1 Introduzione

"[...] soggiogate la Terra
e dominate sui pesci del mare
e sugli uccelli del cielo
e su ogni essere vivente,
che striscia sulla terra"
Gen 1,8 [1]

Il presente elaborato tratta il modello preda-predatore, noto anche come modello Lotka-Volterra, dal nome dei due matematici che per primi e in modo indipendente hanno studiato tali equazioni[2].
Lo stesso Volterra, nell'introduzione di [1], racconta di come, "a seguito di alcune conversazioni con il Sign. D'Ancona[3], che chiedeva se fosse possibile studiare le variazioni della composizione di un ecosistema per via matematica, egli abbia cominciato le sue ricerche su tale soggetto"[4]. Più precisamente D'Ancona aveva raccolto dai registri dei porti di Venezia, Trieste e Fiume dei dati statistici relativi alla pesca nel periodo 1905-1923, nel tentativo di capire se una pesca eccessiva potesse essere causa di estinzione del pesce. Egli rilevò che durante la guerra, periodo in cui la pesca era stata meno intensa, si era verificato un notevole aumento della percentuale di pesci appartenenti alla classe dei selaci; questi si nutrivano di altri pesci, e il loro valore commerciale era basso. Concluse quindi che la riduzione delle attività dei pescherecci al fine della ripopolazione del pesce sarebbe risultata dannosa; non gli era chiaro però se tali conclusioni fossero corrette e ne chiese a Volterra una giustificazione dal punto di vista matematico. Volterra rispose a D'Ancona nel modo che segue

2 Presentazione del modello

Consideriamo un ecosistema composto da H prede[5] e P predatori[6], e studiamo come queste due quantità variano nel tempo. Supponiamo per semplic-

[1] nonostante non sia uso comune, è doveroso chiarire il significato di tale citazione: l'autore di questi versi, vissuto attorno al 500 aC, esprime l'impressione che l'uomo, fra tutte le creature della Terra, goda di una posizione di dominio; di questa idea tiene conto il modello che presenteremo, dove l'uomo occupa un ruolo di predatore del predatore; del resto, l'idea stessa di fare un modello, di studiare una situazione in astratto, è un tentativo umano di dominio sulla realtà

[2] Alfred J. Lotka (1880-1949), matematico e chimico-fisico statunitense di origini ucraine, Vito Volterra (1860-1940), personaggio di spicco nella matematica italiana del primo dopoguerra; si veda anche [6]

[3] Umberto D'Ancona (1896-1964), biologo marino e futuro genero di Volterra

[4] trad. liberamente dal francese e adattato

[5] dall'inglese *host*, ospite

[6] ing: *parasite*, parassita; in effetti un rapporto di predazione fra due specie è equivalente, dal punto di vista dello studio matematico, a un rapporto di parassitismo

ità che l'ambiente studiato non contenga altre specie, che le prede godano di risorse illimitate, e che i predatori si nutrano esclusivamente delle prede. Sotto queste ipotesi in assenza di predatori le prede crescono in modo esponenziale, e in assenza di prede i predatori decrescono esponenzialmente, in quanto le prede sono la loro unica risorsa; scriviamo quindi

$$\begin{cases} \dot{H} = aH \\ \dot{P} = -bP \end{cases} \tag{1}$$

con a e b coefficienti reali positivi, detti coefficienti biotici. Considerando poi l'interazione fra preda e predatore, l'influenza reciproca fra le due specie è direttamente proporzionale al numero di incontri fra prede e predatori, quindi al prodotto HP.
Il modello che studieremo è quindi il seguente:

$$\begin{cases} \dot{H} = aH - \alpha HP \\ \dot{P} = -bP + \beta HP \end{cases} \tag{2}$$

con α e β sono costanti reali positive, delle quali α rappresenta la probabilità di un incontro fra una preda e un predatore che risulti vincente per il predatore, e $\beta = \gamma\alpha$, dove γ è il fattore di conversione preda-predatore[7].
Tale equazione è detta **modello di Lotka-Volterra**.

3 Le traiettorie nel piano della fasi

Lo studio delle varazioni nel tempo di queste due popolazioni ci ha condotto quindi a equazioni differenziali[8] , più precisamente a un sistema di due equazioni differenziali in due incognite, del quale tracceremo le orbite nel piano delle fasi.
Il sistema ha due punti di equilibrio: $E_1 = (0,0)$ e $E_2 = (\frac{b}{\beta}, \frac{a}{\alpha}) = (H^*, P^*)$.
Osserviamo immdiatamente che, come già accennato,

$$\begin{cases} H(t) \equiv 0 \\ P(t) = P_0\, e^{-bt} \end{cases} \quad \text{e} \quad \begin{cases} H(t) = H_0\, e^{at} \\ P(t) \equiv 0 \end{cases}$$

sono soluzioni di (2); ciò ci dice immediatamente che il punto di equilibrio (0,0) è una sella, quindi instabile. Inoltre risulta che le soluzioni che

[7] per capire la necessità di utilizzare un fattore di conversione che misuri come un predatore "trasforma la preda in sé", basta pensare alla differenza fra una balena che mangia krill e un lupo che divora un cinghiale, anche solo in termini di apporto calorico dato dalla singola preda

[8] "[...] cerchiamo di esprimere con parole come procede all'ingrosso il fenomeno; quindi traduciamo le parole in linguaggio matematico. Questa traduzione conduce ad equazioni differenziali."
V. Volterra, "*Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*"

partono da due condizioni iniziali $H_0, P_0 > 0$ rimangono di segno positivo[9]; questo garantisce che il modello non ci porta all'assurdo di una popolazione negativa.

3.1 Un integrale primo del moto

Notiamo per prima cosa che lo slôpe $P'(H) = \frac{\dot{P}}{\dot{H}} = \frac{P(-b+\beta H)}{H(a-\alpha P)}$ è un'espressione a variabili separabili, che si può integrare nel seguente modo:

$$\dot{P}\frac{a - \alpha P}{P} = \dot{H}\frac{-b + \beta H}{H}$$

integrando rispetto al tempo ottengo

$$c + \int \dot{P}\frac{a - \alpha P}{P}\,dt = \int \dot{H}\frac{-b + \beta H}{H}\,dt$$

con c costante reale,

$$c + \int \frac{a - \alpha P}{P}\,dP = \int \frac{-b + \beta H}{H}\,dH$$

infine

$$c = \alpha P + \beta H - a\ln P - b\ln H \tag{3}$$

Ciò significa che le traiettorie nel piano delle fasi delle soluzioni si trovano nelle curve di livello della funzione $f(H, P) = \alpha P + \beta H - a\ln P - b\ln H$.

3.2 Le traiettorie sono tutte chiuse

Procediamo dimostrando che tali curve di livello sono tutte curve chiuse, ossia che E_2 è un centro globale. Passando all'esponenziale la (3) otteniamo

$$C = e^{-c} = P^a H^b e^{-\alpha P} e^{-\beta H} \tag{4}$$

che, introducendo le variabili

$$X(H) = H^{-b} e^{\beta H} \qquad \text{e} \qquad Y(P) = P^a e^{-\alpha P}$$

diventa l'equazione della retta $Y = CX$ nel piano (X, Y). Ora, $X(H)$ ha il suo unico minimo in H^*, così come $Y(P)$ ha un solo massimo in P^*; inoltre

$$X(0^+) = X(+\infty) = +\infty \qquad \text{e} \qquad Y(0) = Y(+\infty) = 0$$

costruendo "per punti" il grafico della curva di livello risulta dunque che le traiettorie sono tutte chiuse[10], come si evince dalla figura, per ogni valore di C tale che $0 < C < \exp(-f(H^*, P^*))$.

[9] ciò avviene perché l'equazione (2) è autonoma, quindi le orbite non si possono incrociare

[10] ad eccezione ovviamente delle due soluzioni banali già mostrate, e delle soluzioni costanti nei due punti di equilibrio E_1 e E_2

3.3 Le soluzioni sono periodiche

Dimostriamo ora che una qualsiasi soluzione percorre tutto il proprio ciclo in un tempo finito[11]. Per prima cosa, $\dot{H}$ e $\dot{P}$ si annullano contemporaneamente solo in E_2, dunque i cicli vengono percorsi tutti in verso antiorario. Chiamiamo ora

$$\vec{r}(t) = (H(t)-H^*, P(t)-P^*) \quad \text{il raggio vettore e} \quad \varphi(t) = \arctan\left(\frac{P(t)-P^*}{H(t)-H^*}\right)$$

l'angolo spazzato dal raggio vettore rispetto all'asse orizzontale $\{P = P^*\}$. Derivando φ otteniamo

$$\dot{\varphi} = \frac{1}{r^2}\left((H-H^*)\dot{P} - (P-P^*)\dot{H}\right) = \frac{1}{r^2}\vec{r}\cdot(\dot{P}, -\dot{H})$$

Una forma equivalente della (2) è la seguente

$$\begin{cases} \dot{H} = \alpha H(P^* - P) \\ \dot{P} = \beta P(H - H^*) \end{cases}$$

e, sostituendo $\dot{H}$ e $\dot{P}$ nell'equazione di $\dot{\varphi}$ ne ricaviamo

$$r^2\dot{\varphi} = (H-H^*)^2\beta P + (P-P^*)^2\alpha H \geq r^2 \min\{\beta P, \alpha H\}$$

dunque $\dot{\varphi} \geq \min\{\beta P, \alpha H\} > 0$, che vuol dire che il raggio vettore compie un giro in un tempo finito[12]. Le soluzioni $H(t)$ e $P(t)$ sono dunque periodiche e sfasate di un quarto di periodo.

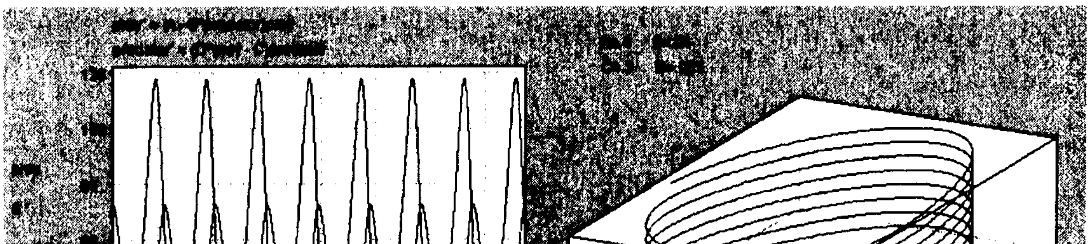

Figura 1: Un esempio numerico: i grafici di $H(t)$ e $P(t)$ con accanto la rispettiva curva integrale

[11] ciò basta a dire che le soluzioni sono periodiche in quanto la (2) è autonoma

[12] questa dimostrazione contiene un piccolo *bug*: in effetti l'angolo φ, non é definito in $H = H^*$; possiamo aggirare questo problema considerando l'angolo che il raggio vettore forma con l'asse verticale girando in senso antiorario, ossia $\vartheta = -\arctan\left(\frac{H-H^*}{P-P^*}\right)$, e ripetendo gli stessi conti fatti per φ, con il medesimo risultato

3.4 La media su un ciclo

In generale il periodo di oscillazione non é uguale per ogni orbita[13]; tuttavia, conoscendo il periodo di un ciclo posso calcolare i valori medi di H e P su quel ciclo

$$\overline{H} = \frac{1}{T}\int_0^T H(t)\,dt \qquad \text{e} \qquad \overline{P} = \frac{1}{T}\int_0^T P(t)\,dt$$

Osserviamo che, su qualunque ciclo vale

$$\int_0^T \frac{\dot{P}}{P}\,dt = \ln P(T) - \ln P(0) = 0$$

per la periodicità delle soluzioni. Dunque possiamo dire immediatamente che

$$0 = \int_0^T \frac{\dot{P}}{P}\,dt = \int_0^T (-b + \beta H)\,dt = -bT + \beta \int_0^T H\,dt$$

quindi risulta

$$\frac{b}{\beta} = \frac{1}{T}\int_0^T H(t)\,dt = \overline{H}$$

cioè il valor medio di H non dipende dal periodo T, quindi é indipendente anche dal ciclo sul quale ci troviamo.
Analogamente troviamo che

$$0 = \int_0^T \frac{\dot{H}}{H}\,dt = \int_0^T (a - \alpha P)\,dt = aT - \alpha \int_0^T P\,dt$$

quindi $\overline{P} = \frac{a}{\alpha}$ in ogni ciclo. Dunque il punto di equilibrio E_2 é anche la media di $(H(t), P(t))$ su qualunque ciclo.

