little detective work it is possible to extend the local pictures and obtain a global picture of the trajectories in the phase plane. First, we are only interested in x and y positive. Since trajectories cannot cross other trajectories and since the x and y axes are trajectories it follows that a trajectory that starts in the first quadrant must stay in the first quadrant, and a trajectory that starts in any other quadrant cannot enter the first quadrant. Second, we accept without proof two facts that follow from advanced theory: (i) the system (2) does not have any periodic solutions, that is, trajectories that are closed curves (see Theorem 9.7 of Section 9.7); and (ii) a trajectory that is not a closed curve must either enter a critical point or go off to infinity as $t \to \infty$. But consider what is happening for x and y large. The nonlinear terms $-(x^2 + xy)$ and $-\frac{1}{4}(y^2 + 3xy)$ in the first and second of Eqs. (2), respectively, outweigh the linear terms. Since they are negative, dx/dt and dy/dt are negative for x and y large. Thus for x and y large the direction of motion on any trajectory is inward. The trajectories cannot escape to infinity! Eventually they must head toward one or the other of the two stable nodes. The *schematic*

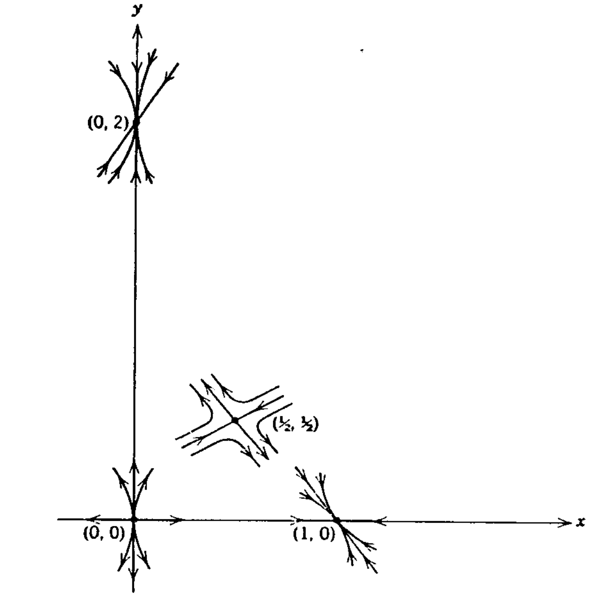


FIGURE 9.27a

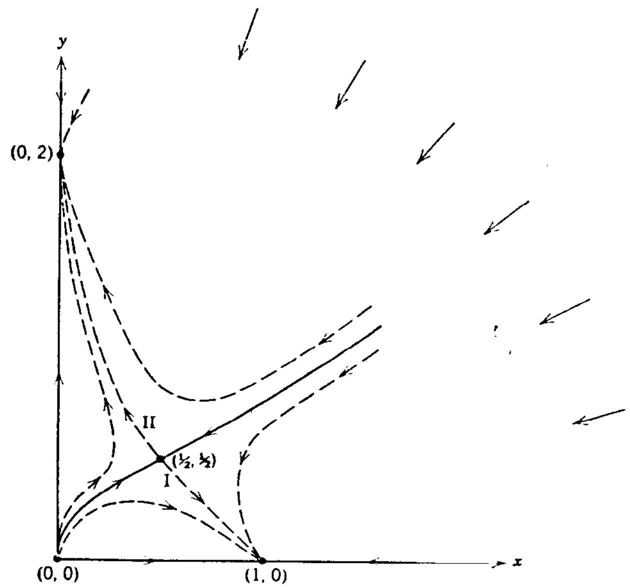


FIGURE 9.27b

sketch shown in Figure 9.27b is not an unreasonable representation* of what must be happening in the first quadrant. If the initial values of x and y are in region I of Figure 9.27b, then x wins the competition; if the initial values are in region II, then y wins. "Peaceful coexistence" is not possible unless the initial point lies exactly on the dividing trajectory (separatrix). Of particular interest would be the determination of the dividing trajectories that enter the saddle point $(\frac{1}{2}, \frac{1}{2})$ which separate regions I and II.

Now let us return to the general system (1). Recall from Section 9.2 that four cases must be considered depending on the relative orientation of the lines

$$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0 \quad \text{and} \quad \epsilon_2 - \sigma_2 y - \alpha_2 x = 0, \tag{7}$$

as shown in Figure 9.28. Let (X_E, Y_E) denote any critical point in any one of the four cases. To study the system (1) in the neighborhood of this critical point we let

$$x = X_E + u, \qquad y = Y_E + v, \tag{8}$$

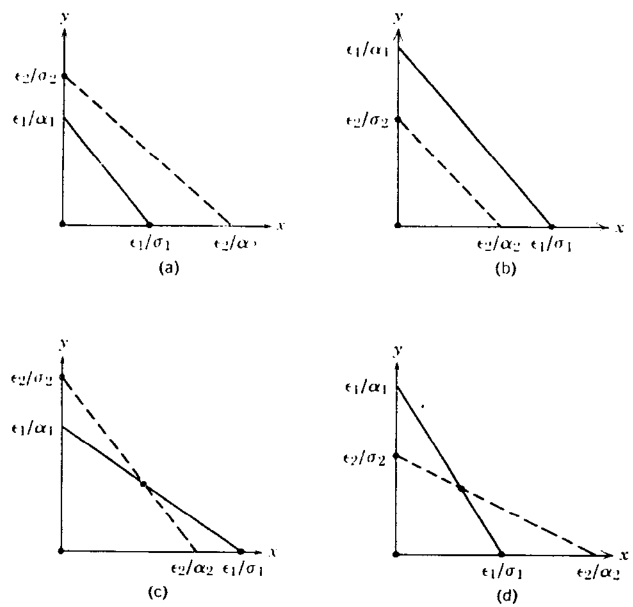


FIGURE 9.28

* The dotted trajectories that appear to go, for example, from (0, 0) to (1, 0) actually correspond to trajectories through particular initial points. As $t \to \infty$ they approach the stable node (1, 0); as $t \to -\infty$ they approach the unstable node (0, 0).

and substitute in Eqs. (1):

$$\frac{d}{dt}(X_E + u) = (X_E + u)[\epsilon_1 - \sigma_1(X_E + u) - \alpha_1(Y_E + v)],$$
$$\frac{d}{dt}(Y_E + v) = (Y_E + v)[\epsilon_2 - \sigma_2(Y_E + v) - \alpha_2(X_E + u)]. \tag{9}$$

Since $dX_E/dt = dY_E/dt = 0$, we have

$$\frac{du}{dt} = (X_E + u)[(\epsilon_1 - \sigma_1 X_E - \alpha_1 Y_E) - \sigma_1 u - \alpha_1 v], \tag{10a}$$

$$\frac{dv}{dt} = (Y_E + v)[(\epsilon_2 - \sigma_2 Y_E - \alpha_2 X_E) - \sigma_2 v - \alpha_2 u]. \tag{10b}$$

The right-hand side of Eq. (10a) has the form $X_E(\epsilon_1 - \sigma_1 X_E - \alpha_1 Y_E) + (\ \ )u + (\ \ )v + (\ \ )u^2 + (\ \ )uv$. The constant term is zero since either $X_E = 0$ or $\epsilon_1 - \sigma_1 X_E - \alpha_1 Y_E = 0$. Similarly, the right-hand side of Eq. (10b) reduces to the form $(\ \ )u + (\ \ )v + (\ \ )uv + (\ \ )v^2$. Next, it is clear that Eqs. (10) are almost linear, so we consider the corresponding linear system

$$\frac{du}{dt} = [(\epsilon_1 - \sigma_1 X_E - \alpha_1 Y_E) - \sigma_1 X_E]u - \alpha_1 X_E v,$$
$$\frac{dv}{dt} = -\alpha_2 Y_E u + [(\epsilon_2 - \sigma_2 Y_E - \alpha_2 X_E) - \alpha_2 Y_E]v. \tag{11}$$

Equations (11), along with Theorem 9.2 of Section 9.4, can be used to determine the type and stability of any critical point (X_E, Y_E) of the original system (1). These linear equations are referred to as the linearized equations for small perturbations in the neighborhood of the critical point (X_E, Y_E). The process of deriving them is called linearization.

