Chapter 1

A survival kit in phase plane analysis: some basic models and problems

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Abstract The aim of this note is to show how phase plane analysis is a strong tool for the study of a mathematical model, in view of its application in water waves theory. This because only in recent work such method was actually used in water waves theory and people working in this field area might be interested in a discussion of the basic ideas of phase plane analysis, which we call "a survival kit".
In this light, at first we review some classical results in Dynamics of Population and Epidemiology, and then we investigate more carefully the phase portrait of the classical Liénard equation. In particular, starting from the Van Der Pol equation, the problem of existence and uniqueness of limit cycles will be treated and the methods used to attack this problem will be presented. Finally we come back to water waves theory and present in details the results of a joint paper with A. Constantin [1] in which, as far as we know, for the first time phase plane analysis was used in this kind of problems

1.1 Dynamics of population

Starting from the well known logistic equation, we will briefly discuss the competition between two species in an achological niche and examine in more details the classical predator-pray model.

1.2 The logistic equation

The simplest equation which models the growth of an isolated population is

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$$\dot{x} = \varepsilon x \tag{1.1}$$

Clearly in this case the rate of growth is proportional to the size of the population, and actually the growth rate is the positive coefficient ε. But this model it is not realistic, because gives an exponential growth. We must take in account limited resources as well as limited space. Hence, it is necessary an inhibitory effect when the population is large, and this leads to the so called logistic equation

$$\dot{x} = \varepsilon x - \sigma x^2 \tag{1.2}$$

which was proposed by the Belgian mathematician Pierre Francois Verhulst, who in 1838 introduced the solution of such equation in the theory of dynamics of population [33]. Here the inhibitory effect is given by the negative coefficient $-\sigma$. We can easily solve this equation, being actually a Bernoulli equation, but we prefer a simpler and qualitative approach. We just study the sign of the right member, namely $x(\varepsilon - \sigma x)$; it is trivial to see that $x = 0$ and $x = \varepsilon/\sigma$ are constant solutions, while for $0 < x < \varepsilon/\sigma$ solutions are monotone increasing and monotone decreasing for $x > \varepsilon/\sigma$. Hence we proved that the logistic equation has an asymptotically stable solution at ε/σ, and we know the qualitative behavior of all its solutions, without actually having solved the equation. The asymptotically stable solution $x = \varepsilon/\sigma$ is called saturation level or total carrying capacity.

For small values of x the curve solution is S-shaped and it is called sigmoid.

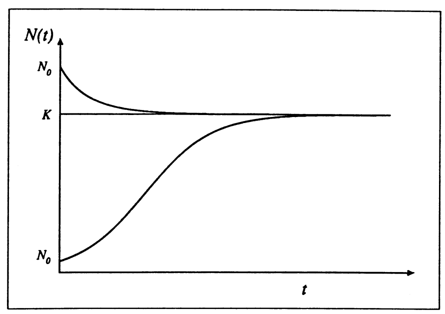

Fig. 1.1 The logistic equation

There are numerous experimental data for the growth of protozoa and bacteria that fit a sigmoid curve (see for instance the data concerning the Glaucoma Scintillans or the Paramecium Aurelia, cfr.Gause [12]).

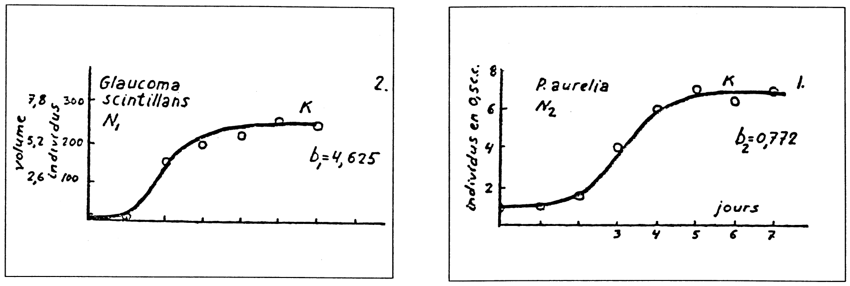

Fig. 1.2 Data fitting the model for Glaucoma Scintillans and P. Aurelia

1.3 Two species in competition and the "Principle of competitive exclusion"

The next step is the study of the differential system

$$\begin{cases} \dot{x} = \varepsilon_1 x - \sigma_1 x^2 - \alpha_1 xy = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ \dot{y} = \varepsilon_2 y - \sigma_2 y^2 - \alpha_2 xy = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases} \tag{1.3}$$

which describes the competition between two similar species for a limited resources as, for instance, food supply. We assume also that the two species share a limited territory. This environmental situation is called ecological niche. The inhibitory effect to the growth of a species due to the presence of the other is given by the coefficients α_i. In order to understand what will happen we must study the phase portrait of this planar nonlinear dynamical system in the first quadrant.