[13]dato che $\dot{H}(0) = \dot{P}(0) = 0$, le soluzioni tendono a girare tanto più lentamente quanto più la soluzione è vicina agli assi $H = 0$ e $P = 0$

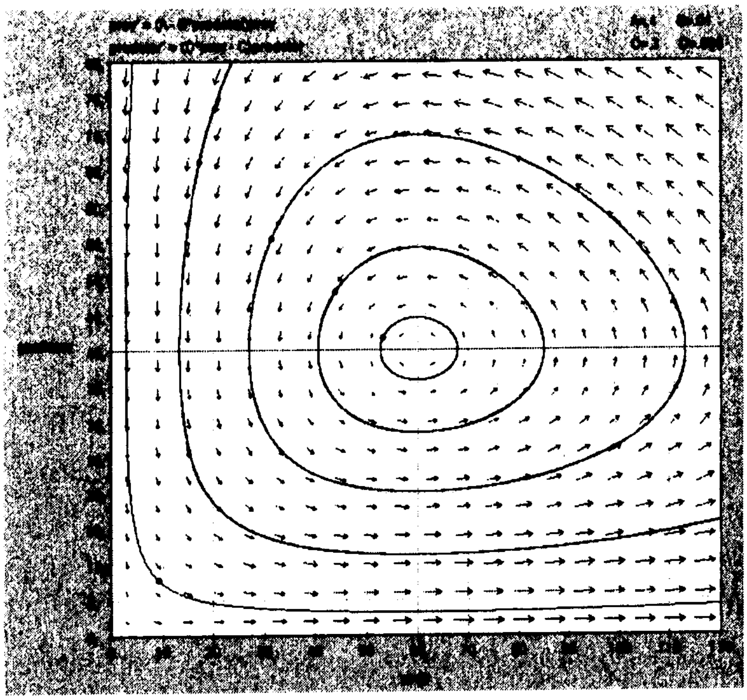

Figura 2: Un esempio numerico: alcune orbite nel piano delle fasi; su ognuna di queste orbite il numero medio di prede e predatori è lo stesso

4 Prede, predatori e umani: modelli con pesca

Introduciamo adesso nel modello un termine che rappresenti un prelievo costante da entrambe le specie, causato, ad esempio, da un'azione di pesca, caccia, o la diffusione di un antiparassitario da parte dell'uomo. Il modello diventa quindi

$$\begin{cases} \dot{H} = aH - \alpha HP - AH \\ \dot{P} = -bP + \beta HP - BP \end{cases} \tag{5}$$

con A e B, al solito, costanti positive.

4.1 Il principio di Volterra

Notiamo subito che, se $a > A$, la (5) è ancora un modello di Lotka-Volterra[14] con nuovi coefficienti biotici $a' = a - A$ e $b' = b + B$. Quindi la media su un ciclo di H e P è la seguente

$$\overline{H}' = \frac{b+B}{\beta} > \overline{H} \qquad \text{e} \qquad \overline{P}' = \frac{a-A}{\alpha} < \overline{P}$$

[14]in caso contrario il prelievo risulterebbe eccessivo, e porterebbe all'estinzione della preda, seguita dall'estinzione del predatore

che vuol dire che la presenza di condizioni meno favorevoli a entrambe le specie causano un incremento del numero medio di prede e un decremento del numero medio di predatori. Questo fenomeno è noto come **effetto Volterra**, ed è un risultato non intuibile e non ottenibile se non con tale ragionamento matematico[15]. Supponendo di poter pescare con una rete selettiva, cioè che prenda solo prede, quindi $B = 0$, o solo predatori, $A = 0$, ci accorgiamo che la pesca va ad alterare solo il numero medio della specie "scartata"; se il risultato generale appariva inconsueto, questo caso particolare sembra veramente paradossale.

4.2 Prelevamento *una tantum*

Consideriamo il caso in cui il prelievo non è costante, ma occasionale. In tal caso la (2) non cambia, tuttavia dall'istante t_1 in cui è avvenuto il prelievo le orbite cambiano ciclo, ossia ripartono dalle nuove condizioni iniziali

$$(\tilde{H}(t_1), \tilde{P}(t_1)) = (H(t_1) - A, P(t_1) - B)$$

ciò vuol dire che il prelevamento *una tantum* non varia il numero medio delle due specie, ma ne altera l'ampiezza delle oscillazioni (si veda la figura (4)).

[15]tuttavia, come spesso accade, un ragionamento intuitivo fatto a posteriori può giustificare l'effetto Volterra: le prede infatti sono già soggette a prelievo, e si sono adattate a subire perdite continue, cosa che non vale per i predatori. Ad esempio le lepri o i conigli, la cui velocità di riproduzione si è rilevata determinante alla sopravvivenza della specie

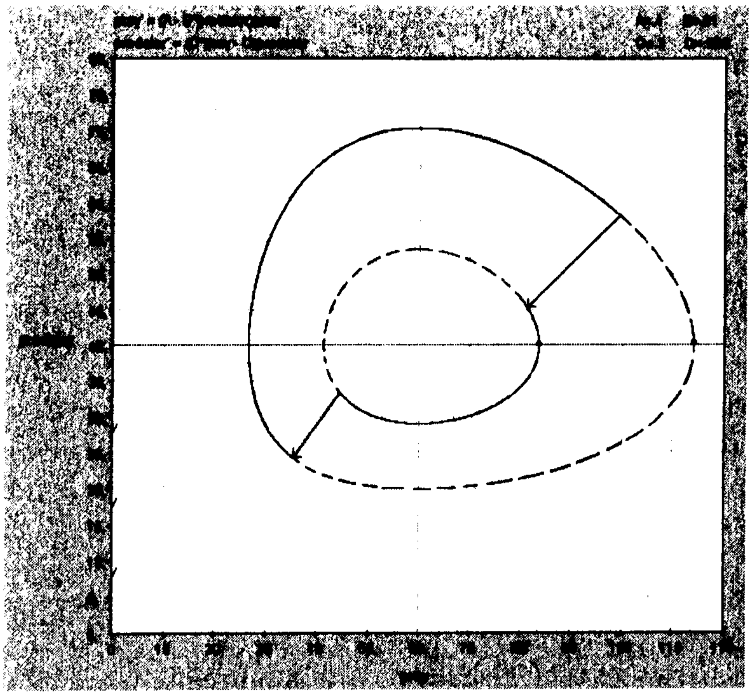

Figura 3: Un esempio di prelevamento *una tantum*: un primo prelievo diminuisce l'ampiezza delle oscillazioni portando l'orbita su un ciclo più interno; un secondo prelievo effettuato in un altro punto può aumentare l'ampiezza e riportare la traiettoria sul ciclo originale

5 Un modello più realistico

Ricostruiamo il modello escludendo dalle ipotesi l'illimitatezza delle risorse a disposizione delle prede; senza tale ipotesi dobbiamo correggere l'equazione (2) sottraendo un termine logistico, direttamente proporzionale alla probabilità di incontro fra due prede

$$\begin{cases} \dot{H} = aH(1 - \frac{H}{K}) - \alpha HP \\ \dot{P} = -bP + \beta HP \end{cases} \tag{6}$$

dove K è, al solito, una costante positiva.
Anche in questo caso ho due soluzioni sugli assi cartesiani,una logistica per le prede in assenza di predatori e una malthusiana decrescente per i predatori, quindi le soluzioni $H(t)$ e $P(t)$ sono sempre positive a patto di prendere condizioni iniziali positive, cosa essenziale perché il modello non perda di significato fisico.
Studiamo i punti di equilibrio del sistema: $E_1 = (0,0)$, che, come prima, è una sella, $E_2 = (K,0)$, e $E_3 = (\frac{b}{\beta}, \frac{a}{\alpha}(1 - \frac{b}{K\beta})) = (H_\infty, P_\infty)$; quest'ultimo

punto ha ascissa positiva solo se $K > \frac{b}{\beta}$. Dunque tratteremo separatamente due casi: il primo caso $K < \frac{b}{\beta}$ in cui le due isocline non si incrociano, il caso limite $K = \frac{b}{\beta}$ dove $E_2 = E_3$, e il secondo caso $K > \frac{b}{\beta}$, in cui l'intersezione fra le due isocline genera il nuovo punto di equilibrio E_3.

5.1 Il primo caso e il caso limite

Studiamo la matrice di Jacobi per capire come sono fatti i punti di equlibrio.

$$J(H,P) = \begin{pmatrix} a - 2\frac{a}{K}H - \alpha P & -\alpha H \\ \beta P & -b + \beta H \end{pmatrix} \tag{7}$$

Come ci aspettavamo, $J(E_1) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$, quindi E_1 è una sella; invece $J(E_2) = \begin{pmatrix} -a & -K\alpha \\ 0 & -b + \beta K \end{pmatrix}$ è una matrice a determinante positivo nel primo caso, nullo nel caso limite, e a traccia negativa. Ne segue che E_2 è un punto di equilibrio stabile in entrambi i casi.
Tracciando il grafico qualitativo notiamo che ogni traiettoria tende al punto E_2; ciò significa estinzione dei predatori[16].

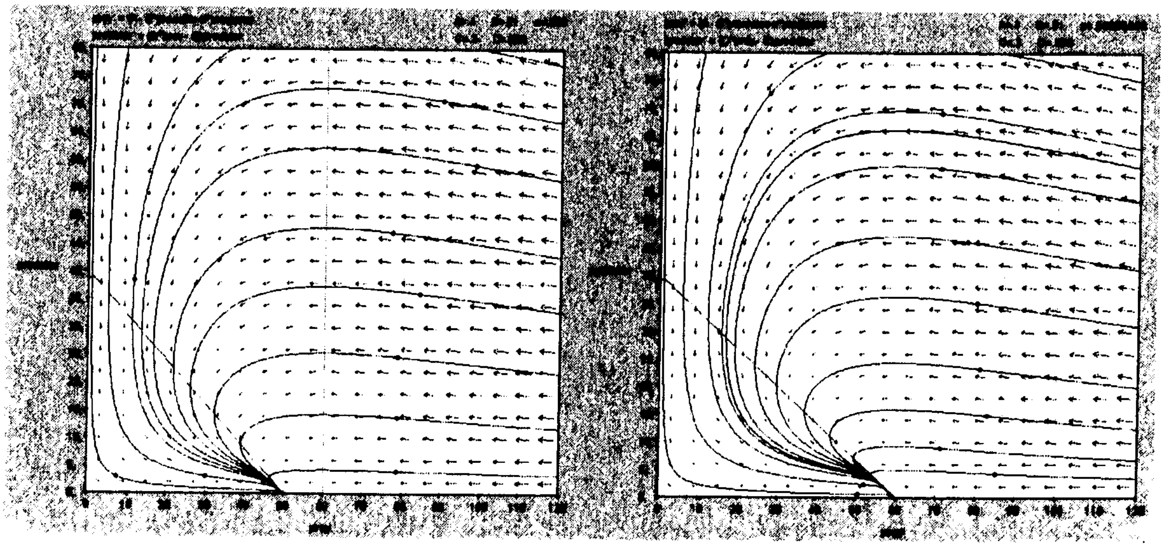

Figura 4: Due esempi numerici: sia il primo caso che il caso limite predicono l'estinzione del predatore

[16]i predatori si estinguono in quanto le prede sono così poche che predatori non riescono a incontrarne; è la stessa cosa che accade nei modelli epidemiologici quando la popolazione è rarefatta: il mancato incontro fra individui rende nullo il contagio e l'epidemia cessa

5.2 Il secondo caso

Nel secondo caso invece $J(E_2)$ ha determinante negativo, quindi E_2 è un punto di sella. Calcolando $J(E_3) = \begin{pmatrix} -\frac{ab}{K\beta} & -\alpha\frac{b}{\beta} \\ \frac{a\beta}{\alpha}(1-\frac{b}{K\beta}) & 0 \end{pmatrix}$ risulta che

$$det(J(E_3)) = ab(1-\frac{b}{K\beta}) > 0 \qquad e \qquad tr(J(E_3)) = -\frac{ab}{K\beta} < 0$$

E_3 è dunque un punto di equilibrio stabile.