We will use Eqs. (11) to determine whether the model given by the system (1) can ever lead to coexistence for the two species x and y, and if so, under what conditions on the parameters $\epsilon_1, \sigma_1, \ldots, \alpha_2$. The four possible situations are shown in Figure 9.28; coexistence is possible only in cases (c) and (d). The values of $X_E \neq 0$ and $Y_E \neq 0$ are obtained by solving the system of simultaneous linear algebraic equations (7). We readily obtain

$$X_E = \frac{\epsilon_1\sigma_2 - \epsilon_2\alpha_1}{\sigma_1\sigma_2 - \alpha_1\alpha_2}, \qquad Y_E = \frac{\epsilon_2\sigma_1 - \epsilon_1\alpha_2}{\sigma_1\sigma_2 - \alpha_1\alpha_2}. \tag{12}$$

Moreover, since $\epsilon_1 - \sigma_1 X_E - \alpha_1 Y_E = 0$ and $\epsilon_2 - \sigma_2 Y_E - \alpha_2 X_E = 0$, Eqs. (11) immediately reduce to

$$\frac{du}{dt} = -\sigma_1 X_E u - \alpha_1 X_E v,$$
$$\frac{dv}{dt} = -\alpha_2 Y_E u - \sigma_2 Y_E v. \tag{13}$$

The corresponding characteristic equation is

$$r^2 + (\sigma_1 X_E + \sigma_2 Y_E)r + (\sigma_1\sigma_2 X_E Y_E - \alpha_1\alpha_2 X_E Y_E) = 0. \tag{14}$$

Thus

$$r = \frac{-(\sigma_1 X_E + \sigma_2 Y_E) \pm \sqrt{(\sigma_1 X_E + \sigma_2 Y_E)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)X_E Y_E}}{2}. \tag{15}$$

If $\sigma_1\sigma_2 - \alpha_1\alpha_2 < 0$ then the radicand of Eq. (15) is positive and greater than $(\sigma_1 X_E + \sigma_2 Y_E)^2$. Thus the roots are real with one positive and one negative. The critical point (X_E, Y_E) is an unstable saddle point. Coexistence is not possible. This is the case in the specific example given by Eqs. (2); there $\sigma_1 = 1$, $\alpha_1 = 1$, $\sigma_2 = \frac{1}{4}$, $\alpha_2 = \frac{3}{4}$ and $\sigma_1\sigma_2 - \alpha_1\alpha_2 = -\frac{1}{2}$. On the other hand, if $\sigma_1\sigma_2 - \alpha_1\alpha_2 > 0$, then the radicand of Eq. (15) is less than $(\sigma_1 X_E + \sigma_2 Y_E)^2$. Thus the roots are real, negative, and unequal, or complex with negative real parts. A simple analysis of the radicand of Eq. (15) shows that the roots cannot be complex (see Problem 2). Thus the critical point is an asymptotically stable improper node. Coexistence is possible if $\sigma_1\sigma_2 > \alpha_1\alpha_2$.

Let us relate this result to Figures 9.28c and 9.28d. In Figure 9.28c we have

$$\frac{\epsilon_1}{\sigma_1} > \frac{\epsilon_2}{\alpha_2} \quad \text{or} \quad \epsilon_1\alpha_2 > \epsilon_2\sigma_1 \quad \text{and} \quad \frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1} \quad \text{or} \quad \epsilon_2\alpha_1 > \epsilon_1\sigma_2. \tag{16}$$

These inequalities coupled with the condition that X_E and Y_E given by Eqs. (12) be positive yield the inequality $\sigma_1\sigma_2 < \alpha_1\alpha_2$. Hence in this case the mixed state is an unstable saddle point. Corresponding to Figure 9.28d, we have

$$\frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1} \quad \text{or} \quad \epsilon_2\sigma_1 > \epsilon_1\alpha_2 \quad \text{and} \quad \frac{\epsilon_1}{\alpha_1} > \frac{\epsilon_2}{\sigma_2} \quad \text{or} \quad \epsilon_1\sigma_2 > \epsilon_2\alpha_1. \tag{17}$$

Now, the condition X_E and Y_E positive yields $\sigma_1\sigma_2 > \alpha_1\alpha_2$. Hence this mixed state is asymptotically stable. For this case we can also show that the critical points $(0, 0)$, $(\epsilon_1/\sigma_1, 0)$ and $(0, \epsilon_2/\sigma_2)$ are unstable. Thus no matter what the initial values of $x \neq 0$ and $y \neq 0$ are, the two species approach an equilibrium state of coexistence given by Eqs. (12).

Equations (1) provide the biological interpretation of the result that $\sigma_1\sigma_2 > \alpha_1\alpha_2$ leads to coexistence and $\alpha_1\alpha_2 > \sigma_1\sigma_2$ does not allow coexistence. The σ's are a measure of the inhibitory effect the growth of each species has on its own growth rate, while the α's are a measure of the inhibiting effect the growth of each species has on the other species (interaction). Thus when $\sigma_1\sigma_2 > \alpha_1\alpha_2$ interaction is "small" and the species can coexist; when $\alpha_1\alpha_2 > \sigma_1\sigma_2$ interaction (competition) is "large" and the species cannot coexist—one must die out.

Predator-Prey. As a second example we consider the classical predator-prey problem. We study an ecological situation involving two species, one of which preys on the other (does not compete with it for food but actually preys on it) while the other lives on a different source of food. An example is foxes and rabbits in a closed forest; the foxes prey on the rabbits, the rabbits live on the vegetation in the forest. Other examples are bass in a lake as predators and redear (sunfish) as prey, and lady bugs as predators and aphids (insects that suck the juice of plants) as prey. Let $H(t)$ and $P(t)$ be the populations of prey and predator, respectively, at time t.

We build as simple a model of the interaction as possible. We make the following assumptions:

1. In the absence of the predator the prey grows without bound; thus $dH/dt = aH$, $a > 0$ for $P = 0$.

2. In the absence of the prey the predator dies out, thus $dP/dt = -cP$, $c > 0$, for $H = 0$.

3. The increase in the number of predators is wholly dependent on the food supply (the prey) and the prey are consumed at a rate proportional to the number of encounters between predators and prey. Thus, for example, if the number of prey is doubled the number of encounters is doubled. Encounters decrease the number of prey and increase the number of predators. A fixed proportion of prey is killed in each encounter, and the rate of population growth of the predator is enhanced by a factor proportional to the amount of prey consumed.

As a consequence, we have the equations

$$\begin{aligned} dH/dt &= aH - \alpha HP = H(a - \alpha P), \\ dP/dt &= -cP + \gamma HP = P(-c + \gamma H). \end{aligned} \tag{18}$$

The constants a, c, α, and γ are positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, and α and γ are measures of the effect of the interaction between the two species. Equations (18) are known as the Lotka-Volterra equations. They were developed in papers by Lotka* in 1925 and Volterra† in 1926. Although these equations are simple, they do characterize a wide class of problems. Ways of making them more realistic are discussed at the end of this section and in the problems.

What happens for given initial values of $P > 0$ and $H > 0$? Will the predators eat all of their prey and in turn die out, will the predators die out because of a too low level of prey and then the prey grow without bound, will an equilibrium state be reached, or will a cyclic fluctuation of prey and predator occur?

* A. J. Lotka (1880–1949), an American biophysicist born in Austria, was the author of the first book on mathematical biology published in 1924; it is now available as *Elements of Mathematical Biology*, Dover, New York, 1956.

† Vito Volterra (1860–1940) was a very distinguished Italian mathematician. In mathematics he is particularly famous for his work on integral equations. Indeed, one of the major classes of integral equations has been given his name in honor of his extensive investigations. He also made valuable contributions to mathematical biology. His theory of competing species was motivated by data collected by a friend, D'Ancona, covering fish catches in the Adriatic Sea. A translation of his 1926 paper can be found in an appendix of Chapman, R. N., *Animal Ecology with Special Reference to Insects*, McGraw-Hill, New York, 1931.

The critical points of Eqs. (18) are the solutions of

$$\begin{aligned} H(a - \alpha P) &= 0, \\ P(-c + \gamma H) &= 0. \end{aligned} \tag{19}$$

These solutions are

$$H = 0, P = 0 \quad \text{and} \quad H = c/\gamma, P = a/\alpha. \tag{20}$$

We will examine the predator-prey model (18) in the neighborhood of each critical point. It is very easy to show that the critical point (0, 0) is an unstable saddle point. Entrance to the saddle point is along the line $P = 0$; all other trajectories recede from the critical point.

To study the critical point $(c/\gamma, a/\alpha)$ we let

$$H = \frac{c}{\gamma} + u, \qquad P = \frac{a}{\alpha} + v. \tag{21}$$

Substituting for H and P in Eqs. (18), we obtain

$$\begin{aligned} du/dt &= -\alpha\frac{c}{\gamma}v - \alpha uv, \\ dv/dt &= \gamma\frac{a}{\alpha}u + \gamma uv. \end{aligned} \tag{22}$$

This is an almost linear system, and the corresponding linear system is

$$\begin{aligned} du/dt &= -\alpha\frac{c}{\gamma}v, \\ dv/dt &= \gamma\frac{a}{\alpha}u. \end{aligned} \tag{23}$$

The characteristic equation is

$$r^2 + ac = 0 \quad \text{so} \quad r = \pm i\sqrt{ac}. \tag{24}$$

Since the roots of the characteristic equation are pure imaginary, the critical point is a (stable) center of the linear system. The trajectories of the linear system are closed curves corresponding to solutions that are periodic in time. They neither approach nor recede from the critical point. In particular, the trajectories can readily be shown to be ellipses in the following way. From Eqs. (23), we have

$$\frac{dv}{du} = -\frac{(\gamma a/\alpha)u}{(\alpha c/\gamma)v}, \tag{25}$$

$$\frac{\gamma a}{\alpha}u\,du + \frac{\alpha c}{\gamma}v\,dv = 0,$$

$$\frac{\gamma a}{\alpha}u^2 + \frac{\alpha c}{\gamma}v^2 = C, \tag{26}$$

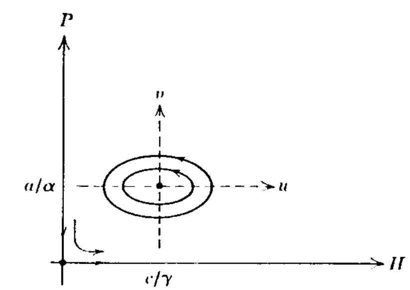


FIGURE 9.29 Trajectories for the linearized predator-prey problem.

where C is an arbitrary nonnegative constant of integration. Several of these trajectories are sketched in Figure 9.29.