As usual we consider the critical points, the 0-isocline (that is the points in which $\dot{y} = 0$) and the ∞-isocline (that is the points in which $\dot{x} = 0$), because they play a crucial role in the construction of the phase portrait. At first we consider a phase portrait with 3 singular points, namely the origin, the point $A(\varepsilon_1/\sigma_1, 0)$ which gives the saturation level of the first species in absence of the second one, and the point $B(0, \varepsilon_2/\sigma_2)$ which gives the saturation level of the second species in absence of the first one. The 0-isocline is formed by the x-axis and the segment joining the points $A(\varepsilon_1/\sigma_1, 0)$ and $C(0, \varepsilon_1/\alpha_1)$, while the ∞-isocline is formed by the y-axis and the segment joining the points $B(0, \varepsilon_2/\sigma_2)$ and $D(\varepsilon_2/\alpha_2, 0)$.

Under the condition that we have 3 singular points, these 2 segments do not intersect each other and divide the first quadrant in 3 regions. It easy to see, with standard sign computations, that the ω-limit of any positive trajectory in the first quadrant is the singular point on the external segment. This means that only a species will survive tending to his saturation level, while the other must die out. This phenomenon is known as the "Principle of

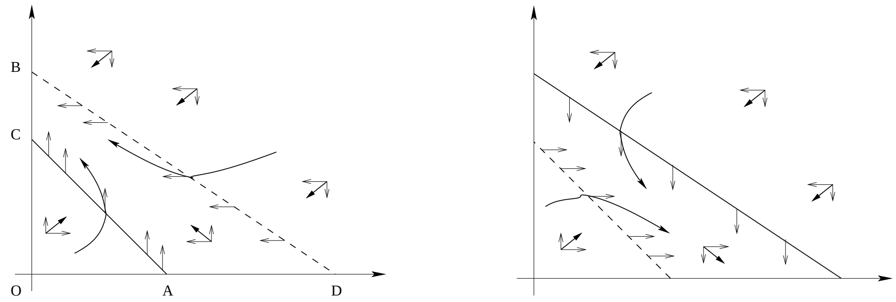

Fig. 1.3 The principle of exclusion. Only a species will survive, while the other must die out.

competitive exclusion". It was first enunciated, in a slight different form, by C. Darwin in 1859 in his celebrated paper "The origin of species by natural selection" [7], and it is a kind of axiom in ecology.

But if the above defined two segments intersect each other at a point P, it appears a fourth singular point P.

In order to understand what it is going on in this new situation it is necessary to investigate the stability of the point P. Using a standard linearization argument and after some straightforward calculations which we omit for sake of simplicity (a survival kit, by definition, cannot be "heavy") it is possible to see that if $\sigma_1\sigma_2 > a_1a_2$ the point P is asymptotically stable,and this leads to coexistence. On the other hand if $\sigma_1\sigma_2 > a_1a_2$ the point P is a saddle, and the separatrices of the saddle divide the first quadrant in two regions. Coexistence it is not possible and in one region the first species will survive and the other must die out, while in the other region the second species will survive, and the first must die out.

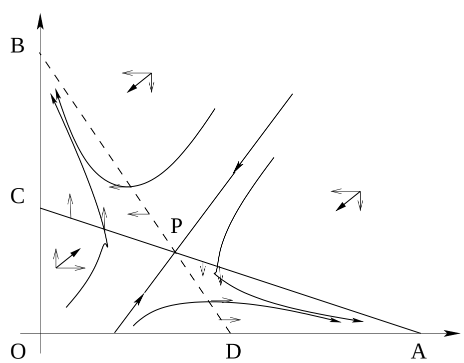

Fig. 1.4 A case in which only one species will survive depending from the initial conditions. But ecologists do not like it.

However coexistence, as well as the existence of different regions of supremacy, is in contrast with the "Principle of competitive exclusion" and therefore it is not accepted by the ecologists.
How can we fix this problem? We just observe that this is a problem, because the validation of a mathematical model depends on his applicability in the study of the real phenomenon. For this reason the first simple model was not acceptable because was giving an exponential growth of the population, while the logistic equation his supported by many experimental data. The answer to this question lies in the fact that we consider two species which are similar! "Mathematically" speaking this leads to the fact that the coefficients ε_i, σ_i and a_i are similar and therefore the two segments joining respectively the points $A(\varepsilon_1/\sigma_1, 0)$ and $C(0, \varepsilon_1/\alpha_1)$ and the points $B(0, \varepsilon_2/\sigma_2)$ and $D(\varepsilon_2/\alpha_2, 0)$ tends to be parallel and hence do not intersect.
In the model of the ecological niche the fourth singular point does not exist and the "Principle of competitive exclusion" has a rigorous mathematical proof, while if we consider the interaction between two species which are not similar the fourth singular point may arise leading the model to coexistence or different regions of supremacy.

1.4 The predator-pray model

This model plays a crucial role in the theory of dynamics of population and his surprising outcome was giving a decisive contribute in showing the importance of the application of phase portrait analysis in a mathematical model of a real phenomenon. We can say that after this intriguing result the whole theory of mathematical models was finally recognized in his importance and applicability, and for this reason we present this model in more details, giving also a brief story of the problem.