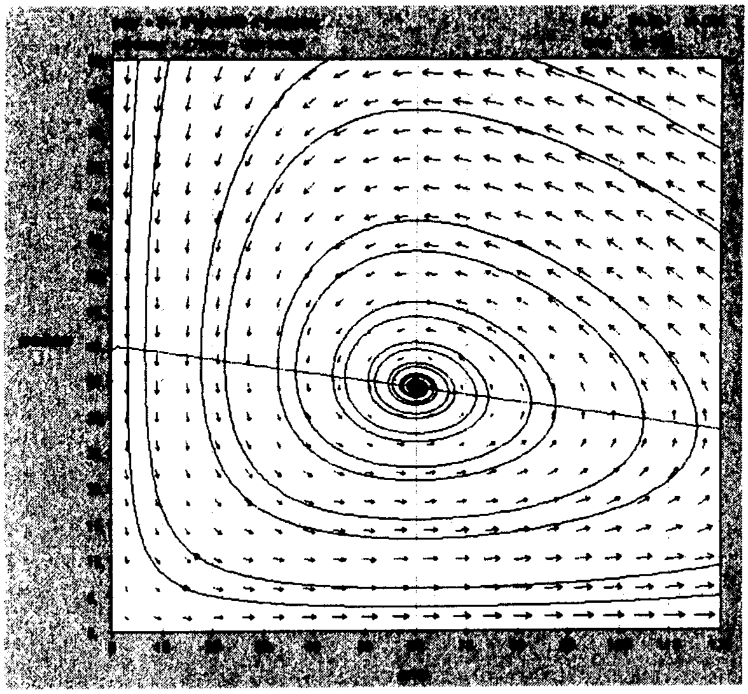

Figura 5: Un esempio numerico: il nuovo punto di equilibrio è un attrattore

5.2.1 Una funzione di Lyapunov

Ciò non basta per dire che E_3 è un attrattore globale: non è esclusa infatti la presenza di traiettorie che si svolgono, quindi di uno o più cicli limite. Abbiamo quindi bisogno di una funzione di Lyapunov per la (6) relativa a E_3.
Proviamo a confrontare le orbite della (2) centrate in E_3 con le traiettorie della (6).

$$K = \alpha P + \beta H - a\left(1 - \frac{b}{K\beta}\right)\ln P - b\ln H$$

derivando rispetto al tempo, e immettendo le equazioni del sistema (6) ottengo

$$\dot{K} = \alpha\dot{P} + \beta\dot{H} - a\left(1 - \frac{b}{K\beta}\right)\frac{\dot{P}}{P} - b\frac{\dot{H}}{H} =$$

$$= -\frac{ab^2}{K\beta} + 2\frac{ab}{K}H - \frac{a\beta}{K}H^2 = -\frac{a\beta}{K}\left(H - \frac{b}{\beta}\right)^2 \leq 0$$

L'assenza di soluzioni con $H(t) \equiv \frac{b}{\beta}$ se non quella banale garantiscono che E_3 è un punto di equilibrio asintoticamente stabile. L'effetto che abbiamo ottenuto aggiungendo il termine logistico alla (2) è stato dunque la riduzione progressiva dell'ampiezza delle oscillazioni di $H(t)$ e $P(t)$ fino al loro stabilimento a un valore limite, rispettivamente H_∞ e P_∞.

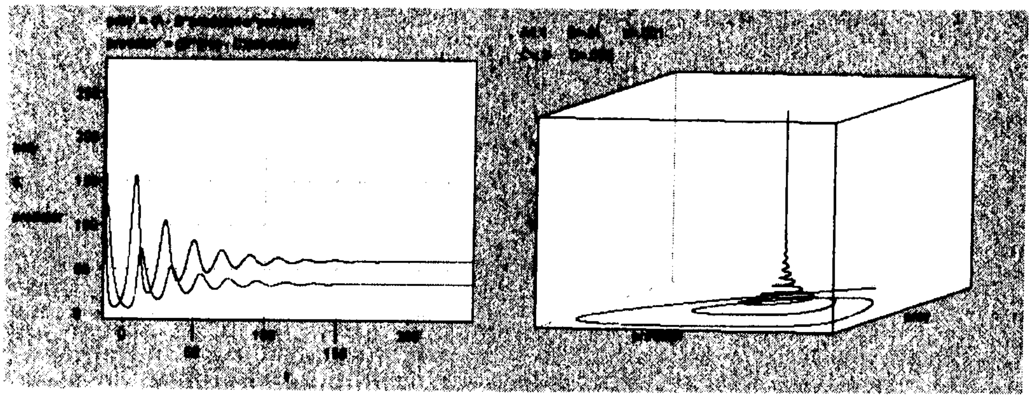

Figura 6: Un esempio numerico: il termine logistico smorza le ampiezze delle oscillazioni fino al loro annullamento

5.2.2 Ancora il principio di Volterra

Inseriamo ancora una volta un termine di prelievo costante al modello; ciò che otteniamo è il sistema:

$$\begin{cases} \dot{H} = aH(1 - \frac{H}{K}) - \alpha HP - AH \\ \dot{P} = -bP + \beta HP - BH \end{cases} \tag{8}$$

Questo modello, sempre a condizione che A sia più piccolo di a, è equivalente al sistema (6) con

$$H'_\infty = \frac{b+B}{\beta} > H_\infty \quad \text{e} \quad P'_\infty = \frac{a-A}{\alpha}(1 - \frac{b}{K\beta}) < P_\infty$$

quindi vale sempre il principio di Volterra, non più nel senso di media su un ciclo, ma nel senso di valore limite verso il quale le soluzioni si stabilizzano.

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Can the drinking bird explain economic cycles ? (A history of auto-oscillations and limit cycles)

par Jean Mawhin
Membre de la Classe

Dedicated to the memory of my thesis adviser, Paul Ledoux

1 Introduction

The aim of this work is to sketch and, sometimes, to refine the history of the notion of self-sustained oscillations or auto-oscillations in science and technology, and that of the mathematical concept of limit cycle of a system of ordinary differential equations. Devices producing self-sustained oscillations, i.e. able to transform a continuous source of energy into a periodic motion exist since Antiquity, and the most ludic one can be the drinking bird, a scientific toy patented by an American inventor in 1946. Those self-sustained oscillations are of the greatest importance in clock-making, musical instruments, wireless telegraphy, friction dynamics, chemistry, biology and many other fields.

Almost simultaneously and independently in the beginning of the years 1880, LORD RAYLEIGH proposed a second order differential equation with a suitable nonlinear friction term, to modelize self-sustained oscillations, and HENRI POINCARÉ showed the presence, in planar systems of nonlinear differential equations, of periodic solutions to which neighboring solutions are asymptotic for t going to infinity, named *limit cycles*.

Despite of his interest and his contributions in wireless telegraphy, POINCARÉ did not apparently connect the auto-oscillations with his limit cycles, a relation only made in 1929 by ALEKSANDR A. ANDRONOV. It was the starting point for a new interdisciplinary research domain called *theory of nonlinear oscillations* or sometimes *nonlinear mechanics*.

The development of this area in the XXth and the beginning of the XXIst centuries has been spectacular and the scope of its domains of applications constantly widened, including attempts in human sciences like economics. This explains the somewhat cryptic title of this essay.

2 Linear models for oscillators

2.1 Celestial harmony

HOOKE's law tells us that the restoring force of a spring is proportional to its elongation with respect to its equilibrium position. Mathematically speaking, if the displacement of a unit mass attached to a spring, with respect to its equilibrium position, is denoted by x, the restoring force F is given by

$$F = -x$$

and NEWTON's fundamental law of dynamics gives the linear differential equation

$$x'' + x = 0 \tag{1}$$

for the motion of the unit mass, where $'$ denotes the derivative with respect to time t. Eq. (1), called the *harmonic oscillator equation*, also modelizes the small oscillations of the pendulum, of oscillating circuits made of a self and a capacity in electricity and even, after a clever change of variables, of the two-body problem in astronomy. Its solutions, known at least since the time of LEONHARD EULER, are given by

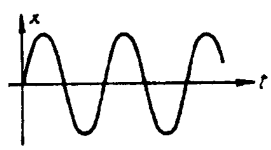
$$x(t) = A\sin(t + \phi) \quad (A \geq 0, \quad 0 \leq \phi < 2\pi), \tag{2}$$

where $A \geq 0$ is called the *amplitude* of the motion, and ϕ its *phase*. It is also easy to show, by multiplying both members of eq. (1) by x' and integrating, that eq. (1) has a first integral, namely the *energy integral*

$$\frac{x'^2}{2} + \frac{x^2}{2} = C \quad (C \geq 0), \tag{3}$$

expressing that the sum of the kinetic energy and of the potential energy remain constant during the motion.

To picture the solutions of eq. (1), one can use a time-picture given by (2)



HARMONIC OSCILLATOR : TIME EVOLUTION

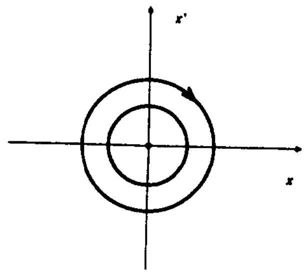
showing that the time-evolution of any solution is sinusoidal, but also a geometric-picture in the so-called *phase plane* of coordinates (x, x') deduced from eq. (3), namely

$$x'^2 + x^2 = 2C,$$

showing that the *orbits* of the motion, namely the curves

$$\{(x(t), x'(t)) : -\infty < t < +\infty\}$$

corresponding to the solutions $x(t)$ of eq. (1) are circles centered at the origin, which is a *stable* equilibrium position. They are described for increasing time in the sense given by the arrows.



HARMONIC OSCILLATOR : PHASE PLANE

So, periodic solutions with respect to time correspond, in the phase plane, to *closed orbits* or *cycles*, and equilibria to *fixed points*.

2.2 Terrestrial friction

The model of the previous section does not easily apply to terrestrial phenomena because of the almost unavoidable and often positive presence of *friction* or *damping* in those phenomena. A real pendulum will never exhibit oscillations with constant amplitude, but with decreasing one. The harmonic oscillator eq. (1), only valid in a world without friction, is just an equation for motions in heavens.

The simplest model for a damped oscillation is given by

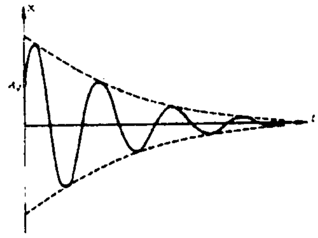
$$x'' + \epsilon x' + x = 0 \quad (\epsilon > 0), \tag{4}$$

and referred as the *damped linear oscillator*. If $\epsilon > 0$ is sufficiently small (namely $0 < \epsilon < 2$), its solutions are given by

$$x(t) = Ae^{-\frac{\epsilon}{2}t} \sin\left(t\sqrt{1 - \frac{\epsilon^2}{4}} + \phi\right) \quad (A \geq 0, \quad 0 \leq \phi < 2\pi). \tag{5}$$

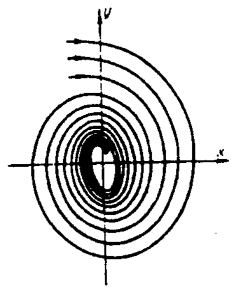
and consist in oscillations whose amplitude $Ae^{-\frac{\epsilon}{2}t}$ decreases exponentially to the *asymptotically stable equilibrium position* $x(t) \equiv 0$.

Hence, the time-picture of the motions is given by a sinusoid with decreasing amplitude,



DAMPED OSCILLATOR : TIME EVOLUTION

from which one can deduce (there is no energy integral anymore !) that the corresponding orbits in the phase plane are spirals around the origin (0,0), described in increasing time in the sense shown by the arrows.