While the critical point is a stable center of the linear system, we need to assess its character for the almost linear system. Here, as we know, our theory for almost linear systems fails. The effect of the nonlinear terms may be to change the center into a stable spiral point, or into an unstable spiral point, or it may remain as a stable center. Fortunately, for the predator-prey problem we can actually solve the nonlinear equations (18) and determine what happens. Dividing the second of Eqs. (18) by the first equation, we obtain

$$\frac{dP}{dH} = \frac{P(-c + \gamma H)}{H(a - \alpha P)}. \tag{27}$$

Upon separating the variables in Eq. (27), we have

$$\frac{a - \alpha P}{P}\,dP = \frac{-c + \gamma H}{H}\,dH,$$

from which it follows that

$$a \ln P - \alpha P = -c \ln H + \gamma H + \ln C, \tag{28}$$

where C is a constant of integration. We cannot solve Eq. (28) explicitly for P in terms of H or for H in terms of P, but it can be shown that the graph of this equation for a fixed value of C is a closed curve (not an ellipse, of course) enclosing the critical point $(c/\gamma, a/\alpha)$*. Thus the predator and prey have a cyclic variation about the critical point.

We can analyze this cyclic variation in more detail when the deviation from the point $(c/\gamma, a/\alpha)$ is small; that is, when it is permissible to linearize the perturbation equations (22) for u and v. As we have noted, the trajectories are the

* Volterra gave a clever elementary geometric proof of this result. Shorter (but mathematically more advanced) proofs have also been discovered.

family of ellipses given by Eq. (26). We can also verify, either by solving Eqs. (23) or by direct substitution, that the solution of Eqs. (23) is

$$u(t) = \frac{c}{\gamma} K \cos(\sqrt{ac}\, t + \phi), \qquad v(t) = \frac{a}{\alpha}\sqrt{\frac{c}{a}}\, K \sin(\sqrt{ac}\, t + \phi), \tag{29}$$

where the constants K and ϕ are determined by the initial conditions. Thus

$$H(t) = \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac}\, t + \phi),$$
$$P(t) = \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}}\, K \sin(\sqrt{ac}\, t + \phi). \tag{30}$$

These equations are valid for the elliptical trajectories close to the critical point $(c/\gamma, a/\alpha)$. We can use them to draw several conclusions about the cyclic variation of the predator and prey on such trajectories.

1. The size of the predator and prey populations varies sinusoidally with period $2\pi/\sqrt{ac}$. This period of oscillation is independent of the initial conditions.

2. The predator and prey populations are out of phase by one quarter of a cycle. The prey leads and the predator lags as one might expect. This is discussed in Problem 5.

3. The amplitude of the oscillations are Kc/γ for the prey and $a\sqrt{c}K/\alpha\sqrt{a}$ for the predator and hence depend on the initial conditions as well as the parameters of the problem.

4. The average numbers of predators and prey over one complete cycle are c/γ and a/α, respectively. These are the same as the equilibrium populations (see Problem 6).

Cyclic variations of predator and prey as predicted by Eqs. (18) are often observed in nature. One striking example is described by Odum (pp. 191–192): based on the records of the Hudson Bay Company of Canada, the abundance of lynx and snowshoe hare as indicated by the number of pelts turned in over the period 1845–1935 shows a distinct periodic variation with a period of nine to ten years. The peaks of abundance are followed by very rapid declines, and the peaks of abundance of the lynx and hare are out of phase, with that of the hare preceding that of the lynx by a year or more.

The Volterra-Lotka model of the predator-prey problem has revealed a cyclic variation that was perhaps intuitively expected. On the other hand, the use of the Volterra-Lotka model in other situations can lead to conclusions that are not intuitively obvious. An example revealing a possible danger in using insecticides is given in Problem 8.

One criticism of the Volterra-Lotka predator-prey model is that in the absence of the predator the prey will grow without bound. This can be corrected by allowing for the natural inhibiting effect that an increasing population has on the growth rate of the population; for example, by modifying the first of Eqs. (18) so that when $P = 0$ it reduces to a logistic equation for H (see Problem 9). The models of two competing species and predator-prey discussed here can be modified to allow for the effect of time delays; probabilistic and statistical effects can also be included. Finally, we mention that there are *discrete* analogs of each of the problems we have discussed corresponding to species that breed only at certain times. The mathematics of the discrete problems are often interesting and some of the results are unexpected. These generalizations are discussed in the references given at the end of this chapter as well as in other books on mathematical biology and ecology.

We conclude with a final warning. Using only elementary phase plane theory for one and two nonlinear ordinary differential equations, we have been able to illustrate several of the fundamental principles of simple biological systems. But one should not be misled—ecology is not this simple.

PROBLEMS

1. Consider the linear system of equations

$$du/dt = -u/2 - v/2, \qquad dv/dt = -3u/8 - v/8$$

associated with the critical point $(\frac{1}{2}, \frac{1}{2})$ of Figure 9.27a.

(a) Determine the general solution of this system of equations.

(b) The trajectories that enter the saddle point correspond to the solution in which only the negative exponential is present. Show for this solution that $dv/du = (dv/dt) \div (du/dt) \to (\sqrt{57} - 3)/8$ as $t \to \infty$.

2. Show that

$$(\sigma_1 X_E + \sigma_2 Y_E)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)X_E Y_E = (\sigma_1 X_E - \sigma_2 Y_E)^2 + 4\alpha_1\alpha_2 X_E Y_E.$$

Hence conclude that the roots given by Eq. (15) can never be complex.

3. Two species of fish that compete with each other for food, but do not prey on each other, are bluegill and redear. Suppose that a pond is stocked with bluegill and redear and let x and y be the populations of bluegill and redear, respectively, at time t. Suppose further that the competition is modeled by the equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y),$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

(a) If $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$ show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?

(b) If $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?

4. Consider the competition between bluegill and redear mentioned in Problem 3. Suppose that $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, so, as shown in the text, there is a stable equilibrium point at which both species can coexist. It is convenient to rewrite the equations of Problem 3 in terms of the carrying capacities of the pond for bluegill ($B = \epsilon_1/\sigma_1$) in the absence of redear and for redear ($R = \epsilon_2/\sigma_2$) in the absence of bluegill.

(a) Show that the equations of Problem 3 take the form

$$\frac{dx}{dt} = \epsilon_1 x\left(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y\right),$$

$$\frac{dy}{dt} = \epsilon_2 y\left(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x\right),$$

where $\gamma_1 = \alpha_1/\sigma_1$ and $\gamma_2 = \alpha_2/\sigma_2$. Determine the coexistence equilibrium point (X_E, Y_E) in terms of B, R, γ_1, and γ_2.

(b) Now suppose that a fisherman fishes only for bluegill with the effect that B is reduced. What effect does this have on the equilibrium populations? Is it possible, by fishing, to reduce the population of bluegill to such a level that they will die out?

5. In this problem we examine the phase difference between the cyclic variations of the predator and prey populations as given by Eqs. (30) of the text. Suppose we assume that $K > 0$ and that T is measured from the time that the prey population (H) is a maximum; then $\phi = 0$. Show that the predator population (P) is a maximum at $t = \pi/2\sqrt{ac} = T/4$, where T is the period of the oscillation. When is the prey population increasing most rapidly, decreasing most rapidly, a minimum? Answer the same questions for the predator population. Draw a typical elliptic trajectory enclosing the point $(c/\gamma, a/\alpha)$, and mark these points on it.

6. The average sizes of the prey and predator populations are defined as

$$\bar{H} = \frac{1}{T}\int_A^{A+T} H(t)\,dt, \qquad \bar{P} = \frac{1}{T}\int_A^{A+T} P(t)\,dt,$$

respectively, where T is the period of a full cycle and A is any nonnegative constant. Show for trajectories near the critical point that $\bar{H} = c/\gamma$, $\bar{P} = a/\alpha$.

7. Suppose that the predator-prey equations (18) of the text govern foxes (P) and rabbits (H) in a forest. A trapping company is engaging in trapping foxes and rabbits for their pelts. Explain why it is reasonable for the company to conduct its operation in such a way as to move the population of each species closer to the center $(c/\gamma, a/\alpha)$. When is it best to trap foxes? Rabbits? Rabbits and foxes? Neither?
Hint: See Problem 5. A mathematical argument is not required.

8. Suppose that an insect population (H) is controlled by a natural predator population (P) according to the model (18), so that there are small cyclic variations of the populations about the critical point $(c/\gamma, a/\alpha)$. Show that it is self-defeating to employ an insecticide if the insecticide also kills the predator. Assume that the insecticide kills both prey and predator at rates proportional to each population, respectively. To ban insecticides on the basis of this very simple model would certainly be ill-advised. On the other hand, it is also rash to ignore the possible genuine existence of a phenomenon suggested by a simple model.

9. As was mentioned in the text, one improvement in the predator-prey model is to modify the equation for the prey so that it has the form of a logistic equation in the absence of the predator. Thus in place of Eqs. (18) we consider the model system

$$dH/dt = H(a - \sigma H - \alpha P),$$
$$dP/dt = P(-c + \gamma H),$$

where a, σ, α, c, and γ are positive constants. Determine all critical points and discuss their nature and stability. Assume that $a/\sigma \gg c/\gamma$. What happens for initial data $H \neq 0$, $P \neq 0$?

9.6 LIAPOUNOV'S SECOND METHOD

In Section 9.4 we showed how the stability of a critical point of an almost linear system can usually be determined from a study of the corresponding linear system. However, no conclusion can be drawn when the critical point is a center of the corresponding linear system. Examples of this situation are the undamped pendulum, Eqs. (1) and (2), and the predator-prey problem discussed in Section 9.5. Also, for an asymptotically stable critical point it may be important to investigate the region of asymptotic stability; that is, the domain such that all solutions starting within that domain approach the critical point. Since the theory of almost linear systems is a local theory, it provides no information about this question.