Right after the World War I, the Italian biologist Umberto D'Ancona was studying the population variation of varies species of fishes that were interacting each other in the Mediterranean Sea. He observed the total catch of selachians (as for instance Sharks or similar fishes) versus the total catch of food fish, taking some data from the port of Fiume (a city at that time in Italy and now in Croatia with the name of Rijeka). There was a clear large increase in the percentage of selachians during the war, as well as a decrease in the percentage of the food fish. How this strange phenomenon could be explained? As selachians are predators and their pray was the food fish, we are in front of a struggle for life between competing species. It was reasonable to assume the fact that the war was reducing fishing and therefore the number of selachians was actually increasing. But why this is not happening for the pray, namely the food fish. And this is an important question because the food fish is actually also our food. No one, with very few exceptions, likes to eat sharks. The father in low of D'Ancona was the famous Italian mathematician

Vito Volterra, and was natural to discuss this issue with him. And now it is time to present his model, which appeared in 1927 not in a mathematical journal but in a journal of marine biology [39]. We also note that in the introduction of this paper Volterra gives probably the best definition of a mathematical model and why this leads to differential equations.

Consider the following system, in the HP-plane

$$\begin{cases} \dot{P} = -bP + \beta HP \\ \dot{H} = aH - \alpha HP \end{cases} \tag{1.4}$$

where P denotes the predator and H the pray and the coefficients α, β, γ, δ are all positive.

Let us briefly discuss this system. In absence of predator the pray grows without limit, because there is not a logistic term. This is not a problem, because the number of pray is reduced by the predators with the rate of predation given by the coefficient α. On the other hand, in absence of pray, the predator die out, but this cannot occur in virtue of the positive coefficient β which measures the rate of growth due to the predation. We observe that actually $\beta = \gamma\alpha$, were γ is the so called "biological gain". There is a clear difference if the struggle for life involves lions and zebras instead of whales and plankton.

Again, we consider the critical points, the 0-isocline and the ∞-isocline. There are 2 critical points: the origin and the point $E(b/\beta,\, a/\alpha)$. The 0-isocline is formed by the H-axis and the line $H = b/\beta$.

The ∞-isocline is formed by the y-axis and the line $P = a/\alpha$. Clearly such lines intersect each other at the point $E(b/\beta,\, a/\alpha)$ and trajectories are counterclockwise around it. Clearly the origin is a saddle points and, after a linearization, the point $E(b/\beta,\, a/\alpha)$ is a center. This is a problem, because it is well known in the theory of linearization of almost linear systems that if the linearization gives a center any arbitrary small perturbation can change the stability.

This means that the point can actually remain a center, but can became asymptotically stable, or asymptotically unstable as well. This fact will play a crucial role when we will prove that for a linear water wave no particle trajectory is closed unless the free surface is flat, which is the main motivation of these notes. Coming back to our system, this problem was actually solved by Volterra with a very elegant and smart proof which we omit, in order to keep the survival kit "light".

Hence we can say that all the trajectories in the first quadrant are cycles and therefore periodic. This result is interesting, but in principle it can be inferred without the help of phase portrait analysis, arguing as follows. Let us start at a generic point, which gives a certain amount of predators and pray (clearly such amount is to be considered in terms of "biomass", otherwise one must explain what π predators or $\sqrt{2}$ prays mean!). Predators are eating the prays and in this way reducing their number. But in this way at a certain moment there will be a lack of pray. Predators are starving and, with no more food, eventually will die out. This will reduce the number of predators and making the life of the food fish more “easy”. For this reason, assuming that the the pray does not compete for the food supply since this is abundant, the pray will increase. Now the predators are in a smaller number and their prays are increasing. It is again easy for them to pray and in this way the number of predators increases again. One may argue that this will give a periodicity in the number of predators and prays. The mathematical analysis of their mutual interaction is just a confirmation. But for a more accurate analysis of the evolution of the model we must come back to the system.

We consider any periodic solution, with a generic period T, and define the mean value $\widehat{P}$ and $\widehat{H}$, namely

$$\widehat{H} = \frac{1}{T}\int_0^T (t)dt, \qquad \widehat{P} = \frac{1}{T}\int_0^T P(t)dt. \tag{1.5}$$

From the equation $\dot{P} = -bP + \beta HP$ we can evaluate

$$\int_0^T \frac{\dot{P}(t)}{P(t)}dt = -bT + \beta\int_0^T H(t)dt.$$

Being $\displaystyle\int_0^T \frac{\dot{P}(t)}{P(t)}dt = \ln P(T) - \ln P(0) = 0$, because we are on a T-periodic solution, we get that $bT = \beta\displaystyle\int_0^T H(t)dt$ and therefore that

$$\frac{b}{\beta} = \frac{1}{T}\int_0^T H(t)dt = \widehat{H}.$$

With a same argument one can see that $\widehat{P} = \dfrac{a}{\alpha}$.

Hence we proved that the mean values $\widehat{H}$ and $\widehat{P}$ do not depend from the period and are actually the coordinates of the critical point $E(b/\beta,\, a/\alpha)$. This last result should not surprise, because once that we proved that the mean values do not depend from the choice of the periodic solution we can read the singular point as a periodic solution and clearly one has

$$\widehat{H} = \frac{1}{T}\int_0^T H(t)\,dt = \frac{1}{T}\int_0^T \frac{b}{\beta}\,dt = \frac{b}{\beta}$$

and

$$\widehat{P} = \frac{1}{T}\int_0^T P(t)\,dt = \widehat{P} = \frac{1}{T}\int_0^T \frac{a}{\alpha}\,dt = \frac{a}{\alpha}.$$

as well.
We just observe that actually the biologist D'Ancona founded a kind of mean values in the port of Fiume. Now we are going to apply this important result.