DAMPED OSCILLATOR : PHASE PLANE

Mathematically, there is no reason to exclude the case of a *negative damping*

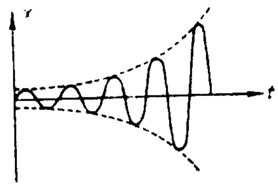
$$x'' - \epsilon x' + x = 0 \quad (\epsilon > 0) \tag{6}$$

and eq. (6) can indeed be associated to the linearization of some models of nonlinear oscillators, as shown later. Its solutions, for $0 < \epsilon < 2$, are given by

$$x(t) = Ae^{\frac{\epsilon}{2}t} \sin\left(t\sqrt{1 - \frac{\epsilon^2}{4}} + \phi\right) \quad (A \geq 0, \quad 0 \leq \phi \leq 2\pi), \tag{7}$$

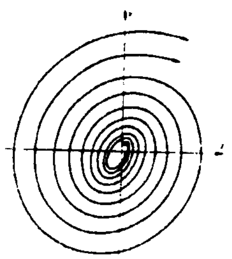
and consist in oscillations with indefinitely increasing amplitude $Ae^{\frac{\epsilon}{2}t}$ around the *unstable equilibrium* $x(t) \equiv 0$.

So the time-picture of the motion is given by a sinusoid of increasing amplitude.



NEGATIVE DAMPING : TIME EVOLUTION

and the corresponding orbits in the phase plane



NEGATIVE DAMPING : PHASE PLANE

are spirals around the origin (0,0), described in increasing time in the sense shown by the arrows.

2.3 Periodic oscillations by forcing

The discussion above show that *periodic motions* in a linear oscillator with one degree of freedom are only possible in the absence of damping, so that this linear model is unable to describe adequately most periodic motions observed on Earth. The harmonic oscillator has also the property of being *structurally unstable*, in the sense that the nature of its orbits (cycles) can be destroyed by arbitrarily small perturbations of the equations (for example by terms $\epsilon x'$ with arbitrary small $\epsilon > 0$). Because of the fact that approximations of various types are always present in the modelization of a physical phenomenon, a reliable model should on the contrary be *structurally stable* in the sense of A.A. ANDRONOV and L.S. PONTRYAGIN [6].

Of course, it is well known that if we force a (positively or negatively) damped harmonic oscillation with an exterior force of some period T, namely if we consider the *forced linear damped oscillator* with $e(t)$ periodic of period T,

$$x'' + \epsilon x' + x = e(t) \tag{8}$$

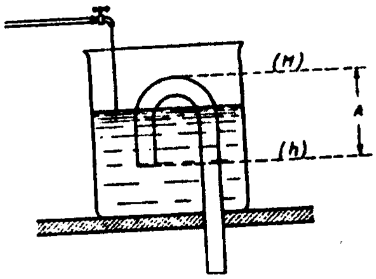
and $\epsilon \neq 0$, there will always be a *unique* solution of period T to which all other solutions tend when t tends to $+\infty$.

This just shifts the question, because we will have in practice to device a physical way to produce the T-periodic forcing term $e(t)$, i.e. to construct another oscillator having $e(t)$ as solution. Physically, the question is how to transform a *constant* source of energy into a *periodic* motion ? Such periodic motions are generally called *self-sustained oscillations* or *auto-oscillations*. All the linear models described above fail in providing an answer.

3 Self-sustained oscillations

3.1 Early technical examples

As it has often happened in the evolution of science, technology did not wait for a scientific explanation or for a mathematical model to produce devices exhibiting self-sustained oscillations. Maybe the eldest one is the so-called *vase of Tantale* already known in Antiquity : a continuous flow of liquid provides a periodic oscillation of the level of the liquid in the vase, which never reaches the top of the vase, because of the presence of the syphon (hence the name of 'Tantale' because of the torment of a potential drinker having his lips at the top of the vase).



VASE OF TANTALE

Musical instruments with rubbed string, or wind instruments are other examples, as well as pendulum clocks with weight or WATT's double action steam engine.

Another example was exhibited by the French engineer GÉRARD-LESCUYER in 1880 [52, 155] : if a direct current produced by a dynamo is sent through a magneto-electrical engine, this one starts to move, with increasing velocity depending upon the intensity of the sent electrical current, but then slows down, stops and restarts in opposite sense, to stop again and restart in the original sense, and so on. The source of electrical direct current of the first machine produces a periodic motion in the second one.

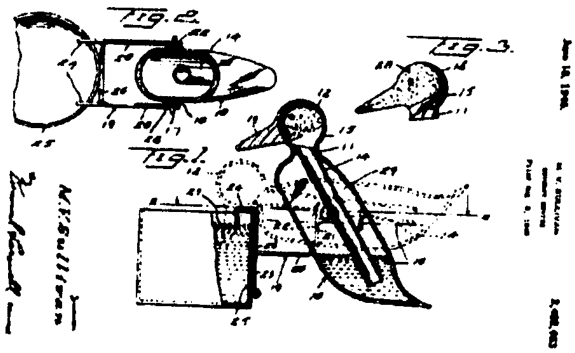
The development of radio-electricity strongly depends upon devices producing self-sustained oscillations of high frequency, whose amplitude will be modulated to produce the sound. One of the first devices to reach this aim was the *singing arc* invented around 1900 by the British electrical engineer WILLIAM DU BOIS DUDDELL (1872-1917) [45] : a circuit made of a self L and a capacity C is placed on an electrical arc alimented by continuous current. Under some conditions, the arc light flashes in a periodic way, producing a very pure sound of frequency $T = 2\pi\sqrt{LC}$. This is the prototype of electrical musical instruments. See [19, 75] for more details and explanations. The singing arc was later replaced in radio transmission by an oscillating circuit containing a *triode* or lamp with three electrodes, invented by the American LEE DE FOREST (1873-1961) in 1906 under the name of *audion*.

A more recent and more popular example is the *drinking bird* (or *dunking bird*, or *dippy bird* or *dipping bird*), a scientific toy invented and patented in 1946 by the American MILES V. SULLIVAN (born in 1917).

MILES V. SULLIVAN (B. IN 1917)

DRINKING BIRD

The bird is constructed in glass and its head and body consist in balls emptied from the air and joined by a tube going in the body ball to a small distance of the bottom, in order to produce a siphon effect. The body is partly filled with a volatile liquid and the beak is covered with a spongy material. The bird is posed in such a way that its beak can plunge into a glass full of water. The bird freely oscillates on a support. When his beak has been watered in this way, evaporation of the water diminishes the temperature inside the head and the volatil liquid moves up in the neck, modifiying the center of gravity of the bird, whose head moves down and enters the water again, causing a return of the volatile liquid in the bird's bottom. The same motion then repeats itself periodically. Of course it is not an example of perpetual motion, but a water engine ! See [66, 93] and references therein for more detailed physical explanations and experiments.



US PATENT 2,402.463, JUNE 18, 1946

Other examples of auto-oscillations can be found in [30, 72, 86, 104, 123].

3.2 The first mathematical model for self-sustained oscillations

In 1883, the British physicist and Nobel prize JOHN WILLIAM STRUTT (LORD RAYLEIGH) (1842-1919) introduced the first *mathematical* model for self-sustained oscillations [120]. The paper is reproduced with slight modifications in his famous treatise *Theory of Sound.*

LORD RAYLEIGH (1842-1919)

RAYLEIGH realized that self-sustained oscillations could be mathematically modelled by replacing the *linear* damping $\pm\epsilon x'$ in eq. (4) or (6) by a suitable nonlinear one $f(x')$ such that $f(0) = 0$, namely by considering the differential equation

$$x'' + f(x') + x = 0, \tag{9}$$

and choosing f in such a way that $f(x')$ behaves qualitatively like a negative damping $-\epsilon x'$ when $|x'|$ is small, and like a positive damping $\epsilon x'$ when $|x'|$ is large. The simplest choice (from a Taylor expansion) of $f(x') = \epsilon(-1 + x'^2)x'$ ($\epsilon > 0$) provides the so-called *Rayleigh equation*

$$x'' + \epsilon(-1 + x'^2)x' + x = 0, \tag{10}$$

whose damping term corresponds to a negative damping producing oscillations of increasing amplitude when $|x'| < 1$, and to a positive damping producing oscillations of decreasing amplitude when $|x'| > 1$. Heuristically, RAYLEIGH conjectured the existence of a self-sustained oscillation separating those two regimes of motion. He gave some approximate justification and some estimate of the amplitude of the self-oscillation when ϵ is small. In his own words [120] :

When a vibrating system is subjected to dissipative forces, the vibrations cannot be permanent, since they are dependent upon an initial store of energy which suffers gradual exhaustion. In the usual equation

$$\frac{d^2\theta}{dt^2} + \kappa\frac{d\theta}{dt} + n^2\theta = 0 \tag{11}$$

κ is positive, and the solution indicates the progressive decay of the vibrations in accordance with the exponential law. In order that the vibrations may be maintained, the vibrating body must be in connexion with a source of energy. This condition being satisfied, two principal classes of maintained vibrations may be distinguished. [...] The first class is by far the more extensive, and includes vibrations maintained by wind (organ-pipes, harmonium-reeds, aeolian harps,etc.), by heat (singing flames, Rijke's tubes, etc.), by friction (violin-strings, finger-glasses, etc.), as well as the slower vibrations of clock-pendulums and of electromagnetic tuning-forks. When the amplitude is small, the force acting upon the body may be divided into two parts, one proportional to the displacement θ (or to the acceleration), the second proportional to the velocity $\frac{d\theta}{dt}$. The inclusion of these forces does not alter the *form* of (11). [...] When κ is negative, so that small vibrations tend to increase, a point is of course reached after which the approximate equations cease to be applicable. We may form an idea of the state of things which then arises by adding to equation (11) a term proportional to a higher power of the velocity. Let us take

$$\frac{d^2\theta}{dt^2} + \kappa\frac{d\theta}{dt} + \kappa'\left(\frac{d\theta}{dt}\right)^3 + n^2\theta = 0 \tag{12}$$

in which κ and κ' are supposed to be small. The approximated solution of (12) is

$$\theta = A\sin nt + \frac{\kappa' n A^3}{32}\cos 3nt \tag{13}$$

in which A is given by

$$\kappa + \frac{3}{4}\kappa' n^2 A^2 = 0. \tag{14}$$

From (14) we see that no steady vibration is possible unless κ and κ' have different signs. [...] If κ be negative and κ' positive, the vibration becomes steady and assumes the amplitude determined by (14). A smaller vibration increases up to this point, and a larger vibration falls down of it. If, on the other hand, κ be positive, while κ' is negative, the steady vibration abstractedly possible is unstable, a departure in either direction from the amplitude given by (14) tending always to increase.

3.3 Poincaré's limit cycles

In a paper of 1881 anounced by a note to the *Comptes rendus* of the French Academy of Science in 1880 [116], the French mathematician, physicist and astronomer Henri Poincaré (1854-1912) initiated a study of the geometry of the *orbits*

$$\{(x(t), y(t)) : -\infty < t < +\infty\}$$

described by the solutions $(x(t), y(t))$ of (possibly) nonlinear planar differential systems of the form

$$x' = P(x,y), \quad y' = Q(x,y), \tag{15}$$

where P and Q are polynomials.

HENRI POINCARÉ (1854-1912)

Notice that all the cases of unforced oscillations already considered above reduce to eq. (15) (when f is a polynomial) by letting $y := x'$ and writing (9) in the equivalent form of the planar system

$$x' = y, \; y' = -f(y) - x.$$

The plane (x,y) corresponds to the phase plane (x,x') of the corresponding second order differential equation.