In this section we discuss another approach, known as *Liapounov's second method* or *direct method*. The method is referred to as a direct method because no knowledge of the solution of the system of differential equations is required. Rather, conclusions about the stability or instability of a critical point are obtained by constructing a suitable auxiliary function. The technique is a very powerful one that provides a more global type of information; for example, an estimate of the extent of the region of asymptotic stability of a critical point. In addition, Liapounov's second method can also be used to study systems of equations that are not almost linear; however, we will not discuss such problems.

Basically Liapounov's second method is a generalization of the physical principles that for a conservative system, (i) a rest position is stable if the potential energy is a local minimum, otherwise it is unstable, and (ii) the total energy is a constant during any motion. To illustrate these concepts, again consider the undamped pendulum (a conservative mechanical system), which is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{1}$$

The corresponding system of first order equations is

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{g}{l}\sin x, \tag{2}$$

where $x = \theta$ and $y = d\theta/dt$. Omitting an arbitrary constant, the potential energy U is the work done in lifting the pendulum above its lowest position, namely $mgl(1 - \cos\theta)$ (see Figure 9.22 of Section 9.4). Hence

$$U(x, y) = mgl(1 - \cos x). \tag{3}$$

The critical points of the system (2) are $x = \pm n\pi, y = 0$, $n = 0, 1, 2, 3, \ldots$, corresponding to $\theta = \pm n\pi$, $d\theta/dt = 0$. Physically, we expect that the points

$x = 0, y = 0$; $x = \pm 2\pi$, $y = 0$; ... corresponding to $\theta = 0$, $\pm 2\pi, \ldots$ for which the pendulum bob is vertical with the weight down will be stable; and that the points $x = \pm\pi$, $y = 0$; $x = \pm 3\pi$, $y = 0$; ... corresponding to $\theta = \pm\pi$, $\pm 3\pi, \ldots$ for which the pendulum bob is vertical with the weight up will be unstable. This agrees with statement (i), for at the former points U is a minimum equal to zero, and at the latter points U is a maximum equal to $2mgl$.

Next consider the total energy V, which is the sum of the potential energy U and the kinetic energy $\frac{1}{2}ml^2(d\theta/dt)^2$. In terms of x and y

$$V(x, y) = mgl(1 - \cos x) + \tfrac{1}{2}ml^2y^2. \tag{4}$$

On a trajectory corresponding to a solution $x = \phi(t)$, $y = \psi(t)$ of Eqs. (2), V can be considered as a function of t. The derivative of $V[\phi(t), \psi(t)]$ is called the rate of change of V following the trajectory. By the chain rule

$$\begin{aligned}\frac{dV[\phi(t), \psi(t)]}{dt} &= V_x[\phi(t), \psi(t)]\frac{d\phi(t)}{dt} + V_y[\phi(t), \psi(t)]\frac{d\psi(t)}{dt}\\ &= (mgl\sin x)\frac{dx}{dt} + ml^2y\frac{dy}{dt},\end{aligned} \tag{5}$$

where it is understood that $x = \phi(t)$, $y = \psi(t)$. But dx/dt and dy/dt can be obtained in terms of x and y from Eqs. (2). Substituting in Eqs. (5) for dx/dt and dy/dt we find that $dV/dt = 0$. Hence V is a constant along any trajectory of the system (2), which is in agreement with our earlier remark (ii) that the total energy is constant during any motion of a conservative system.

It is important to note that at any point (x, y) the rate of change of V along the trajectory was computed *without actually solving* the system (2). It is precisely this fact that allows us to use Liapounov's second method for systems whose solutions we do not know, and hence makes it such an important technique.

At the stable critical points, $x = \pm 2n\pi$, $y = 0$, $n = 0, 1, 2, \ldots$, the energy V is zero. If the initial state, say (x_1, y_1), of the pendulum is sufficiently near a stable critical point, then the energy $V(x_1, y_1)$ is small, and the motion (trajectory) associated with this energy stays close to the critical point. It can be shown that if $V(x_1, y_1)$ is sufficiently small, then the trajectory is closed and contains the critical point. For example, suppose that (x_1, y_1) is near $(0, 0)$ and that $V(x_1, y_1)$ is very small. The equation of the trajectory with energy $V(x_1, y_1)$ is

$$V(x, y) = mgl(1 - \cos x) + \tfrac{1}{2}ml^2y^2 = V(x_1, y_1).$$

For x small we have $1 - \cos x = 1 - (1 - x^2/2! + \ldots) \simeq x^2/2$. Thus the equation of the trajectory is approximately

$$\tfrac{1}{2}mglx^2 + \tfrac{1}{2}ml^2y^2 = V(x_1, y_1)$$

or

$$\frac{x^2}{2V(x_1, y_1)/mgl} + \frac{y^2}{2V(x_1, y_1)/ml^2} = 1.$$

This is an ellipse enclosing the critical point $(0, 0)$; the smaller $V(x_1, y_1)$ is, the smaller are the major and minor axes of the ellipse. Physically, the closed trajector corresponds to a solution that is periodic in time—the motion is a small oscillation about the equilibrium point.

If damping is present, however, it is natural to expect that the amplitude of the motion decays in time and that the stable critical point (center) becomes an asymptotically stable critical point (spiral point). See the sketch of the trajectories for the damped pendulum in Figure 9.25 of Section 9.4. This can almost be argued from a consideration of dV/dt. For the damped pendulum, the total energy is still given by Eq. (4), but now from Eqs. (12) of Section 9.4 $dx/dt = y$ and $dy/dt = -(g/l)\sin x - (c/lm)y$. Substituting for dx/dt and dy/dt in Eq. (5) gives $dV/dt = -cly^2 \leq 0$. Thus the energy is nonincreasing along any trajectory and, except for the line $y = 0$, the motion is such that the energy decreases, and hence each trajectory must approach a point of minimum energy—a stable equilibrium point. If $dV/dt < 0$ instead of $dV/dt \leq 0$, it is reasonable to expect that this would be true for all trajectories that start sufficiently close to the origin.

To pursue these ideas further, consider the autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y), \tag{6}$$

and suppose that the point $x = 0$, $y = 0$ is an asymptotically stable critical point. Then there exists some domain D containing $(0, 0)$ such that every trajectory that starts in D must approach the origin as $t \to \infty$. Suppose that there exists an "energy" function V such that $V(x, y) \geq 0$ for (x, y) in D with $V = 0$ only at the origin. Since each trajectory in D approaches the origin as $t \to \infty$, then following any particular trajectory, V decreases to zero as t approaches infinity. The type of result we want to prove is essentially the converse: if, on every trajectory, V decreases to zero as t increases, then the trajectories must approach the origin as $t \to \infty$, and hence the origin is asymptotically stable. First, however, it is necessary to make several definitions.

Let V be defined on some domain D containing the origin. Then V is said to be *positive definite* on D if $V(0, 0) = 0$ and $V(x, y) > 0$ for all other points in D. Similarly, V is said to be *negative definite* on D if $V(0, 0) = 0$ and $V(x, y) < 0$ for all other points in D. If the inequalities $>$ and $<$ are replaced by $\geq$ and $\leq$, then V is said to be *positive semidefinite* and *negative semidefinite*, respectively. We emphasize that in speaking of a positive definite (negative definite, . . .) function on a domain D containing the origin, the function must be zero at the origin in addition to satisfying the proper inequality at all other points in D.

Example 1. The function

$$V(x, y) = \sin(x^2 + y^2)$$

is positive definite on $x^2 + y^2 < \pi/2$ since $V(0, 0) = 0$ and $V(x, y) > 0$ for $0 < x^2 + y^2 < \pi/2$. However, the function

$$V(x, y) = (x + y)^2$$

is only positive semidefinite since $V(x, y) = 0$ on the line $y = -x$.

We will also want to consider the function

$$\dot{V}(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y). \tag{7}$$

We choose this notation because $\dot{V}(x, y)$ can be identified as the rate of change of V along the trajectory of the system (6) that passes through the point (x, y). That is, if $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (6), then

$$\begin{aligned} \frac{dV[\phi(t), \psi(t)]}{dt} &= V_x[\phi(t), \psi(t)]\frac{d\phi(t)}{dt} + V_y[\phi(t), \psi(t)]\frac{d\psi(t)}{dt} \\ &= V_x(x, y)F(x, y) + V_y(x, y)G(x, y) \\ &= \dot{V}(x, y). \end{aligned} \tag{8}$$

The function $\dot{V}$ is sometimes referred to as the derivative of V with respect to the system (6).

We now state two Liapounov theorems, the first dealing with stability, the second with instability.

Theorem 9.3. *Suppose that the autonomous system* (6) *has an isolated critical point at the origin. If there exists a function V that is continuous and has continuous first partial derivatives, is positive definite, and for which the function $\dot{V}$ given by Eq.* (7) *is negative definite on some domain D in the xy plane containing* $(0, 0)$, *then the origin is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite, then the origin is a stable critical point.*

Theorem 9.4. *Let the origin be an isolated critical point of the autonomous system* (6). *Let V be a function that is continuous and has continuous first partial derivatives. Suppose that* $V(0, 0) = 0$ *and that in every neighborhood of the origin there is at least one point at which V is positive (negative). Then if there exists a domain D containing the origin such that the function $\dot{V}$ as given by Eq.* (7) *is positive definite (negative definite) on D, then the origin is an unstable critical point.*

The function V is called a *Liapounov function*. Before sketching geometric proofs of Theorems 9.3 and 9.4, we note that the difficulty in using these theorems is that they tell us nothing about how to construct a Liapounov function, assuming that one exists. In cases where the autonomous system (6) represents a physical problem, it is natural first to consider the actual total energy function of the system as a possible Liapounov function. However, we emphasize that Theorems 9.3 and 9.4 are applicable in cases where the concept of physical energy is not pertinent. In such cases a judicious trial-and-error approach may be necessary.