Assume that the condition of life becomes harder for both species, as for instance when fishing takes place, because in principle we cannot choose the kind of fish that we will get. the system becomes:

$$\begin{cases} \dot{P} = -bP + \beta HP - BP \\ \dot{H} = aH - \alpha HP - AH \end{cases} \tag{1.6}$$

being A and B the positive coefficients which measures the effect of fishing. This gives

$$\begin{cases} \dot{P} = -(b+B)P + \beta HP \\ \dot{H} = (a+A)H - \alpha HP \end{cases} \tag{1.7}$$

and we can can study this new system as before, getting that the new coordinates of the new critical point E^* are $\left(\dfrac{b+B}{\beta}, \dfrac{a-A}{\alpha}\right)$. We get that $\widehat{H}$ increases, while $\widehat{P}$ decreases. Hence Volterra proved this crucial result, which is known as "Volterra's effect": conditions more difficult advantage the pray! And vice versa when, because of the war, a lot of people was obliged to stop fishing, this produced an increasing of the predators. The validation of the model was soon given by the fact that in the following years the percentage of selachians and food fish returned to the expected levels. As already mentioned, this spectacular result was giving a strong impulse to the whole theory of mathematical models.

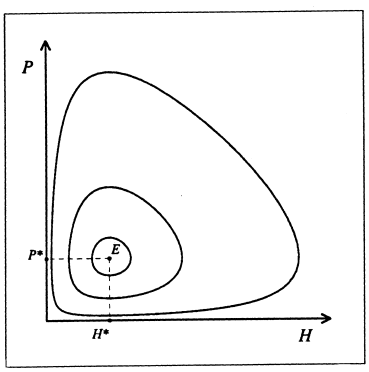

Fig. 1.5 The classical predator-pray model.

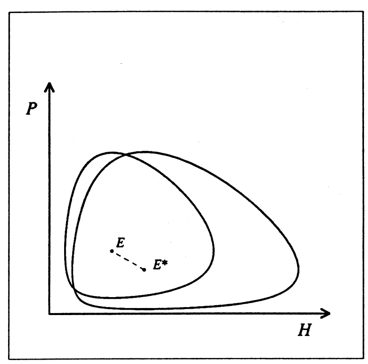

Fig. 1.6 Hard condition of life: the Volterra's effect.

Finally we consider a situation in which the evolution equation of the pray has a logistic term. The system becomes

$$\begin{cases} \dot{P} = -bP + \beta HP \\ \dot{H} = aH - \alpha HP - \sigma H^2 \end{cases} \tag{1.8}$$

and we get a new critical point $Q(a/\sigma, 0)$ on the H-axis. We do not go in details, but we just note that it is not difficult to prove the following.

If $a/\sigma > b/\beta$ the singular point R still is present but changes its stability becoming asymptotically stable. This means that in a long term we will have coexistence. Observe that if we consider fishes and ocean, it is natural to assume that in this situation a/σ must be very large, and therefore $a/\sigma > b/\beta$.

There are several predator-pray models in which the logistic term must be considered and actually we can have that $a/\sigma < b/\beta$. Now the evolution of the system changes dramatically. The critical point R disappears and eventually the predator must die out, just because there is not enough food for survive. And the pray will tend to the saturation level. It is not necessary to stress the fact that this scenario is very important in biology and gives a clear support to the ecologists that remind us how dangerous is to change the environment.

1.5 A mathematical descriptions of epidemics

We are now presenting some mathematical models in epidemiology. Following a work of M.Iannelli [17], we can say that *"A main point to stress when introducing mathematical models of epidemics (and of any other population phenomenon as well) is that we must consider oversimplified phenomenological situations, so that the basic mechanism that we want to investigate can be looked at freed from the encumbrance of a highly detailed description which – though justified by the need of realism – would actually lead to impossible mathematics and/or to the hiding of the essential features of the phenomenon"*

Nevertheless we observe that often simple models work as fundamental prototypes which show the basic aspect of the phenomena. This is actually the case of the two models, namely the SIS model and the SIR model, that we are going to discuss.

At first we consider an isolated population and we assume that the size of such population is constant, say of N individuals. This is not restrictive as it can appear: indeed for epidemics usually one consider "the rapid spread of infectious disease to a large number of persons in a given population within a short period of time, usually two weeks or less".