Among many other things, POINCARÉ observed that system (15) could have closed orbits (cycles) to which all neighboring orbits would spiral when $t \to +\infty$ or $t \to -\infty$. He called such closed orbits *limit cycles*. In his own words [116] :

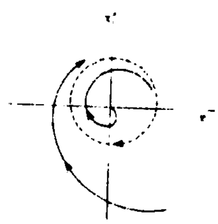
> Ce Mémoire a pour but l'étude géométrique des courbes définies par une équation différentielle de la forme $\frac{dx}{X} = \frac{dy}{Y}$, où X et Y sont des polynomes entiers en x et y. [Je les] appelle *caractéristiques*. [...] Parmi ces courbes fermées, les unes ne sont pas caractéristiques et ne touchent une caractéristique en aucun point : je les appelle *cycles sans contact*; les autres sont des caractéristiques : je les appelle *cycles limites*, parce qu'elles sont asymptotes aux caractéristiques voisines.
> [This memoir is devoted the geometric study of the curves defined by a differential equation of the form $\frac{dx}{X} = \frac{dy}{Y}$, where X and Y are integer polynomials in x and y. I call them *characteristics*. [...] Among those closed curves, some are not characteristics and touch a characteristic in no point : I call them *cycles without contact*; other ones are characteristics: I call them *limit cycles*, because they are asymptotic to the neighboring characteristics.]

POINCARÉ gave some rigorous methods for their detection, in particular the so-called *Poincaré-Bendixson criterion* [16], implying the existence of a limit cycle in an annular region free of constant solutions and containing a solution defined for all $t \geq t_0$ for some t_0.

POINCARÉ also gave a slight variant of the following simple example of differential equation admitting such a limit cycle

$$x'' + \epsilon(-1 + x^2 + x'^2)x' + x = 0 \quad (\epsilon > 0). \tag{16}$$

This example differs from the one given in 1883 by RAYLEIGH (without any reference to POINCARÉ's paper) by the presence of the term x^2x' in the damping. Here, when $x^2+x'^2 < 1$, the nonlinear term behaves like a negative damping, and the solutions are oscillations of increasing amplitude (or spirals going away from $(0,0)$ in the phase plane). When $x^2 + x'^2 > 1$, the nonlinear term behaves like a positive damping, and the solutions are oscillations of decreasing amplitude (or spirals approching $(0,0)$ in the phase plane). Now, the circle of equation $x^2 + x'^2 = 1$ in the phase plane is itself an orbit for eq. (16), corresponding to the periodic solutions $x(t) = \sin(t + \phi)$ of eq. (16), as immediately checked.



POINCARÉ'S LIMIT CYCLE

In his own words POINCARÉ's example goes as follows [116] :

Soit l'équation

$$\frac{dx}{x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1)} = \frac{dy}{y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1)}.$$

Il n'y a qu'un point singulier [...]; c'est le point $x = y = 0$ qui est un foyer. [...] Tous les cercles sont des cycles sans contact, excepté le cercle de rayon 1 qui est un cycle limite. Il n'y a pas d'autre cycle limite.
[Consider the equation

$$\frac{dx}{x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1)} = \frac{dy}{y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1)}.$$

There is only one singular point [...]; it is the point $x = y = 0$ which is a focus. [...] All the circles are cycles without contact, except the circle of radius 1 which is a limit cycle. There is no other limit cycle.]

3.4 Poincaré and the singing arc

POINCARÉ's scientific interests were very wide and he contributed to the early development of the theory of Hertzian waves. At this occasion, he wrote in 1907 a popular account of the recent progress of wireless telegraphy entitled *Théorie de Maxwell et les oscillations Hertziennes. La télégraphie sans fil* [118].

In this book, POINCARÉ described DUDDELL' singing arc mentioned above, first used in wireless telegraphy by VALDEMAR POULSEN (1869-1942) in 1902, before the introduction of the triode. POINCARÉ noticed the similarity of this device with a classical example of self-sustained oscillations when he wrote :

> Ces oscillations sont *entretenues* comme le sont celles du balancier de nos horloges.
> [Those oscillations are *sustained* as those of the pendulum of our clocks].

However, POINCARÉ did not make any connection between those auto-oscillations and the limit cycles he had introduced some twenty-five years before. As we will see, another twenty-five years will be necessary to make it !

4 Modeling self-sustained oscillations

4.1 Janet and electrical machines in series

The French physicist PAUL JANET (1863-1937) had included for many years the experiment of GÉRARD-LESCUYER in his lectures on electricity.

PAUL JANET (1863-1937)

In a note of 1919 at the *Comptes rendus* of the Academy of Science of Paris [76], he proposed a model for a direct current dynamo excited in series and connected to an electrical motor with permanent magnets or series excitation. He also observed interesting analogies :

> Il m'a semblé intéressant de signaler les analogies inattendues que présente cette expérience avec les oscillations entretenues si largement utilisées aujourd'hui en télégraphie sans fil, par exemple avec celles qui se produisent dans l'arc de Duddell ou dans les lampes à trois électrodes employées comme oscillateurs. [...] La dynamo-série génératrice se comporte comme

une résistance négative, et le moteur à excitation séparée se comporte comme un condensateur.
[It seemed to me interesting to mention the unexpected analogies of this experiment with the sustained oscillations so widely used to-day in wireless telegraphy, for example with those produced in Duddell's arc or in the lamps with three electrodes used as oscillators. [...] The dynamo-series acts as a negative resistance, and the engine with separated excitation acts as a capacity.]

He deduced from the fundamental laws of electricity, that, if $e = F(i)$ represents the electro-mechanical force of the dynamo, the intensity of the current in the electrical motor satisfies a differential equation of the form

$$Li'' + \left[R - \frac{dF}{di}(i)\right] i' + \frac{k^2}{K} i = 0, \qquad (17)$$

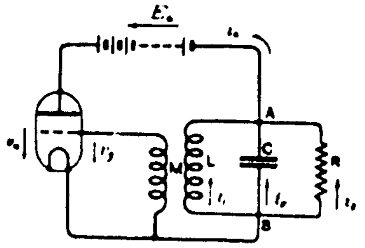
quite similar to RAYLEIGH's one.

4.2 Van der Pol and the oscillating circuit with triode

One year later, the Dutch physicist BALTHAZAR VAN DER POL (1889-1959) used the fundamental laws of electricity and electronics to modelize an oscillating circuit containing a triode [142].

BALTHAZAR VAN DER POL (1889-1959)

Before him, with the exception of JANET's paper mention above, the study of this circuit was limited to its linear approximation and the search of the ranges of its coefficients leading to oscillations with increasing amplitudes (see e.g. [20]).



In VAN DER POL's own words :

> As a given triode oscillator, with definite settings of the circuit constants, will generate oscillatory currents with harmonics all having a definite amplitude, it may be worth while to put forward a theory of the oscillating triode having regard to the non-linear terms in the equation.

A triode has three active electrodes, the plate (anode), the grid, and the (heated) filament (cathode). Its work depends upon a linear combination $u = v_g + Dv_a$ (with $0 < D < 1$) of the anode potential v_a and grid potential v_g with respect to the filament. The plate current i_p is represented by an S-shape function of u, $i_a = \phi(u)$, called the *characteristic* of the triode. Its has an inflection point at the unstable steady value (v_{a_0}, i_{a_0}). VAN DER POL obtained first, for the instantaneous deviation i of the total plate current from i_{a_0} and for the instantaneous deviation v of the plate potential from v_{a_0}, the differential system

$$i' + Cv'' + \frac{1}{R}v' + \frac{1}{L}v = 0, \quad i = \psi(kv), \tag{18}$$

where $\psi(kv) = \phi(v_{a_0} - kv) - \phi(v_{a_0})$. From this he deduced, by assuming (according to Taylor's expansion) that

$$i = \psi(kv) = -\alpha v + \beta v^2 + \gamma v^3 \tag{19}$$

the second order differential equation for v

$$Cv'' + \left(\frac{1}{R} - \alpha\right) v' + \frac{1}{L}v + \beta(v^2)' + \gamma(v^3)' = 0, \tag{20}$$

which he reduced to

$$x'' - \epsilon(1 - x^2)x' + x = 0, \tag{21}$$

by taking, for simplicity, $\beta = 0$, and normalizing the coefficients.

He gave two heuristic methods for showing the existence of a periodic oscillation of amplitude close to 2 when $\epsilon > 0$ is small, which is the situation occuring in the radio problem. He did not refer to JANET's work [76] mentioned above, but, in a footnote to his first method, he noticed that :

> we follow closely a method of solution given by Prof. Lorentz in a series of lectures at Leiden University.

The second method consisted in multiplying the first equation in (18) by a function V such that $V' = v$ and integrating over the period of the limit cycle to get, after some integrations by parts,

$$\int_0^T iv\, dt + \frac{1}{R}\int_0^T v^2\, dt = 0, \tag{22}$$

Now, eq. (19) implies that

$$\int_0^T iv\,dt = \int_0^T [-\alpha v^2 + \beta v^3 + \gamma v^4]\,dt.$$

so that eq. (22) becomes

$$\int_0^T \left[(\frac{1}{R} - \alpha)v^2 + \beta v^3 + \gamma v^4 \right] dt = 0. \tag{23}$$

The approximate formula $v(t) = a\cos\omega t$ with $\omega = \frac{2\pi}{T}$ for the limit cycle introduced in eq. (23) gives, after simple calculations the approximate expression

$$a^2 = \frac{4}{3}\frac{\alpha - \frac{1}{R}}{\gamma}$$

for the approximate amplitude a of the limit cycle. This reduces to $a^2 = 4$ in the case of eq. (21). This method was developed by DARIO GRAFFI [62] in 1942.

One should also mention that VAN DER POL quoted RAYLEIGH's paper [120] on p. 708, but only when discussing the values of the coefficients leading to an unstable limit cycle. He did not comment on the close relationship between his equation (21) and RAYLEIGH's one (10).

Six years later, VAN DER POL [143] studied the orbits of his equation (21), written as a system

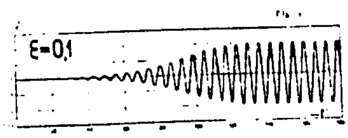
$$x' = y, \quad y' = \epsilon(1 - x^2)y - x \tag{24}$$

in the phase plane, for the three values $\epsilon = 0.1, 1$ and 10 [143], through the graphical method of *isoclines*, i.e. drawing from the equation of the orbits of (24) in the phase plane (x, y)

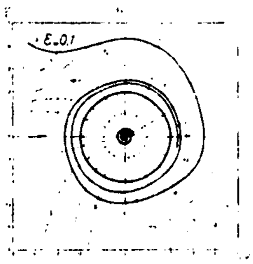
$$\frac{dy}{dx} = \epsilon(1 - x^2) - \frac{x}{y}$$

the curves of constant slope $\frac{dy}{dx}$, and connecting small lines on them having the corresponding slope to obtain the orbits.

When $\epsilon = 0.1$, this shows the presence of a unique cycle approximately circular, of radius close to 2, approached from inside and from outside by spiraling orbits, and corresponding to a unique stable quasi-sinusoidal time-periodic solution of eq. (21).



$\epsilon = 0.1$: TIME EVOLUTION



$\epsilon = 0.1$: PHASE PLANE

VAN DER POL's analytical approach given in [143] to deduce heuristically this result is a version of the method of variation of constants in a nonlinear context, successfully developed by N.M. KRYLOV and N.N. BOGOLIUBOV [83] and their school, and called to-day the *averaging method*. Writing the solution in the form $v = a\sin(t+\phi)$, with a and ϕ supposed to depend also upon t, assuming that a and ϕ are slowly varying function of time, VAN DER POL found for a^2 the approximate differential equation

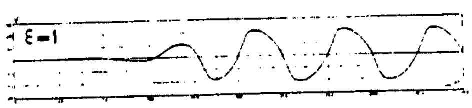
$$\frac{1}{\epsilon}(a^2)' - a^2 + \frac{1}{4}a^4 = 0$$

whose solutions

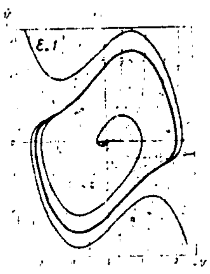
$$a^2 = \frac{4}{1 + e^{-\epsilon(t+C)}}$$

provide oscillating amplitudes with limiting value 2.