Now consider the second part of Theorem 9.3; that is, the case $\dot{V}(x, y) \leq 0$. Let $c \geq 0$ be a constant and consider the curve in the xy plane given by the equation $V(x, y) = c$. For $c = 0$ the curve reduces to the single point $x = 0$, $y = 0$. However, for $c > 0$ and sufficiently small, it can be shown by using the continuity of V that we will obtain a closed curve containing the origin as illustrated

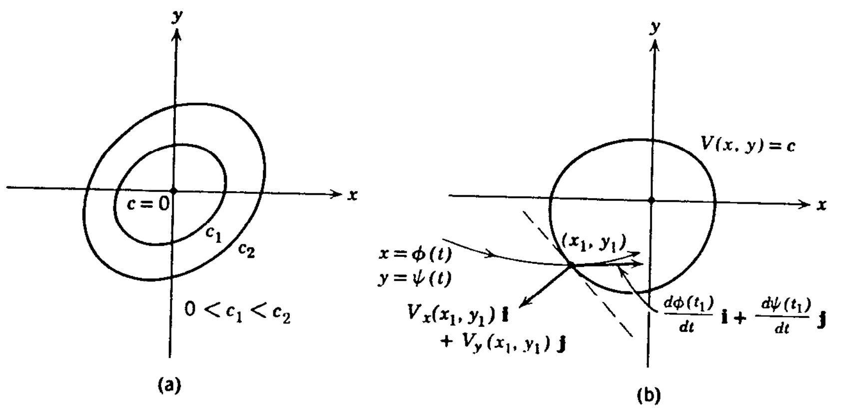


FIGURE 9.30

in Figure 9.30a. There may, of course, be other curves in the xy plane corresponding to the same value of c, but these are not of interest. Further, again by continuity, as c gets smaller and smaller the closed curves $V(x, y) = c$ enclosing the origin shrink to the origin. We will show that a trajectory starting inside of a closed curve $V(x, y) = c$ cannot cross to the outside. Thus, given a circle of radius ϵ about the origin, by taking c sufficiently small we can ensure that every trajectory starting inside of the closed curve $V(x, y) = c$ stays within the circle of radius ϵ; indeed, it will stay within the closed curve $V(x, y) = c$ itself. Thus the origin will be a stable critical point.

It is shown in the calculus that the vector*

$$V_x(x, y)\mathbf{i} + V_y(x, y)\mathbf{j}, \tag{9}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the positive x and y directions, is normal to the curve $V(x, y) = c$. Furthermore, since $V(x, y)$ increases outward from the origin, the vector (9) will point away from the origin, as indicated in Figure 9.30b. Next consider the angle of intersection of a trajectory corresponding to a solution $x = \phi(t)$, $y = \psi(t)$ of the system (6) and a closed curve $V(x, y) = c$. Suppose that the point of intersection is $x_1 = \phi(t_1)$, $y_1 = \psi(t_1)$. From Eq. (7) and the fact that $d\phi(t_1)/dt = F(x_1, y_1)$, $d\psi(t_1)/dt = G(x_1, y_1)$ we have

$$\dot{V}(x_1, y_1) = V_x(x_1, y_1)\frac{d\phi(t_1)}{dt} + V_y(x_1, y_1)\frac{d\psi(t_1)}{dt}$$

$$= [V_x(x_1, y_1)\mathbf{i} + V_y(x_1, y_1)\mathbf{j}] \cdot \left[\frac{d\phi(t_1)}{dt}\mathbf{i} + \frac{d\psi(t_1)}{dt}\mathbf{j}\right],$$

where $[d\phi(t_1)/dt]\mathbf{i} + [d\psi(t_1)/dt]\mathbf{j}$ is the tangent vector to the trajectory $x = \phi(t)$, $y = \psi(t)$ at the point (x_1, y_1) (see Figure 9.30b). The right-hand side of the above equation is the scalar product of two vectors and hence can be interpreted as the

* The vector $V_x(x, y)\mathbf{i} + V_y(x, y)\mathbf{j}$ is known as the gradient of V or grad V.

product of their magnitudes times the cosine of the angle between them. Since, by hypothesis, $\dot{V}(x_1, y_1) \le 0$, it follows that the cosine of the angle is less than or equal to zero so the angle lies between 90° and 270°. Hence the direction of motion on the trajectory must be inward or, at worst, tangent to the curve. Trajectories starting inside a closed curve $V(x, y) = c$ (no matter how small c is) cannot escape, so the origin is a stable point. If $\dot{V}(x, y) < 0$, then the trajectories passing through points on the curve are actually pointed inward. As a consequence, it can be shown that trajectories starting sufficiently close to the origin must approach the origin, and hence the origin is asymptotically stable.

A geometric proof of Theorem 9.4 follows by somewhat similar arguments. Briefly, suppose that $\dot{V}$ is positive definite, and suppose that given any circle about the origin there is an interior point (x_1, y_1) at which $V(x_1, y_1) > 0$. Consider a trajectory that starts at (x_1, y_1). Along this trajectory it follows from Eq. (8) that V must increase, since $\dot{V}(x, y) > 0$; furthermore, since $V(x_1, y_1) > 0$ the trajectory cannot approach the origin because $V(0, 0) = 0$. This shows that the origin cannot be asymptotically stable. By further exploiting the fact that $\dot{V}(x, y) > 0$, it is possible to show that the origin is an unstable point; however, we will not pursue this argument.

To illustrate the use of Theorem 9.3 we consider the question of the stability of the critical point (0, 0) of the undamped pendulum equations (2). While the system (2) is almost linear, the point (0, 0) is a center of the corresponding linear system, so no conclusion can be drawn from Theorem 9.2. Since the mechanical system is conservative, it is natural to suspect that the total energy function V given by Eq. (4) will be a Liapounov function. For example, if we take D to be the domain $-\pi/2 < x < \pi/2$, $-\infty < y < \infty$, then V is positive definite. As we have seen $\dot{V}(x, y) = 0$, so it follows from the second part of Theorem 9.3 that the critical point (0, 0) of Eqs. (2) is a stable critical point. The damped pendulum is discussed in Problem 4.

From a practical point of view one is usually interested more in asymptotic stability. For example, for the automatic control for the wing flap mentioned in Section 9.1, it is not sufficient that the control keep the flap close to a certain setting—rather, it must damp out deviations from the correct setting. Clearly it is desirable to know the allowable deviations from the set position that the automatic control can damp out; that is, to know the region of asymptotic stability. One of the simplest results dealing with this question is given by the following theorem.

Theorem 9.5. *Let the origin be an isolated critical point of the autonomous system (6). Let the function V be continuous and have continuous first partial derivatives. If there is a bounded domain D_K containing the origin where $V(x, y) < K$, V is positive definite, and $\dot{V}$ is negative definite, then every solution of Eqs. (6) that starts at a point in D_K approaches the origin as t approaches infinity.*

In other words, Theorem 9.5 says that if $x = \phi(t)$, $y = \psi(t)$ is the solution of Eqs. (6) for initial data lying in D_K then (x, y) approaches the critical point

$(0, 0)$ as $t \to \infty$. Thus D_K gives a region of asymptotic stability; of course, it may not be the entire region of asymptotic stability. This theorem is proved by showing that (i) there are no periodic solutions of the system (6) in D_K, and (ii) there are no other critical points in D_K. It then follows that trajectories starting in D_K cannot escape and, hence, must tend to the origin as t tends to infinity.

Theorems 9.3 and 9.4 give sufficient conditions for stability and instability, respectively. However, these conditions are not necessary, nor does our failure to determine a suitable Liapounov function mean that there is not one. Unfortunately, there are no general methods for the construction of Liapounov functions; however, there has been extensive work on the construction of Liapounov functions for special classes of equations. One simple result from elementary algebra, which is often useful in constructing positive definite or negative definite functions, is stated without proof in the following theorem.

Theorem 9.6. *The function*

$$V(x, y) = ax^2 + bxy + cy^2 \tag{10}$$

is positive definite if, and only if,

$$a > 0 \quad \textit{and} \quad 4ac - b^2 > 0, \tag{11}$$

and is negative definite if, and only if,

$$a < 0 \quad \textit{and} \quad 4ac - b^2 > 0. \tag{12}$$

The use of Theorem 9.6 is illustrated in the following example.

Example 2. Show that the critical point $(0, 0)$ of the autonomous system

$$\frac{dx}{dt} = -x - xy^2, \qquad \frac{dy}{dt} = -y - yx^2 \tag{13}$$

is asymptotically stable.

We try to construct a Liapounov function of the form (10). Then $V_x(x, y) = 2ax + by$, $V_y(x, y) = bx + 2cy$ so

$$\dot{V}(x, y) = (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - yx^2)$$
$$= -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + yx^3) + 2c(y^2 + x^2y^2)].$$

If we choose $b = 0$, and a and c to be any positive numbers, then $\dot{V}$ is negative definite and V is positive definite by Theorem 9.6. Thus by Theorem 9.3 the origin is an asymptotically stable critical point.