And in a short period of time the population of Florence, as well as the population of London or New York can clearly considered as a constant. The spread of an epidemic is usually described by dividing the population into

three main subclasses:
– Susceptible: those individuals who are not sick and can be infected
– Infective : those individuals who have the disease and can transmit it to others
– Removed: those individuals who have been infective and now are immune, or dead (or isolated).
If we call $S(t)$, $I(t)$, $R(t)$ respectively the number of individuals that, at the time t, belong to the above defined subclasses, we get the relation

$$S(t) + I(t) + R(t) = N \tag{1.9}$$

The SIS mode describes a disease that is not lethal and does not gives immunity, therefore a susceptible individual can be infected, but eventually recovers and becomes again susceptible, namely $S(t) \to I(t) \to S(t)$ and we do not have the class $R(t)$. In general the pathogen is a bacterium, as for instance in the case of gonorrhea, because a virus gives immunity. But we include in this model also common cold and flu, because the virus causing this kind of disease is unstable and roughly speaking changes is structure. Therefore the acquired immunity does not "work" for the new virus. The evolution system for the SIS model is the following

$$\begin{cases} \dot{S} = -\gamma IS + \delta I \\ \dot{I} = \gamma IS - \delta I \end{cases} \tag{1.10}$$

where γ measures the rate at which susceptible catch the disease entering in contact with individuals of the infective class (this can occur just in a bus if the pathogen is resistant in the contact with air as the case of common cold, or in a sexual relationship if the pathogen is fleeting, as in the case of gonorrhea), while δ measures the rate at which infective recover and leave their class. Being $R(t) = 0$, from (1.14) we have that $S(t) = N - I(t)$, and (1.16) becomes

$$\dot{I} = +\gamma I(N - I) - \delta I = (\gamma N - \delta)I - \gamma I^2 \tag{1.11}$$

If $\gamma N - \delta \leq 0$ the disease cannot spread and epidemic goes extinct. If $\gamma N - \delta > 0$ equation (1.17) is just the logistic equation which has been previous presented. The saturation level, namely $N - \delta/\gamma$ is now called endemic level. It is crucial to take any possible action in order to reduce such level, or even to have $N \leq \delta/\gamma$ so that the disease cannot spread. Being N fixed we must work on the ratio δ/γ in order to have it as large as possible. This can be done increasing δ and reducing γ. The meaning of this is very clear: more prevention and better treatments! It is amazing to see how this oversimplified model can give us what we expect.

On the other hand the SIR model describes a disease that is or lethal, as for instance HIV infection, or gives immunity, as for instance measles, mumps and other childhood diseases. A susceptible individual can be infected, but

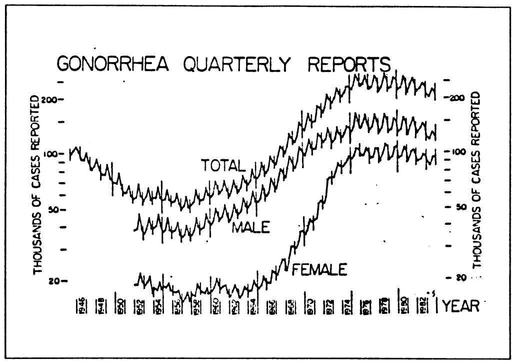

Fig. 1.7 A confirmation of the SIS model for the gonorrhea. One can see a relevant increase of infected in the women curve after year 1968. The reason is not what you might think, namely the sexual revolution, but it is due to the fact that in most cases gonorrhea is asymptomatic in women and after 1968 the medical checks for this kind of disease, as well as for most venereal diseases, drastically increased.

eventually or dies out or recovers with acquired immunity, namely $S(t) \rightarrow I(t) \rightarrow R(t)$. The evolution system for the SIR model is the following

$$\begin{cases} \dot{S} = -\gamma IS \\ \dot{I} = \gamma IS - \delta I \\ \dot{R} = \delta I \end{cases} \tag{1.12}$$

where γ is defined as before, while δ is now called the removal rate.

As far as we know, this model was presented by W.O. Kermack and A.G. McKendrick [18] when they were monitoring the pest which occurred in Bombay in 1926. We can study only the planar system given by the first two equations, because once that we solve the system, knowing $S(t)$ and $I(t)$, we can find $R(t)$ in virtue of (1.14). The main difference of this system compared with the previous ones lies in the fact that now we do not have isolated critical points, but the set of critical points is given by the whole positive x-axis. This is not a big issue because if an epidemics is occurring there must be infected individuals and we do not consider point of the positive x-axis as initial points. Is easy to see that $S(t)$ is always decreasing, and that, a part from the positive x-axis, the 0-isocline is given by the line $x = \delta/\gamma$. This gives the fact that $I(t)$ increases for $x > \delta/\gamma$ and decreases for $x < \delta/\gamma$. Therefore if we consider the positive semi-trajectory starting from the initial point $P(x, y)$ with $x > \delta/\gamma$, we will intersect the 0-isocline at a level which is called "acme" of the epidemic, because it gives the maximum number of infected individuals. From this moment $I(t)$ decreases and eventually tends to 0: the epidemics goes extinct, and with some more not difficult calculation it is possible to see that, as expected, at the end of the epidemics the

population is not depleted. On the other hand, if we consider the positive semi-trajectory starting from the initial point $P(x, y)$ with $x < \delta/\gamma$, $I(t)$ decreases and eventually tends to 0.
Arguing as before it is necessary to have $N \le \delta/\gamma$, and again this happens increasing δ and reducing γ. But now there is a significative difference. We can always reduce γ, that is to have more prevention, but we can increase δ only if we are in the case of a disease which gives immunity, because this means as before better treatments. In case of a lethal disease increasing δ, means to kill the infected individuals, and clearly this is not acceptable! But if the disease affects animals instead of humans, this is precisely what we are doing. In this light consider how was defeated the MCD (mad cow disease), or how is flighted the virus HPAI (high pathogenic avian influenza).