When $\epsilon = 1$, one still observes a unique cycle of less regular shape, approached from inside and from outside by spiraling orbits. This corresponds to a unique stable time-periodic solution of eq. (21) containing higher harmonics.



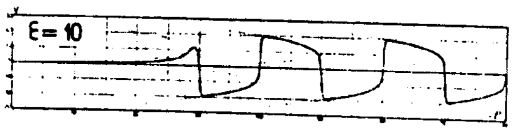
$\epsilon = 1$: TIME EVOLUTION



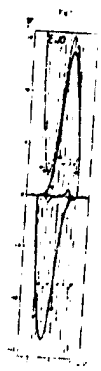
$\epsilon = 1$: PHASE PLANE

Finally, when $\epsilon = 10$, a unique cycle of irregular shape is observed, rapidely approached from inside and from outside by the other (non-spiraling) orbits.

It corresponds to a unique stable time-periodic solution of eq. (21) close to a repeating sequence of quasi-stationary states separated by fast transitions. VAN DER POL called this type of solutions *relaxation oscillations*. The reason of this terminology is that the period of the limit cycle for $\varepsilon > 0$ large is shown to be approximately equal to ϵ, i.e., if one returns to the original equation (20), to RC, which is a time of relaxation.



$\epsilon = 10$: TIME EVOLUTION



$\epsilon = 10$: PHASE PLANE

VAN DER POL observed in [143] that the type of oscillation described in 1919 by HENRI ABRAHAM and EUGÈNE BLOCH [2] was of relaxation type and provided a mathematical analysis of their *multibrator*. He concluded by noticing that :

> finally, it seems quite likely that, when the total characteristic [...] is taken in account, the well-known vibration of a neon-tube connected to a resistance and condenser in shunt may be similarly treated under the heading of relaxation-oscillations. Similarly, (though no detailed investigation has been carried out) it is likely that the oscillation of a "Wehnelt" interrupter belongs to the general class of relaxation-oscillations and perhaps also heart-beats.

As we will see later, VAN DER POL has soon refined his last prediction. In 1927 [144], he showed that relaxation oscillations occur also in a tetrode-multivibrator, in some circuits considered by E. FRIEDLANDER [50], and in GÉRARD-LESCUYER experiment with dynamo and engine. At this occasion, VAN DER POL quoted for the first time, without comment, the papers of JANET [77] and of CARTAN [28] mentioned below, as well a a book by BUSCH [25].

5 Proving the existence of self-sustained oscillations

5.1 Cartan's family and Janet's equation

In 1925, the French mathematicians ELIE CARTAN (1869-1951) and HENRI CARTAN (1904-2008) (father and son) published their unique joint paper as well as their unique paper devoted to a question of applied mathematics [28]. At this time, HENRI CARTAN was still a student at the *École Normale Supérieure*.

E.CARTAN (1869-1951) H.CARTAN (1904-2008)

The paper appeared in the *Annales des Postes et Télégraphes* (where very few mathematicians ever published) without any reference, and proved the existence of at least one self-sustained oscillation for a differential equation of the form

$$Li'' + [R - \varphi(i)]i' + \frac{1}{C}i = 0, \tag{25}$$

where L, C, R are positive constants, $\varphi(0) > R$, φ is even and decreasing to 0 when i *to* $+\infty$. They did not use phase plane techniques but an analytical method based upon the study of the oscillatory behavior of the solutions. From the structure of the equation, they first showed that any non-equilibrium solution must oscillate, with positive maximums and negative minimums. They showed also that, given a minimum $-i_1 < 0$ of a solution of eq. (25), the next maximum i_2 is an increasing continuous function of i_1, with $i_2 > i_1$ for sufficiently small i_1 and $i_2 < i_1$ for sufficiently large i_1. The intermediate value theorem implies the existence of at least one i_1^* such that $i_2^* = i_1^*$, which, because of the evenness of φ, corresponds to a periodic solution of eq. (25).

If the corresponding i_1^* are ordered by increasing values, E. and H. CARTAN proved that the self-sustained oscillations with an odd order number are stable and the other ones unstable. They found lower and upper bounds for the amplitudes and the periods and leaved as an open problem the question of the *uniqueness* of the self-sustained oscillation. Their approach was extended in 1940 by DARIO GRAFFI [61] to equations of the form

$$x'' + h(x, x') + g(x) = 0. \tag{26}$$

More general results for eq. (26) were obtained in 1942 by NORMAN LEVINSON and O.K. SMITH [91] using Poincaré-Bendixson's criterion. The same approach was used in 1949 by JOSEPH P. LASALLE [84] to study the relaxation oscillations associated to Liénard equation. For higher order equations, for example third order ones important in electronics and astrophysics, Poincaré-Bendixson's criterion has been replaced by BROUWER fixed point theorem [27, 33, 51, 119].

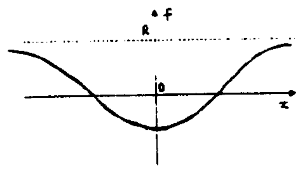
The total absence of references in [28] is explained by the fact that the paper was preceded, in the same issue of the *Annales*, by a short introductory one of JANET [77] starting as follows :

> L'intéressant travail mathématique de MM. Cartan que l'on va lire a pour origine une très ancienne expérience d'électricité appliquée qui a été réalisée dès 1880. [...] L'expérience montre que dans ces conditions il s'établit un régime oscillatoire très stable et parfaitement déterminé. Il serait intéressant de prédéterminer, en fonction des données du problème, la période et l'amplitude des oscillations. C'est là le but du travail de MM. Cartan.
>
> [The interesting mathematical work of MM. Cartan that one will read originates from a very old experiment of applied electricity realized in 1880. [...] The experiment show that, in those conditions, a very stable and perfectly determined oscillatory regime is established. I would be interesting to determine in advance, in function of the data of the problem, the period and the amplitude of the oscillations. This is the aim of the work of MM. Cartan.]

After recalling his equation (17) of 1919, JANET precised the shape of the function $f(i)$, an assumption retained by É. and H. CARTAN who wrote $\varphi(i) = \frac{dF}{di}(i)$. In JANET's own words :

> La fonction dF/di, comme il est facile de le voir, jouit des propriétés suivantes : Elle ne change pas quand on change i en $-i$, passe, pour $i = 0$ par un maximum que nous supposerons plus grand que $R > 0$ (ce qu'on peut toujours obtenir en faisant R suffisamment petit et en donnant à la dynamo-série une vitesse angulaire suffisante) et s'annule pour $i = \pm\infty$.
> [The function dF/di, as can easily be seen, has the following properties : it does not change when on replaces i by $-i$, has a maximum at $i = 0$ that we assume larger than $R > 0$ (which can always be obtained by taking R sufficiently small and giving to the dynamo-series a sufficiently large angular velocity) and vanishes for $i = \pm\infty$.]

The picture below gives the shape of the function $f(x) := R - \varphi(x) = R - \frac{dF}{dx}(x)$.



JANET'S DAMPING COEFFICIENT FUNCTION

JANET concluded as follows :

> On sait que l'on trouve des équations de la même forme que (17) dans un grand nombre de phénomènes d'oscillations entretenues (arc chantant, oscillateur à lampe, etc...). Toute contribution à l'étude de cette équation sera donc fort utile, et nous devons remercier MM. Cartan de lui avoir consacré l'intéressant travail que l'on va lire.
> [It is known that one find equations of the same form as (17) in a large number of phenomena of self-sustained oscillations (singing arc, lamp oscillator, etc ...). Any contribution to the study of this equation will therefore be very useful, and we must thank MM. Cartan to have devoted to it the interesting work that we will read.]

Notice that JANET did not quote any of the earlier contributions of VAN DER POL in this direction.

5.2 Liénard : unification and generalization

JANET's wish was already realized in 1928, by the French mathematical physicist ALFRED LIÉNARD (1869-1958). His important contribution in the *Revue générale d'électricité* [92] starts as follows :

> Dans une note publiée dans les "Annales des Postes, Télégraphes et Téléphones" [77], M. P. Janet a attiré l'attention sur l'intérêt que présente l'étude d'une certaine équation différentielle qui régit plusieurs phénomènes physiques oscillatoires (entre autres, arc chantant, lampes à plusieurs électrodes, etc.). [...] L'étude de l'équation qui régit les oscillations entretenues a déjà fait l'object de plusieurs mémoires, entre autres un de MM. Élie et Henri Cartan [28] et un autre de M. van der Pol [143]. Les hypothèses faites par ces deux auteurs ne sont pas les mêmes. [...] Je me suis proposé, dans cette étude, de pousser plus loin les résultats obtenus par les précédents auteurs et d'obtenir des déductions valables pour des cas plus généraux. [...] J'emploie une méthode géométrique qui se rapproche de celle de M. van der Pol en conservant la rigueur des raisonnements de MM. É. et H. Cartan.
> [In a note published in the 'Annales des Postes, Télégraphes et Téléphones' [77], M. P. Janet has called the attention on the interest to study some

differential equation which describes several oscillatory physical phenomena (among others, singing arc, lamp with several electrodes, etc...). [...] The study of the equation describing the sustained oscillations has already been the object of several memoirs, among others one of MM. Élie and Henri Cartan [28] and another one of M. van der Pol [143]. The assumptions made by those authors are not the same ones. [...] I proposed myself in this study to push further the results obtained by the preceding authors and to obtain deductions valid for more general cases. [...] I make use of a geometrical method close to that of M. van der Pol but keeping the rigor of the reasonings of MM. É. and H. Cartan.]

ALFRED LIÉNARD (1869-1958)

The geometric method is the use of some phase plane analysis (without any reference to POINCARÉ) and the considered equation is

$$x'' + f(x)x' + x = 0, \tag{27}$$

where f is even, $F(x) := \int_0^x f(s)\,ds < 0$ for $0 \leq x < r$ and some $r > 0$, and $F(x) > 0$ is increasing for $x > r$. It is easy to show that those assumptions cover the cases considered by VAN DER POL and JANET-CARTAN.

LIÉNARD proved that, under those assumptions, eq. (27) has a unique closed orbit. His idea consisted in writing eq. (27) in the form

$$[x' + F(x)]' + x = 0,$$

leading immediately to the equivalent planar system

$$x' = y - F(x), \quad y' = -x$$

whose orbits in the plane (x, y) (now refered often as *Liénard's plane*) are given by the differential equation

$$\frac{dy}{dx} + \frac{x}{y - F(x)} = 0.$$

Like É. and H. CARTAN, LIÉNARD also obtained a number of estimates for the amplitude and period of the corresponding periodic solution, including the case of relaxation oscillations.

The physical interpretation of the geometric approach of LIÉNARD is well described by NICHOLAS MINORSKY in [107] :

> A physical interpretation of Liénard's criterion resulted in the relationship between the periodicity of a solution and a special condition of energy exchanges. In fact, the fundamental point of Liénard's theory reduces the question of periodicity to the vanishing of a certain curvilinear integral. This integral turns out to be the one which specifies the energy exchanges between the oscillating system and the outside sources and. on this basis, Liénard's criterion acquired the very simple interpretation that a stationary state is reached when the energies absorbed and dissipated during one period cancel out.

LIÉNARD's paper has been the starting point of a very vast literature (see for example [30, 121, 131]). Eq. (27) is called now *Liénard equation*, and LIÉNARD's approach was developed, among others, by the French mathematician JULES HAAG (1882-1953) [67], with special emphasis upon relaxation oscillations and their asymptotic developments. See also [43, 49, 129].

5.3 Self-sustained oscillations and limit cycles

One year after the publication of LIÉNARD's paper, the Russian mathematician and physicist ALEKSANDR A. ANDRONOV (1901-1952) published an important note in the *Comptes rendus* of the French Academy of science [5].