PROBLEMS

1. By constructing suitable Liapounov functions of the form $ax^2 + cy^2$, where a and c are to be determined, show that for each of the following systems the critical point at the origin is of the indicated type.

(a) $dx/dt = -x^3 + xy^2$, $\quad dy/dt = -2x^2y - y^3$; $\quad$ asymptotically stable

(b) $dx/dt = -\frac{1}{2}x^3 + 2xy^2$, $\quad dy/dt = -y^3$; $\quad$ asymptotically stable

(c) $dx/dt = -x^3 + 2y^3$, $\quad dy/dt = -2xy^2$; $\quad$ stable (at least)

(d) $dx/dt = x^3 - y^3$, $\quad dy/dt = 2xy^2 + 4x^2y + 2y^3$; $\quad$ unstable

2. Consider the system of equations

$$\frac{dx}{dt} = y - xf(x, y), \qquad \frac{dy}{dt} = -x - yf(x, y),$$

where f is continuous and has continuous first partial derivatives. Show that if $f(x, y) > 0$ in some neighborhood of the origin, then the origin is an asymptotically stable critical point, and if $f(x, y) < 0$ in some neighborhood of the origin, then the origin is an unstable critical point.
Hint: Construct a Liapounov function of the form $c(x^2 + y^2)$.

3. A generalization of the undamped pendulum equation is

$$\frac{d^2u}{dt^2} + g(u) = 0, \tag{i}$$

where $g(0) = 0$, $g(u) > 0$ for $0 < u < k$, and $g(u) < 0$ for $-k < u < 0$; that is, $ug(u) > 0$ for $u \neq 0$, $-k < u < k$. Notice that $g(u) = \sin u$ has this property on $(-\pi/2, \pi/2)$.

(a) Letting $x = u$, $y = du/dt$, write Eq. (i) as a system of two equations, and show that $x = 0$, $y = 0$ is a critical point.

(b) Show that

$$V(x, y) = \tfrac{1}{2}y^2 + \int_0^x g(s)\, ds, \qquad -k < x < k, \tag{ii}$$

is positive definite, and use this result to show that the critical point $(0, 0)$ is stable. Note that the Liapounov function V given by Eq. (ii) corresponds to the energy function $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$ for the case $g(u) = \sin u$.

4. By introducing suitable dimensionless variables, the system of nonlinear equations for the damped pendulum [Eqs. (12) of Section 9.4] can be written as

$$dx/dt = y, \qquad dy/dt = -y - \sin x.$$

(a) Show that the origin is a critical point.

*(b) Show that while $V(x, y) = x^2 + y^2$ is positive definite, $\dot{V}(x, y)$ takes on both positive and negative values in any domain containing the origin, so that V is not a Liapounov function.
Hint: $x - \sin x > 0$ for $x > 0$ and $x - \sin x < 0$ for $x < 0$. Consider these cases with y positive but y so small that y^2 can be ignored compared to y.

(c) Using the energy function $V(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$ obtained from Eq. (ii) of Problem 3, show that the origin is a stable critical point. Note, however, that even though there is damping and we can expect that the origin is asymptotically stable, it is not possible to draw this conclusion using the present Liapounov function.

*(d) To show asymptotic stability it is necessary to construct a better Liapounov function than the one used in part (c). Show that $V(x, y) = \frac{1}{2}(x + y)^2 + x^2 + \frac{1}{2}y^2$ is such a Liapounov function, and conclude that the origin is an asymptotically stable critical point.
Hint: From Taylor's formula with a remainder it follows that $\sin x = x - \alpha x^3/3!$, wher

α depends on x but $0 < \alpha < 1$ for $-\pi/2 < x < \pi/2$. Then letting $x = r\cos\theta$, $y = r\sin\theta$ show that $\dot V(r\cos\theta, r\sin\theta) = -r^2[1 + h(r,\theta)]$ where $|h(r,\theta)| < 1$ if r is sufficiently small.

In Problems 5 and 6 we will prove part of Theorem 9.2: if the critical point (0, 0) of the almost linear system

$$\frac{dx}{dt} = ax + by + F_1(x,y), \qquad \frac{dy}{dt} = cx + dy + G_1(x,y) \tag{i}$$

is an asymptotically stable critical point of the corresponding linear system

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy, \tag{ii}$$

then it is an asymptotically stable critical point of the almost linear system (i).

5. Consider the linear system (ii).

(a) Since (0, 0) is an asymptotically stable critical point, show that $a + d < 0$ and $ad - bc > 0$. (See Problem 17 of Section 9.3.)

(b) Construct a Liapounov function $V(x,y) = Ax^2 + Bxy + Cy^2$, such that V is positive definite and $\dot V$ is negative definite. One way to insure that $\dot V$ is negative definite is to choose A, B, and C so that $\dot V(x,y) = -x^2 - y^2$. Show that this leads to the result

$$A = -\frac{c^2 + d^2 + (ad - bc)}{2\Delta}, \quad B = \frac{bd + ac}{\Delta}, \quad C = -\frac{a^2 + b^2 + (ad - bc)}{2\Delta},$$

where $\Delta = (a + d)(ad - bc)$.

(c) Using the result of part (a) show that $A > 0$, and then show (several steps of algebra are required) that

$$4AC - B^2 = \frac{(a^2 + b^2 + c^2 + d^2)(ad - bc) + 2(ad - bc)^2}{\Delta^2} > 0.$$

Thus by Theorem 9.6, V is positive definite.

6. In this problem we show that the Liapounov function constructed in the previous problem is also a Liapounov function for the almost linear system (i). We must show that there is some region containing the origin for which $\dot V$ is negative definite.

(a) Show that

$$\dot V(x,y) = -(x^2 + y^2) + (2Ax + By)F_1(x,y) + (Bx + 2Cy)G_1(x,y).$$

(b) Recall that $F_1(x,y)/r \to 0$ and $G_1(x,y)/r \to 0$ as $r = (x^2 + y^2)^{1/2} \to 0$. This means that given any $\epsilon > 0$ there exists a circle $r = R$ about the origin such that for $0 \le r < R$, $|F_1(x,y)| < \epsilon r$, and $|G_1(x,y)| < \epsilon r$. Letting M be the maximum of $|2A|$, $|B|$, and $|2C|$, show by introducing polar coordinates that R can be chosen so that $\dot V(x,y) < 0$ for $r < R$. *Hint:* Choose ϵ sufficiently small in terms of M.

7. In this problem we prove a part of Theorem 9.2 relating to instability.

(a) Show that if $p = (a + d) > 0$ and $q = ad - bc > 0$ then the critical point (0, 0) of the linear system (ii) is unstable.

(b) The same result holds for the almost linear system (i). Following Problems 5 and 6 construct a positive definite function V such that $\dot V(x,y) = x^2 + y^2$ and hence is positive definite, and then invoke Theorem 9.4.

*9.7 PERIODIC SOLUTIONS AND LIMIT CYCLES

On several occasions we have mentioned the possible existence of periodic solutions of nonlinear autonomous systems. Such solutions, whose trajectories form *closed curves* in the phase plane, play an important role in many physical phenomena. A periodic solution often represents a "final state" toward which all "neighboring" solutions tend as the transients due to the initial conditions die out. In some cases, for example, in an electronic oscillator the flow of current is periodic even though resistance is present. The existence of these periodic solutions can only be explained by a consideration of the nonlinear terms in the governing equations—no linearized mathematical model with a resistance term can predict a periodic solution.

A special case of a periodic solution is a constant solution $x = x_0$, $y = y_0$, corresponding to a critical point of the autonomous system. Such a solution is clearly periodic with any period. In this section when we speak of a periodic solution we mean a nonconstant periodic solution.

Recall that the solutions of the linear autonomous system

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy \tag{1}$$

are periodic if and only if the roots of the characteristic equation

$$r^2 - (a + d)r + (ad - bc) = 0 \tag{2}$$

are pure imaginary. In this case the critical point at the origin will necessarily be a stable critical point, as discussed in Section 9.3. We emphasize that if the roots of Eq. (2) are pure imaginary, then *every* solution of the linear system (1) is periodic, and if the roots are not pure imaginary then Eqs. (1) have *no* periodic solutions.

The situation can be quite different for a nonlinear autonomous system. A standard example is the system

$$\frac{dx}{dt} = y + x - x(x^2 + y^2), \qquad \frac{dy}{dt} = -x + y - y(x^2 + y^2). \tag{3}$$

It is not difficult to show that $x = 0$, $y = 0$ is the only critical point of the system (3), and also that the system is almost linear in the neighborhood of the origin. Further, from a consideration of the corresponding linear system [$a = 1$, $b = 1$, $c = -1$, $d = 1$ in Eq. (1)], it follows that the origin is an unstable spiral point for the system (3). Thus any solution that starts near the origin in the phase plane will spiral away from the origin. Since there are no other critical points, our first thought might be that all solutions of Eqs. (3) have trajectories which spiral out to infinity. However, we will now show that far away from the origin the trajectories are directed inward, so that something else must be happening.

It is convenient to introduce the polar coordinates r and θ where

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{4}$$

Multiplying the first of Eqs. (3) by x, the second by y, and adding gives

$$x\frac{dx}{dt} + y\frac{dy}{dt} = (x^2 + y^2) - (x^2 + y^2)^2. \tag{5}$$

Since $r^2 = x^2 + y^2$ and $r\,dr/dt = x\,dx/dt + y\,dy/dt$, we obtain

$$r\frac{dr}{dt} = r^2(1 - r^2). \tag{6}$$

Thus if $r > 1$, then $dr/dt < 0$, and the direction of motion on a trajectory is inward. Similarly, if $r < 1$, then the direction of motion is outward. Clearly, the circle $r = 1$ in the phase plane has a special significance for this system.