1.6 Phase portrait of the Liénard equation. Existence and uniqueness of limit cycles

We now discuss the problem of existence of periodic solutions for the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1.13}$$

Such a problem has been widely investigated since the first results of Alfred-Marie Liénard, a French physicist and engineer, appeared in 1928 [21] and there is an enormous quantity of papers on this topic.
It is well known that Liénard equation is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x)y - g(x) \end{cases} \tag{1.14}$$

in the phase plane, and to the system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \tag{1.15}$$

in the Liénard plane, where $F(x) = \int_0^x f(x)dx$.
For this reason, the problem of the existence of periodic solutions is bring back to a problem of existence of limit cycles for the previous systems. Among the existence results until 1960, the classical theorems of Filippov [11] , Levinson-Smith [20] and Dragilev [8] may be considered as milestones, while in the last decades the number of results is dramatically increasing. All the results are based on the classical Poincaré-Bendixson Theorem, and in order to fulfill the assumptions of this theorem, once that one has proved that there is a unique singular point, which is unstable, it is necessary to produce a winding trajectory large enough.

Let us discuss this problem in more details, even if we understand that in this way our "survival kit" will be no more "light". The standard assumptions for this kind of problem are the following:

– f is continuous, and g is locally Lipschitz continuous. This guarantees existence and uniqueness of solutions for system (1.15), and hence also for system (1.14).

– $xg(x) > 0$ for all $x \neq 0$. Therefore the origin is the unique singular point and trajectories are clockwise.

At first we consider the well known and paradigmatic example given by the Van Der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \text{ with } \mu > 0 \tag{1.16}$$

which was introduced in 1926 by Dutch electrical engineer and physicist Balthasar Van der Pol while he was working at Philips [32]. Van der Pol found stable oscillations, which he called relaxation-oscillations, in electrical circuits employing vacuum tubes. This is actually the starting point of the whole theory of nonlinear oscillations!

In the phase plane equation (1.16) is written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\mu(x^2 - 1)y - x \end{cases} \tag{1.17}$$

We are going to prove the existence of at least a stable limit cycle, and later we will prove its uniqueness. Clearly the standard assumptions previously introduced are fulfilled, and once again we consider the critical point, the 0-isocline and the ∞-isocline, because as already mentioned they play a crucial role in the construction of the phase portrait. The ∞-isocline is given by the x-axis, while the 0-isocline is given by the function $H(x) = -\frac{x}{\mu(x^2-1)}$.

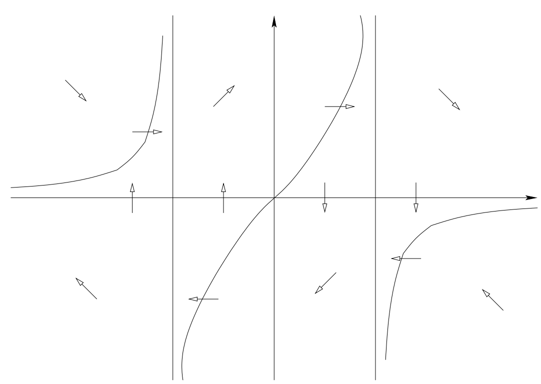

Fig. 1.8 The 0-isocline and the ∞-isocline for the Van der Pol equation in the phase plane

In order to discuss the stability of the origin we consider the function $D(x, y)$ which gives the square distance of a generic point $P(x, y)$ from the origin, namely $D(x, y) = x^2 + y^2$. The rate of change of the distance is given by

$$\dot{D}(x, y) = \frac{1}{2}x\dot{x} + \frac{1}{2}y\dot{y} = -\frac{1}{2}\mu(x^2 - 1)y^2 \tag{1.18}$$

therefore in the strip given by $|x| < 1$ the distance from the origin is increasing, while outside the strip is decreasing, and thus we have that the origin is a source. In order to produce a winding trajectory the 0-isocline will play a crucial role. In this light let Γ be the graph of the function $H(x) = -\frac{x}{\mu(x^2-1)}$ and take a point $P\left(x, -\frac{x}{\mu(x^2-1)}\right)$ with $x < -1$. It is easy to see that the negative semi-trajectory intersecting the point P lies above Γ and therefore is bounded away from the x-axis. On the other hand the positive semi-trajectory intersects at first the line $x = -1$ in a point Q, and then again Γ in $0 < x < 1$. This because the slope of such trajectory, namely