ALEKSANDR A. ANDRONOV (1901-1952)

In this short note, ANDRONOV identified for the first time the self-sustained oscillations in systems with one degree of freedom modeled by equations of the form (15) to the stable limit cycles of POINCARÉ. In his own words :

> Les oscillations dites *auto-entretenues* [...] sont régies par des équations différentielles qui diffèrent de celles qu'étudient la physique mathématique et la mécanique classique. Les systèmes où se produisent ces phénomènes ne sont pas conservatifs et entretiennent leurs oscillations en puisant l'énergie à des sources non périodiques. [...] Considérons le cas le plus simple des oscillations que présente – en mécanique et en physique – un système à un degré de liberté, en chimie une réaction entre deux substances, en biologie deux espèces animales coexistantes. Ces systèmes peuvent être représentés par [...]
>
> $$(A) \qquad \frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y).$$

[...] Exigeons, en nous basant sur l'étude des phénomènes de ce genre effectivement observés, que les mouvements périodiques que nous considérons soient stables, par rapport à des variations arbitraires suffisamment petites : 1) des conditions initiales; 2) des seconds membres des équations (A). On peut facilement montrer qu'aux mouvement périodiques satisfaisant à ces conditions correspondent, sur le plan xy, des courbes fermées *isolées*, dont s'approchent en spirales, de l'intérieur et de l'extérieur (pour t croissant), les solutions voisines. Il en résulte que *les auto-oscillations qui naissent dans les systèmes caractérisés par des équations du type (A) correspondent mathématiquement aux cycles limites stables de Poincaré.*
[The so-called *self-sustained* oscillations [...] are governed by differential equations differing from those studied in mathematical physics and classical mechanics. The systems where those phenomena are produced are not conservative and sustain their oscillations by taking energy from non periodic sources [...]. Let us consider the simplest case of oscillations associated – in mechanics and physics – to a system with one degree of freedom, in chemistry to a reaction between two substances, in biology to two coexisting animal species. Those systems can be represented by [...]

$$(A) \qquad \frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y).$$

[...] Let us impose, following the study of this type of phenomena effectively observed, that the periodic motions that we consider must be stable, with respect to sufficient small arbitrary variations : 1) of the initial conditions; 2) of the right-hand members of equations (A). It is easy to show that to the periodic motions satisfying those conditions correspond, in the plane xy, to *isolated* closed curves, approached, from inside and outside (for increasing t) by spiraling neighboring solutions. It follows that *the auto-oscillations occuring in systems characterized by equations of type (A) mathematically correspond to the stable limit cycles of Poincaré.*]

In 1935, A.G. MAIER [98] proved the existence of limit cycles for the Rayleigh and van der Pol's equations using POINCARÉ's techniques.

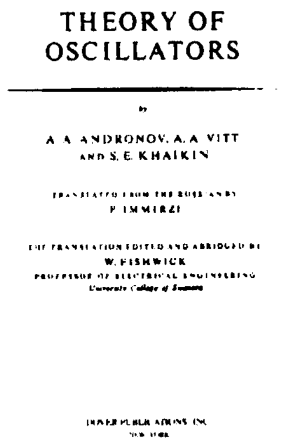
ANDRONOV also introduced the use of the rigorous methods, developed in 1892 by POINCARÉ in his book *Méthodes nouvelles de la mécanique céleste* [117], to prove the existence of limit cycles and to localize them in systems of the form

$$x' = y + \epsilon f(x,y,\epsilon), \quad y' = -x + \epsilon g(x,y,\epsilon)$$

when ϵ is small. Those methods could replace the heuristic ones precedently used by VAN DER POL and others. Notice that, in this direction, ANDRONOV had been preceded by a little noticed paper of W.M.H. GREAVES in 1923 [65]. He has been followed by a large number of authors (see references in [30]).

ANDRONOV's paper was an important impetus for the development, in particular in Soviet Union under the impulsion of LEONID I. MANDELSTAM (1879-1944) (see e.g. [37, 42, 99, 108, 114]) and, during and after the Second World War, in the U.S.A., under the impulsion of SOLOMON LEFSCHETZ (1884-1972) (see e.g. [34]), of the mathematical *theory of nonlinear oscillations*, also sometimes called *nonlinear mechanics*. Both mathematical, physical and technical aspects are nicely presented in the 'bible of nonlinear oscillations', namely the monograph *Theory of oscillators* of ALEKSANDR A. ANDRONOV, ALEKSANDR A. VITT and SEMON E. KHAIKIN [9], whose first edition, published in Soviet Union in 1937 and its English partial translation of 1949 [8], beared only the name of ANDRONOV and KHAIKIN, as VITT was in disgrace under STALIN's regime and died in jail in 1938. One can read, in the preface of the second edition (published in Russian in 1965 and translated in English in 1966), under the signature of KHAIKIN, that

> the writer of this Preface is the only one of the three authors of this book who is still alive. Aleksandr Adol'fovich Vitt, who took part in writing the first edition of this book equally with the other two authors, but who by an unfortunate mistake was not included on the title page as one of the authors, died in 1937. Alexandr Aleksandrovich Andronov died in 1952, i.e. fifteen years after the first edition of the book was published.


THEORY OF
OSCILLATORS

A. A. ANDRONOV, A. A. VITT
AND S. E. KHAIKIN

W. FISHWICK

THE 'BIBLE' OF NONLINEAR OSCILLATIONS

One can consult the papers [12, 40, 41, 53, 63, 74, 100, 105, 130, 145, 149] and the monographs [24, 30, 64, 67, 103, 104, 106, 110, 121, 123, 131, 136, 137] for more technical and bibliographical details about the early history of the theory of nonlinear oscillations. We have also included in the bibliography a number of direct references to contributions made in Western Europe before 1955, to show that, contrary to what has been written in some recent histories, the theory of nonlinear oscillations was also developed in Western Europe, in the first half of the XXth century, by a number of mathematicians, physicists and engineers [1, 10, 14, 21, 26, 29, 38, 41, 46, 54, 85, 102, 105, 111, 115, 132, 133, 134, 147, 150, 152]. The corresponding history remains to be done.

5.4 Science needs time

The milestones of the early history of self-sustained oscillations and limit cycles of nonlinear differential equations can be summarized as follows :

- Devices producing auto-oscillations exist since Antiquity.
- In 1881, POINCARÉ defined and studied the concept of limit cycle of a planar differential system.
- In 1883, RAYLEIGH proposed the second order nonlinear differential equation (10) to model self-sustained oscillations.
- In 1919, JANET modeled some electrotechnical self-sustained oscillations with eq. (17).
- In 1920, VAN DER POL modeled an oscillating circuit containing a triode with eq. (21).
- In 1925, ÉLIE and HENRI CARTAN proved the existence of a self-sustained oscillation for JANET's equation (17).
- In 1928, LIÉNARD proved the existence of a self-sustained oscillation for a class of equations (27) containing both VAN DER POL's and JANET' ones.
- 1929 : ANDRONOV identified the self-sustained oscillations with POINCARÉ's limit cycles.

This shows that it took some fifty years to first class scientists to identify an existing mathematical theory with the solutions of differential equations modeling self-sustained oscillations.

Science needs time and interdisciplinarity cannot be decreed : a good lesson for those modern 'deciders' in science who have a tendency to confuse so often research with development.

6 The universality of self-sustained oscillations

6.1 Heartbeats and relaxation

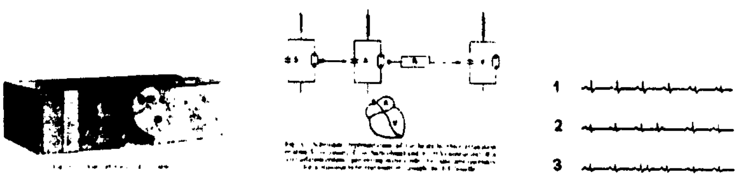
In 1928, VAN DER POL and J. VAN DER MARK, inspired by their work on relaxation oscillations and by the analogy of shape between them and electrocardiograms, constructed an electronic model of the heart, nicely described in their paper [146] :

> We will consider the heart as a system of three degrees of freedom : the *sinus*, the *atrium (auriculum)* and the *ventriculum*. [...] It will be obvious that we exclude at the outset those phenomena of the heart, which in the language of mathematicians can only be described by partial differential equations. [...] We therefore shall not consider flutter, or fibrillation as

> these phenomena are directly connected with progressing and standing waves. [...] We consider each [degree of freedom] able to perform a relaxation oscillation by itself, each of the three having its own natural period. Moreover a coupling exists between the sinus and the auriculum, the former acting on the latter. Another coupling exists between the auriculum and the ventriculum, which coupling is realized through the existence of the bundle of HIS. [...] In the normal heart these couplings have [...] a unidirectional character. [...] In an electrical model [...] these two couplings are therefore represented by two triodes [...] inserted to provide an unidirectional instrument.

VAN DER POL and VAN DER MARK used systems involving intermittent discharge of a neon-tube [50, 132] to obtain the required relaxation oscillations. They obtained responses to their system which exhibit

> a very striking similarity with the electrocardiograms taken from the human heart [...] beating in the normal way, [...] showing a *ventricular extra systole* given shortly before an impulse from the atrium arrives, [...] an *interpolated ventricular systole*, [...] an *auricular extrasystole*, [...] a typical case of *sino-auricular block*.



ELECTRONIC MODEL OF THE HEART, STRUCTURE, OUTPUT

They concluded that

> the very close analogy between the working of our model and the beating of a mammalian heart leaves no doubt that the view expressed [...] regarding the heart beat as a relaxation oscillation is correct. [...] In conclusion we give some further possible disorders mainly obtained mathematically and which were all verified with the aid of our model. Possibly these anomalies either have or will be found in the human heart as well.

Since the pioneering work of VAN DER POL AND VAN DER MARK, more and more sophisticated models of the heart have been developed, including ones with infinitely many degrees of freedom (see e.g. [112, 140, 159, 156] and the references therein).

6.2 They see self-sustained oscillations everywhere

Motivated by their success, VAN DER POL and VAN DER MARK identified self-sustained oscillations in a large number of phenomena [146] :

> Hence, with the Aeolian harp, as with the wind blowing against telegraph wires, causing a whistling sound, [...] many other instances of relaxation oscillations can be cited, such as : a pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the humming noise sometimes made by a watertap, the squeaking of a door, a steam engine with a much too small flywheel, the multivibrator of ABRAHAM and BLOCK [sic], the tetrode multivibrator, the periodic sparks obtaining while turning a WHIMSHURST machine, the WEHNELT interruptor, the intermittent discharge of a condenser through a neon-tube, the periodic reoccurrence of epidemics and of economical crises, the periodic density of an even number of species of animals, living together and the one species serving as food to the other, the sleeping of flowers, the periodic reoccurrence of showers behind a depression, the shivering from cold, the menstruation and finally the beating of the heart.

The authors were just too optimistic concerning the mathematical models for the predator-prey species proposed independently around 1925 by the Italian mathematician VITO VOLTERRA (1860-1940) and the American scientist ALFRED LOTKA (1880-1949). In this case, the planar system of two first-order nonlinear differential equations does not exhibit limit cycles but, because of the presence of a first integral, a family of concentric cycles qualitatively similar to the ones encountered in the undamped harmonic oscillator, with shapes differing from circles or ellipses and periods depending upon the amplitude, because of the presence of the nonlinearities.

Similarly, ANDRONOV identified self-sustained oscillations in the pulsations of Cepheid stars (see also [3, 82, 88, 125, 141, 153, 158]), in FROUDE's pendulum (simple pendulum suspended in a rotating shaft described in [120]) and in periodic reactions in chemistry observed by KREMANN [81] and LOTKA [95] (see also [15, 90, 154, 157]). Auto-oscillations have been shown since to be important, for example, in neurology [48], physiology [151], hydraulics [122], car mechanics [122], control [17, 23, 97, 135], geology [40], chronometry [68], quantum mechanics [4, 139], thermomechanics [109], population dynamics [101], biology [31, 55], and oceane circulation [78].

6.3 Do relaxation oscillations explain the recurrence of economic cycles ?

Since the XIXth century at least, the presence of economic or business cycles (of various periods) had been observed by economists and the tentative explanations were about as numerous as their authors. Astronomical or meteorological explanations were invoked, before trying to find internal economic ones, like over-investment, psychological factors, monetary factors, underconsumption or sudden equilibrium disturbances. Mathematical models involving linear differential equations were unable to provide convincing models.