To obtain an equation for θ, we multiply the first of Eqs. (3) by y, the second by x, and subtract, obtaining

$$y\frac{dx}{dt} - x\frac{dy}{dt} = x^2 + y^2. \tag{7}$$

Upon computing dx/dt and dy/dt from Eqs. (4), we find that the left-hand side of Eq. (7) is $-r^2\,d\theta/dt$, so Eq. (7) reduces to

$$\frac{d\theta}{dt} = -1. \tag{8}$$

The system of equations (6) and (8) for r and θ is equivalent to the original system (3). One solution of the system (6) and (8) is

$$r = 1, \qquad \theta = -t + t_0, \tag{9}$$

where t_0 is an arbitrary constant. As t increases, a point satisfying Eqs. (9) moves clockwise around the unit circle. Thus the autonomous system (3) has a periodic solution. Other solutions can be obtained by solving Eq. (6) by separation of variables; if $r \neq 0$ and $r \neq 1$, then

$$\frac{dr}{r(1 - r^2)} = dt. \tag{10}$$

Equation (10) can be integrated by using partial fractions to rewrite the left-hand side. Omitting these algebraic calculations, we find that the solutions of Eqs. (8) and (10) are

$$r = \frac{1}{\sqrt{1 + c_0 e^{-2t}}}, \qquad \theta = -t + t_0, \tag{11}$$

where c_0 and t_0 are arbitrary constants. In this case the solutions (11) also contain the solution (9), which is obtained by setting $c_0 = 0$ in Eq. (11).

The solution satisfying the initial conditions $r = \rho$, $\theta = \alpha$ at $t = 0$ is given by

$$r = \frac{1}{\sqrt{1 + [(1/\rho^2) - 1]e^{-2t}}}, \qquad \theta = -(t - \alpha). \tag{12}$$

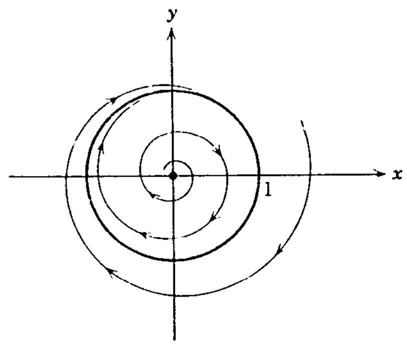


FIGURE 9.31

If $0 < \rho < 1$, $r \to 1$ from the inside as $t \to \infty$; if $\rho > 1$, $r \to 1$ from the outsi as $t \to \infty$. Thus the trajectories spiral toward the circle $r = 1$ as $t \to \infty$. schematic sketch of the trajectories is shown in Figure 9.31.

Not only does the circle $r = 1$ correspond to a periodic solution of t autonomous system (3), but also nonclosed trajectories spiral toward it as $t \to$ c A closed curve in the phase plane which has nonclosed curves spiraling towa it, either from the inside or outside, as $t \to \infty$ is called a *limit cycle*.* Thus t circle $r = 1$ is a limit cycle of the autonomous system (3). If all trajectori that start near a closed trajectory (both inside and outside) spiral toward the clos trajectory as $t \to \infty$, we say that the limit cycle is *stable*. This type of stabili is sometimes referred to as *orbital stability*. If the trajectories on one side spi toward the closed trajectory, while on the other side they spiral away $t \to \infty$, we say that the limit cycle is *semistable*. If the trajectories on both sid of the closed trajectory spiral away as $t \to \infty$, we say that the closed trajectory *unstable*. It is also possible to have closed trajectories that are neither approach nor receded from; for example, the family of periodic solutions of a linear auto omous system. In this case we say that the closed trajectory is *neutrally stable*.

For the example just considered the existence of a stable limit cycle w established by direct calculation; in most cases this is not possible. Howev there are several general theorems concerning the existence or nonexistence limit cycles of a nonlinear autonomous system. While proofs of these theore are too advanced for a book of this level, we can state these theorems and illustr how they may be used. We consider the autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y). \qquad (1$$

Theorem 9.7. *Let the functions F and G have continuous first partial derivati in a domain D in the xy plane. A closed trajectory of the system (13) must necessar enclose at least one critical point. If it encloses only one critical point, the criti point cannot be a saddle point.*

* Slightly different definitions are also used; the present one is sufficient for our purposes.

This theorem is illustrated by Figure 9.31. The closed trajectory $r = 1$ encloses the critical point (0, 0). Also note that each closed trajectory for the linearized pendulum problem (Figure 9.10 of Section 9.2) encloses the critical point (0, 0). One way in which this theorem is extremely useful is a negative one. If a given region contains no critical points, then there can be no closed trajectory (periodic solution) lying entirely in the region. Also, for the competing species problem whose trajectories are sketched in Figure 9.27b of Section 9.5, the theorem tells us that there is no closed trajectory in the first quadrant since the single critical point $(\frac{1}{2}, \frac{1}{2})$ in this quadrant is a saddle point.

A second result about the nonexistence of periodic solutions (closed trajectories) is given by the following theorem.*

Theorem 9.8. *Let the functions F and G have continuous first partial derivatives in a simply connected domain D in the xy plane. If $F_x + G_y$ has the same sign throughout D, then there is no periodic solution of Eqs.* (13) *lying entirely in D.*

Simply connected domains are discussed on page 41; briefly, such a domain is one with no holes. Note that if $F_x + G_y$ changes sign in D, no conclusion is possible; there may or may not be periodic solutions in D. For the example problem (3), a simple calculation shows that

$$F_x(x, y) + G_y(x, y) = 2 - 4(x^2 + y^2). \tag{14}$$

Thus we can conclude that there are no periodic solutions of Eqs. (3) in the domain $0 \le r < 1/\sqrt{2}$. In fact, however, we know that there are none in the larger domain $0 \le r < 1$.

The following theorem gives conditions that ensure the existence of a periodic solution.

Theorem 9.9 (*Poincaré-Bendixson† Theorem*). *Let the functions F and G have continuous first partial derivatives in a domain D in the xy plane. Let D_1 be a bounded subdomain in D, and let R be the region that consists of D_1 plus its boundary (all points in R are in D). Suppose that R contains no critical point of the system* (13). *If there exists a constant t_0 such that $x = \phi(t)$, $y = \psi(t)$ is a solution of the system* (13) *that exists and stays in R for all $t \ge t_0$, then either $x = \phi(t)$, $y = \psi(t)$ is a periodic solution (closed trajectory), or $x = \phi(t)$, $y = \psi(t)$ spirals toward a closed trajectory as $t \to \infty$. In either case, the system* (13) *has a periodic solution in R.*

Note that if R does contain a closed trajectory, then necessarily, by Theorem 9.7, this trajectory must enclose a critical point. However, this critical point cannot be in R. Thus R cannot be simply connected; it must have a hole. As an application of the Poincaré-Bendixson theorem, consider the system (3) again. Since the origin is a critical point, it must be excluded. For example, we might consider the region R defined by $\frac{1}{2} \le r \le 2$. Next we must show that there is a solution whose trajectory stays in R for all t greater than or equal to some t_0. But this follows immediately from Eq. (6). For $r = \frac{1}{2}$, $dr/dt > 0$, so r increases and for $r = 2$, $dr/dt < 0$, so r decreases. Thus any solution of Eqs. (3) starting (the t_0 of Theorem 9.9 can be taken as the initial time) in the region $\frac{1}{2} \le r \le 2$ must stay in that region. Hence, by Theorem 9.9, there is a periodic solution of the system (3) whose trajectory is a closed curve in the region $\frac{1}{2} \le r \le 2$. For this example, we found from Eq. (6) that there are periodic solutions whose trajectories lie on the circle $r = 1$; however, in general we will not be able to solve the autonomous system. The difficulty in using the Poincaré-Bendixson theorem is the determination of a region R not containing critical points and for which a trajectory stays inside.

* For students who have had Green's theorem in the plane, which relates line integrals over closed curves to double integrals over the area inside the curve, the proof of Theorem 9.8 is not difficult (see Problem 7).

† Ivar Otto Bendixson (1861–1935) was a Swedish mathematician. This theorem appeared in a paper published by him in *Acta Mathematica* in 1901.

In Problem 18 of Section 9.4 we introduced the generalized Liénard (1869–1958) equation

$$\frac{d^2u}{dt^2} + f(u)\frac{du}{dt} + g(u) = 0. \tag{15}$$

This equation can be thought of as describing a spring-mass system with a damping term $f(u)\,du/dt$, which is nonlinear unless f is a constant function, and with a spring force $g(u)$, which is nonlinear unless $g(u)$ is proportional to u. A particularly important special case of Eq. (15) is the van der Pol (1889–1959) equation, which governs the flow of current u in a triode oscillator,

$$\frac{d^2u}{dt^2} - \mu(1 - u^2)\frac{du}{dt} + u = 0, \tag{16}$$

where μ is a positive constant. Experimentally it is observed that the flow of current is periodic. This periodic flow can only be explained by a consideration of the nonlinear equation. If we consider the linearized form of Eq. (16) obtained by neglecting the term $u^2\,du/dt$ we have

$$\frac{d^2u}{dt^2} - \mu\frac{du}{dt} + u = 0. \tag{17}$$

Because of the term $-\mu\,du/dt$, Eq. (17) has no periodic solutions. Indeed, since $\mu > 0$, all solutions of Eq. (17) *grow* with increasing time. However, for the nonlinear equation (16) if u becomes greater than one, then the coefficient $-\mu(1 - u^2)$ of du/dt becomes *positive*, and this term damps the current. Thus in the nonlinear equation (16) the "resistance" term $-\mu(1 - u^2)\,du/dt$ amplifies the current if $u < 1$ and damps it if $u > 1$. This alternate action of the resistance term explains how the van der Pol equation can have a periodic solution.