$$y'(x, y) = -\mu(x^2 - 1) - \frac{x}{y} \tag{1.19}$$

is bounded in the strip given by $|x| < 1$ for y bounded away from 0, and therefore there are no vertical asymptotes. Now there are two possibilities: or the trajectory intersects the x-axis in $x < 1$ and then the y-axis in $y < 0$ or intersects the line $x = 1$, again because there are no vertical asymptotes.
In the latter case the trajectory intersects the x-axis in $x > 1$ because, as already noted, in $|x| > 1$ the distance from the origin is decreasing, and moves in $y < 0$ without intersecting again Γ. This because, arguing as above one can see that the negative semi-trajectory intersecting Γ in $x >$ lies below Γ and therefore are bounded away from the x-axis. Hence we proved that in both cases the positive semi-trajectory intersecting P eventually intersects the y-axis in $y < 0$. Now we have again two possibilities: or such trajectory intersects the x-axis in $-1 < x < 0$, and hence it is winding, or intersects the $x = -1$.
Using the fact that the distance from the origin is decreasing we see that the trajectory intersects the x-axis in $x < -1$ and moves in $y > 0$. Being bounded away from Γ the trajectory intersects the line $x = -1$ in a point R below Q and hence it is winding as well. Being the origin a source we can apply the Poincaré-Bendixson Theorem and get the existence of at least a stable limit cycle.

We can obtain the same result with a different approach.

Observe that

$$y(\beta) - y(\alpha) = \int_\alpha^\beta y'(x)\, dx = \int_\alpha^\beta -\mu(x^2 - 1) - \frac{x}{y(x)}\, dx \tag{1.20}$$

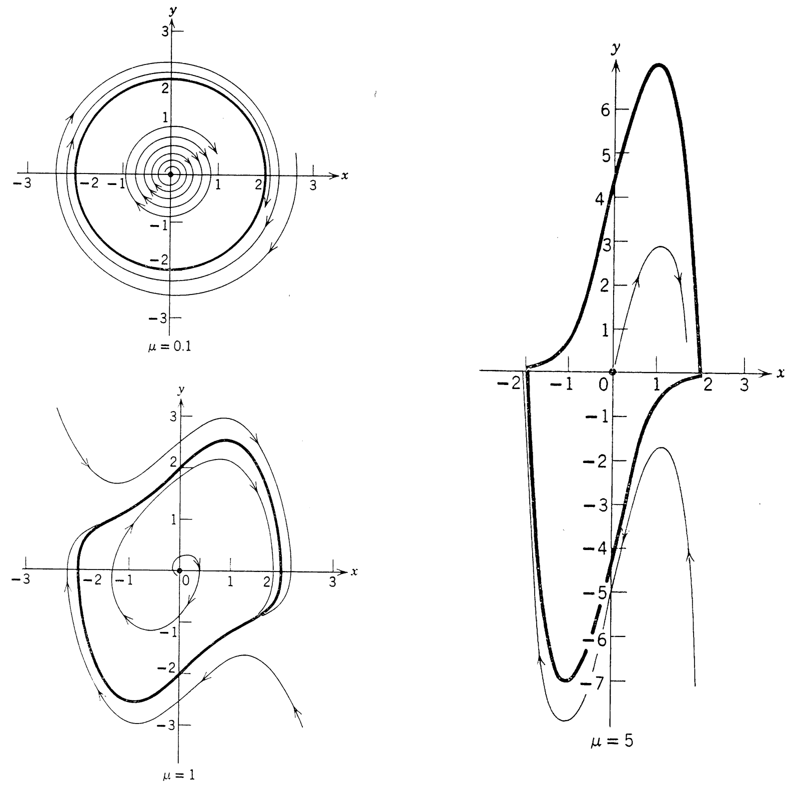

Fig. 1.9 Limit cycles for the Van der Pol equation [1, p. 448] As expected, for μ small the phase portrait is similar to the one of the harmonic oscillator.

It is possible to choose $\alpha < -1 < 1 < \beta$ such that $\displaystyle\int_\alpha^\beta \mu(x^2-1)\,dx > 0$ and if $y(x)$ is large enough we get that in the interval $[\alpha,\beta]$ the quantity $\mu(x^2-1)+\frac{x}{y(x)}$ is positive. Therefore, if we consider a point $P(\alpha, y_1)$ with y_1 large enough and define as $\gamma^+(P)$ the positive semi-trajectory starting from P, we can see in virtue of (1.20) that $\gamma^+(P)$ intersects the line $x=\beta$ in a point $Q(\beta, y_2)$ with $0 < y_2 < y_1$. This because if $\gamma^+(P)$ intersects the x-axis before $x=\beta$, we can consider $0 < y_2 < y_1$ and the negative semi-trajectory starting from $Q(\beta, y_2)$, namely $\gamma^-(Q)$; clearly $\gamma^-(Q)$ intersects the line $x=\alpha$ in a point $P(\alpha, y_1)$ with $y_2 < y_1$. Now we consider $\gamma^+(Q)$. We know that in $|x| > 1$ the distance from the origin is decreasing, hence $\gamma^+(Q)$ intersects the line $x=\beta$ in $y<0$ at a point $R(\beta, y_3)$ with $|y_3| < y_2$. Now we consider $\gamma^+(R)$: arguing as above we can prove that $\gamma^+(R)$ intersects the line $x=\alpha$ in a point $S(\alpha, y_4)$ with $y_3 < y_4 < 0$.