In 1928 already, the Dutch economist LODEWIJK HAMBURGER (1890-?) suggested to associate relaxation oscillations to the periodic recurrence of economic crises. He told the story as follows in [71] :

> The present writer would like to point out that the applicability of the principle of relaxation-oscillations to economic cycles was first emphasized by him in 1928 (May 7, Meeting of the Batavian Society of Logic Empirical Philosophy) in a discussion following a paper read by Messrs. van der Pol and J. Van der Mark on "The Heartbeat considered as a relaxation-oscillation, and an electrical model of the heart" (see e.g. *Arch. Néerland. de Physiologie de l'homme et des animaux*, 3[e] livraison, p. 418 (1929)). This suggestion, at first somewhat ridiculed even by the pioneer in the field of relaxation phenomena, was subsequently corroborated by results indicated in my paper [70]. A French version of this paper appeared January 1931.

HAMBURGER wrote in 1930 [70] :

> Only the conception of economical cycles as relaxation oscillations can give a rational and sufficient basis for an explanation of those important phenomena.

In 1930, VAN DER POL, who, as we have seen, had first rejected HAMBURGER's suggestion, wrote in this respect in the *Onde électrique* [145] :

> The last word surely has not been said, and many questions remain subject to discussions.

In 1933, the French scientist PHILIPPE LE CORBEILLER again suggested in *Econometrica* [87] to apply relaxation oscillations to the theory of economical crises :

> Le problème des crises, et plus généralement des oscillations des prix, est assurément l'un des plus difficiles de l'Économie politique; il ne sera sans doute pas de trop, pour approcher sa solution, de la mise en commun de toutes les ressources de la théorie des oscillations et de la théorie économique. C'est pourquoi j'ai pensé pouvoir vous présenter un compte-rendu succinct d'une avance récente, que je crois importante, de la théorie des oscillations : celle apportée au problème des systèmes autoentretenus par la découverte des oscillations de relaxation, due à un savant hollandais, le Dr. Balth. van der Pol.
> [The problem of the crises, and more generally of the oscillations of prices, is surely one of the most difficult ones of political economics : it will surely not be superfluous, to approach its solution, to put together all the resources of the theory of oscillations and of economic theory. This is why I

thought useful to present a short account of a recent progress, that I believe to be important, of the theory of oscillations : the one made to the problem of self-sustained oscillations through the discovery of relaxation oscillations, due to a Dutch scientist, Dr. Balth. van der Pol.]

LE CORBEILLER's paper was followed by a priority claim of HAMBURGER in the same journal in 1934 [71]. Some early discussions on the use of relaxation oscillations in economics can already be found in the monograph *Théorie des oscillations* of the French physicist YVES ROCARD quoted in [123]. In 1939, the future first Nobel Prize in Economics, PAUL A. SAMUELSON (1915-2009), then a very young researcher, wrote this prophetic statement about the need of nonlinear models leading to auto-oscillations [126] :

> There remains one interesting problem still to be explored. Mathematical analysis of the nonlinear case may reveal that for certain equilibrium values of α and β a periodic motion of definite amplitude will always be approached regardless of initial conditions. Such a relation can never result from systems of difference equations with constant coefficients, involving assumptions of linearity. This illustrates the inadequacy of such assumptions except for the analysis of small oscillations.

Despite of the scepticism expressed by some economists like W. FELLNER [47] in 1956 :

> One cannot expect that a simple model can give a realistic picture of the course of the events in real economics,

the quest for mathematical models for the recurrence of economic crises has continued. We can consider, for example, the series of models successively proposed by the American mathematical economist RICHARD M. GOODWIN (1913-1996) between 1951 and 1990.

RICHARD M. GOODWIN (1913-1996)

In 1951, GOODWIN [57] introduced a model leading to relaxation oscillations and RAYLEIGH's equation, as a refinement of some simpler models all based upon the simple multiplier and accelerator principles :

The first [model] is a threshold oscillator in which the economy once started up continuous until it removes the capital deficiency which started it, and then it goes down until it removes the excess deficiency with which it started downward. The second model introduces a simple linear trend (which is more important to nonlinear systems than to linear) which makes it unnecessary to await the wearing out of all the capital from the preceding boom before beginning the coming one. A third model consists of a combination of a dynamical accelerator and a less crude form of the nonlinear accelerator. This gives a more complicated evolution, but it still contains sudden shifts from declining to rising income and the reverse. This unreality is eliminated in the final model by taking account of the lag between investment decisions and the resulting outlays.

This final model consists of a consumption function, an investment function, and an accounting identity

$$\begin{aligned} c(t) &= \alpha y(t) - \epsilon y'(t) + \beta(t) \\ k(t) &= \phi(y'(t-\theta)) \\ y(t) &= c(t) + k'(t) + l(t), \end{aligned} \tag{28}$$

where $c(t)$ is consumption, $y(t)$ the income, $k'(t)$ is the induced investment (the derivative of that portion of the stock of capital whose change is determined endogenously), $l(t)$ the autonomous investment, and $\beta(t)$ is an autonomous component of consumption expenditure. $\alpha < 1$ is a dimensionless coefficient, $\epsilon > 0$ a constant with dimension of time, t the time, θ a time lag. The induced investment function $\phi(s)$ is the piece-wise linear function defined as follows (with respect to an acceleration coefficient κ, lower limit $\underline{\phi} < 0$ and upper limit $\overline{\phi} > 0$):

$$\phi(s) = \begin{cases} \underline{\phi} & if \quad s < \underline{\phi}/\kappa \\ \kappa s & if \quad \underline{\phi}/\kappa \leq s \leq \overline{\phi}/\kappa \\ \overline{\phi} & if \quad s > \overline{\phi}/\kappa \end{cases}$$

The system (28) reduces to the equation

$$\epsilon y'(t) + (1-\alpha)y(t) = \phi(y'(t-\theta)) + \beta(t) + l(t),$$

and GOODWIN assumed $\beta(t) + l(t)$ to be constant (non-progressive economy) and replaced $y(t)$ by its deviation $z(t)$ with respect to the unstable equilibrium value $\frac{\beta+l}{1-\alpha}$ to obtain, after a shift of time,

$$\epsilon z'(t+\theta) + (1-\alpha)z(t+\theta) = \phi(z'(t)).$$

From this differential-difference equation, he then deduced, by replacing the first two terms by their Taylor first order approximation, the ordinary differential equation

$$\epsilon[z'(t) + \theta z''(t)] + (1-\alpha)[z(t) + \theta z'(t)] = \phi(z'(t))$$

which, as he observed, is of Rayleigh type

$$\epsilon\theta z''(t) + \psi(z'(t)) + (1-\alpha)z(t) = 0, \tag{29}$$

with

$$\psi(s) = [(1-\alpha)\theta + \epsilon]s - \phi(s).$$

Auto-oscillations are expected if the equilibrium is unstable, i.e. if

$$[(1-\alpha)\theta + \epsilon] < \kappa,$$

in which case

$$\psi(s) = (1-\alpha)\theta + \epsilon - \kappa < 0 \quad for \quad \underline{\phi}/\kappa \leq s \leq \overline{\phi}/\kappa,$$

and $\psi(s)$ tends to $(1-\alpha)\theta + \epsilon$ when $s \to \pm\infty$. So we have a negative damping for $|z'|$ small and a positive one for $|z'|$ large. According to GOODWIN :

> the system oscillates with increasing violence in the central region, but as it expands into the outer regions, it enters more and more into an area of positive damping with a growing tendency to attenuation. It is intuitively clear that it will settle down to such a motion as will just balance the two tendencies, although proof requires the rigorous methods developed by Poincaré. It is interesting to note that this is how the problem of the maintenance of oscillation was originally conceived by Lord Rayleigh and that our equation is of the Rayleigh, rather than the van der Pol type. The result is that we get, instead of a stable equilibrium, a stable motion. [...] Therefore, making only assumptions acceptable to most business cycle theoretists, along with two simple approximations, we have been able to arrive at a stable, cyclical motion which is self-generating and self-perpetuating.

An electrical analog of GOODWIN's model was introduced by R.H. STROTZ, J.C. MCANULTY and J.B. NAINES JR in 1953 [138]. Earlier models, like N. KALDOR's one of 1940 [79] have also been reinterpreted in terms of limit cycles [32, 80].

In 1967, GOODWIN introduced a new model [58] to describe the business cycles in the Marxian model of a capitalist economy. This time, the construction of the model led him to a system of two differential equations of the first order

$$u' = u[-\alpha + \beta v], \quad v' = v[\gamma - \delta u] \tag{30}$$

for the share of labor in national income u, and the proportion of labour force employed v. Such a system, as already mentioned, had been introduced independently by LOTKA and VOLTERRA around 1925 to describe the evolution of two animal species, one prey and one predator. Curiously, if we remember some of the arguments given by GOODWIN in his preceding model, such a system does not lead to one limit cycle, but to a family of closed orbits like in the case of the (linear) harmonic oscillator. Hence the amplitude, and here also the frequency, of the corresponding periodic solutions depend upon the initial conditions. In contrast to the preceding one, and like the harmonic oscillator, the structure of the orbits of this model is not *structurally stable* : arbitrary small perturbations can destroy all closed orbits.

This criticism was immediately made by SAMUELSON [127], who observed that GOODWIN's new model contained no notion of diminishing returns, and ceased to admit periodic solutions as soon as diminishing returns were included. A more robust sort of periodicity results is obtained if a highly nonlinear (e.g., cubic) term is added to the original Volterra-Lotka equations, to account for diminishing returns to scale, converting the equations to a Rayleigh-van der Pol-type system, with stable limit cycle independent of the initial conditions or the exact values of the system parameters.

PAUL A. SAMUELSON (1915-2009)

In 1990, GOODWIN [59] returned to the problem in the new setting a chaotic dynamics by adding a control variable z, which could measure the innovation process or public expenditure, to the two equations (30). He obtained a system of three first order differential equations of LORENZ-RÖSSLER type

$$u' = f(u,v), \quad v' = g(u,v,z), \quad z' = h(v,z) \tag{31}$$

for which it is well known that, in contrast to planar systems, the class of attractors can contain, besides equilibria and cycles, more complicated *strange attractors* leading to *chaotic* motions.

The question remains to know if those successive models of GOODWIN, although based upon economic variables and laws, where more influenced by the evolution of macroeconomics, than by some successive fashions in the theory of nonlinear ordinary differential equations : auto-oscillations, population dynamics, chaos theory and their paradigms. Another important question, not restricted to GOODWIN's models, is the real predictive value of such oversimplified models of a very complicated reality.

Of course, we have only concentrated our attention here on some models of business cycles based upon nonlinear ordinary differential equations having periodic or chaotic solutions. There was no claim to give a complete and fair account of the many other theories and approaches, and of their possible mathematical models, proposed in this controversial area. More information can be found, for example, in [18, 56, 94].

We conclude this section on mathematical models for business cycles by a recent wink of SAMUELSON [128] :

> To prove that Wall Street is an early omen of movements still to come in GNP, commentators quote economic studies alleging that market downturns predicted four out of the last five recessions. That is an understatement. Wall Street indexes predicted nine out of the last five recessions! And its mistakes were beauties.

7 Conclusions : the art of modeling

Modeling is a fundamental, unavoidable but difficult and delicate tool in science. Mathematical models should not be taken necessarily like ... models in the usual sense. The French mathematician GEORGES REEB (1920-1993), well known for his colorful pertinent statements, preferred, with less respect, to call a mathematical model, a *mimicry* ('singerie' in French).

A mathematical model always remains, without any pejorative sense, a caricature : it must give a maximum of resemblance in a minimum of strokes, and this is and remains an art. If the model is too complicated, it escapes to any serious mathematical treatment and computer simulation may be hazardous: if it is too simple, it may not represent adequately reality.

It is also important, as we have seen, to distinguish models obtained by analogy from models constructed on scientific laws. One must also resist to the temptation of retaining only the models predicting what is expected. The predictions made with a model describe its own evolution and reality is never forced to follow it. A permanent confrontation with results of reliable experiences or with carefully measured data remains absolutely necessary.

For more reflections and discussions on models and their role in science and society, see [12, 22, 35, 44, 73, 89, 96, 113].

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