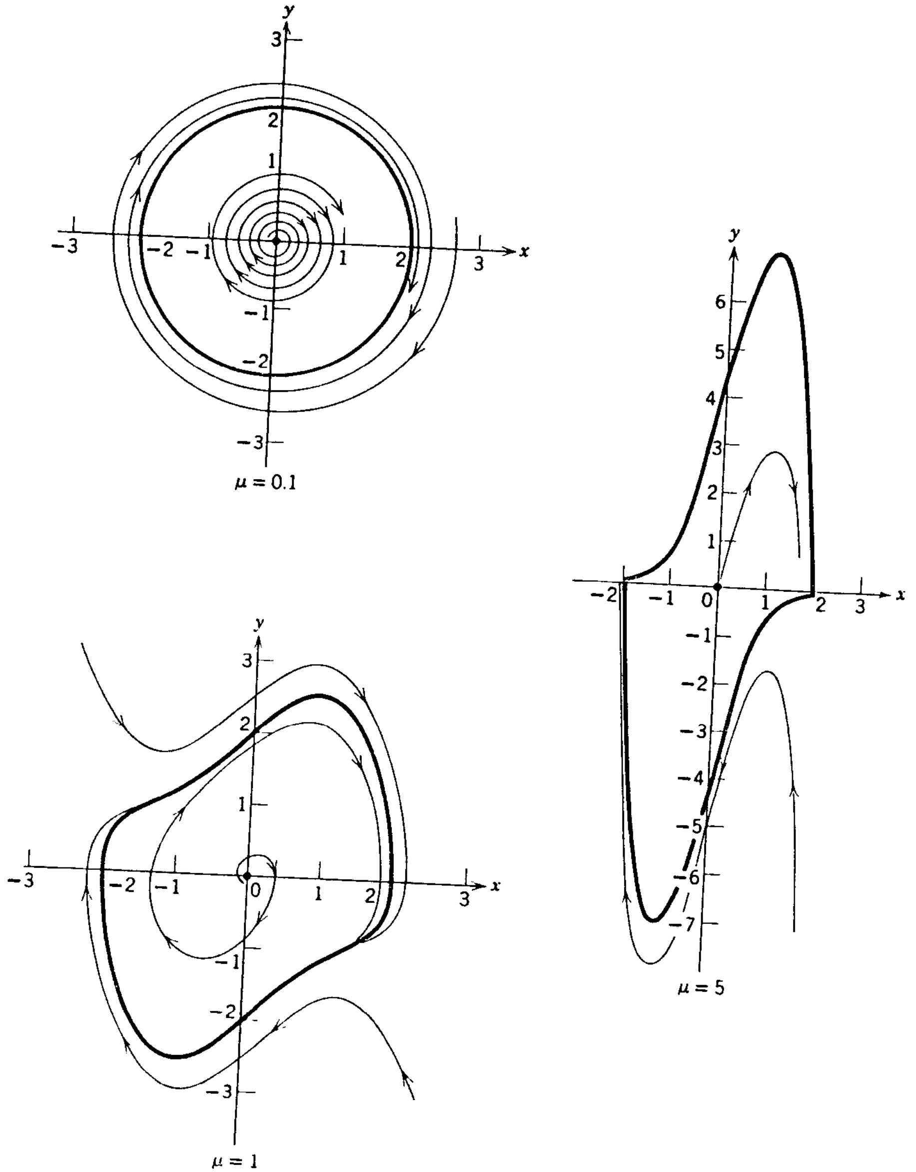


FIGURE 9.32 Limit cycles of the van der Pol equation.

Liénard established the existence of periodic solutions of Eq. (15) for certa types of functions f and g, which included the case of the van der Pol equatic A more general result, proved by Levinson* and Smith in 1942, is the followir

Theorem 9.10. *Let f be even $[f(-u) = f(u)]$ and continuous for all u. Let g odd $[g(-u) = -g(u)]$, with $g(u) > 0$ for all $u > 0$, and have a continuous fi derivative for all u. Let*

$$F(u) = \int_0^u f(s)\, ds, \qquad G(u) = \int_0^u g(s)\, ds. \qquad (1$$

If

(a) *$G(u) \to \infty$ as $u \to \infty$;*

(b) *there exists a positive number u_0 such that $F(u) < 0$ for $0 < u <$ $F(u) > 0$ for $u > u_0$, and $F(u)$ is monotonically increasing for $u > u_0$ with $F(u) \to$ as $u \to \infty$;*

then

(a) *the generalized Liénard equation (15) has a unique periodic solution (uniq in the sense that there are no other closed trajectories);*

(b) *the corresponding trajectory is a closed curve encircling the origin in t phase plane $x = u$, $y = du/dt$;*

(c) *all other trajectories in the xy phase plane, except that corresponding the critical point $(0, 0)$, spiral toward the closed trajectory as $t \to \infty$.*

It is a straightforward calculation to show that the conditions of Theorem 9.1 are satisfied for the van der Pol equation (16). For example, the function satisfies hypothesis (b):

$$F(u) = -\mu\int_0^u (1 - s^2)\, ds = -\mu\left(u - \frac{u^3}{3}\right)$$

is less than zero for $0 < u < \sqrt{3}$, is greater than zero for $u > \sqrt{3}$, and increas monotonically to infinity as u increases to infinity. Thus the van der Pol equatio *has* a periodic solution.

In Figure 9.32 a few of the trajectories in the phase plane $x = u$, $y = du/$ are sketched for the van der Pol equation for the cases $\mu = 0.1$, 1, and 5, respe tively. The limit cycle is shown by a heavy line. For μ small the limit cycle nearly a circle of radius 2 (the trajectories of the linear equation obtained by settin $\mu = 0$ are circles), but for μ large the limit cycle is considerably different.

* Norman Levinson (1912–1975) was a prominent American mathematician who received th Bocher prize from the American Mathematical Society in 1954 for his contributions to the theor of linear, nonlinear, ordinary, and partial differential equations. He received the Chauvenet Pri from The Mathematical Association of America in 1971 for his expository paper, "A Motivate Account for an Elementary Proof of the Prime Number Theorem," *The American Mathematic Monthly*, Vol. 76, 1969, pp. 225–245.

PROBLEMS

1. For each of the following autonomous systems, expressed in polar coordinates, determine all periodic solutions, all limit cycles, and discuss their stability.

(a) $dr/dt = r^2(1 - r^2), \quad d\theta/dt = 1$
(b) $dr/dt = r(1 - r)^2, \quad d\theta/dt = -1$
(c) $dr/dt = r(r - 1)(r - 3), \quad d\theta/dt = 1$
(d) $dr/dt = r(1 - r)(r - 2), \quad d\theta/dt = -1$
(e) $dr/dt = \sin \pi r, \quad d\theta/dt = 1$
(f) $dr/dt = r|r - 2|(r - 3), \quad d\theta/dt = -1$

2. Show that if $x = r\cos\theta$, $y = r\sin\theta$, then $y\,dx/dt - x\,dy/dt = -r^2\,d\theta/dt$.

3. (a) Show that the system

$$dx/dt = -y + xf(r)/r, \qquad dy/dt = x + yf(r)/r$$

has periodic solutions corresponding to the zeros of $f(r)$. What is the direction of motion on these closed curves?

(b) Determine all periodic solutions of the above system and discuss their stability if $f(r) = r(r - 2)^2(r^2 - 4r + 3)$.

4. Determine the periodic solutions, if any, of the system

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \quad \frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2).$$

5. Using Theorem 9.8 show that a sufficient condition for the linear autonomous system

$$dx/dt = ax + by, \qquad dy/dt = cx + dy$$

not to have a periodic solution, other than $x = 0$, $y = 0$, is $a + d \neq 0$.

6. Show that each of the following systems has no periodic solutions other than constant solutions.

(a) $dx/dt = x + y + x^3 - y^2, \quad dy/dt = -x + 2y + x^2y + \frac{1}{3}y^3$
(b) $dx/dt = -2x - 3y - xy^2, \quad dy/dt = y + x^3 - x^2y$

7. Prove Theorem 9.8 by completing the following argument. According to Green's theorem in the plane, if C is a closed curve which is sufficiently "smooth" and if F and G are continuous and have continuous first partial derivatives, then

$$\oint_C [F(x, y)\,dy - G(x, y)\,dx] = \iint_R [F_x(x, y) + G_y(x, y)]\,dA,$$

where R is the region enclosed by C. Assume that $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (13) which is periodic with period T. Let C be the closed curve given by $x = \phi(t)$, $y = \psi(t)$ for $0 \le t \le T$. Show that for this curve the line integral is zero.

8. Show that each of the following differential equations or systems of equations h a periodic solution. A prime is used to denote d/dt.

(a) $u'' + (u^2 - 1)u' + u^3 = 0$ (b) $u'' + (3u^4 - 4u^2)u' + u = 0$
(c) $u'' - (2 - u^2)u' + u^3 + \sin u = 0$
(d) $u'' + \alpha(u^{2m} - k)u' + \beta u^{2n-1} = 0$
(α, β, and k positive constants; n, m positive integers)
(e) $x' = y, \quad y' = -x + y - x^5 - 2x^2y$

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