Finally we consider $\gamma^+(S)$. We are again in $|x| > 1$, and the distance from the origin is decreasing. Therefore $\gamma^+(S)$ intersects the line $x = \alpha$ in $y > 0$ at a point $T(\alpha, y_5)$ with $0 < y_5 < |y_4| < |y_3| < y_2 < y_1$. This proves that $\gamma^+(P)$ is actually winding. As before, being the origin a source we can apply the Poincaré-Bendixson Theorem and get the existence of at least a stable limit cycle.

We called the Van Der Pol equation paradigmatic just because the two methods presented in order to produce the existence of at least a stable limit cycle can applied also for the more general Liénard equation.
More precisely, we have the following theorems, which generalize the above presented results for the Van Der Pol equation .

Theorem 1. *[34] Assume that*

i) *f is continuous, and g is locally Lipschitz continuous. $xg(x) > 0$ for all $x \neq 0$;*
ii) *$f(0) < 0$ and there exist $\delta \in \mathbb{R}$ such that $f(x) > 0$ for $|x| > \delta$;*
iii) $\min \left\{ \limsup_{x\to+\infty} \frac{g(x)}{f(x)}, \limsup_{x\to-\infty} \frac{|g(x)|}{f(x)} \right\} < +\infty$;
iv) *There exists $h > \delta$ and $b > 0$ such that $f(x) + |g(x)| > 0$.*

Then there exists a periodic solution of equation (1.13)

Proof. Let us have a sketch of the proof. Consider the system (1.14). At first we note that the assumption $f(0) < 0$ guarantees that the origin is a source, in virtue of a comparison with the Duffing equation. This is a standard argument and, in any case, the phase portrait of the Duffing equation will be discussed soon. At this point the proof follows the idea used for the Van Der Pol equation, namely to produce a negative semi-trajectory bounded away from the x-axis. The 0-isocline, namely the function $H(x) = -\frac{g(x)}{f(x)}$, will play again a crucial role. Let η be the graph of the function $H(x) = -\frac{g(x)}{f(x)}$ and take a point $P\left(x, -\frac{g(x)}{f(x)}\right)$ with $x < -\delta$, (or $x > \delta$). In virtue of the sign condition on $f(x)$ and assumpion iii), it is easy to see that it is possible to choose a suitable point P in a way that $\gamma^-(P)$ lies above η and therefore is bounded away from the x-axis. Now we follow $\gamma^+(P)$. Assumption iv) avoids the possibility of having horizontal asympotes, therefore $\gamma^+(P)$ intersects the x-axis twice, just as was happening for the Van Der Pol system (1.17). Hence we have a winding trajectory and we can apply the Poincaré-Bendixson Theorem to get the existence of at least a stable limit cycle for system (1.14), that is the existence of a stable periodic solution for equation (1.13).

Theorem 2. *[35] Assume that*

i) *f is continuous and g is locally Lipschitz continuous, $xg(x) > 0$ for all $x \neq 0$, $\lim_{x\to\pm\infty} \int_0^x g(t)dt = +\infty$;*
ii) *$f(0) < 0$, and there exist $\delta \in \mathbb{R}$ such that $f(x) > 0$ for $|x| > \delta$;*

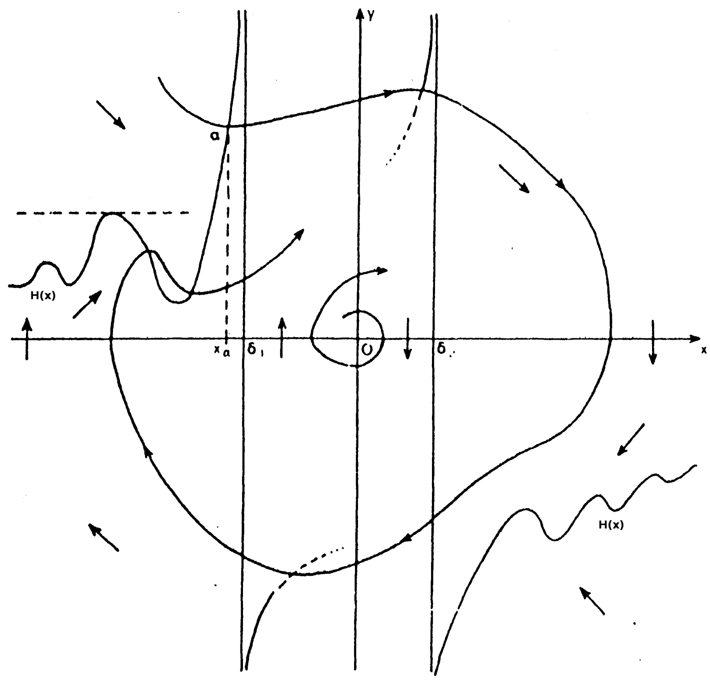

Fig. 1.10 The phase portrait of Liénard equation under the assumption of theorem 1.

iii)there exist $\alpha \leq -\delta$ *and* $\beta \geq \delta$ *such that* $\displaystyle\int_\alpha^\beta f(x)dx > 0.$

Then there exists a periodic solution of equation (1.13)

Proof. Also in this case we give a sketch of the proof, which now is similar to the second approach presented above for the Van Der Pol equation. Consider System (1.14); again, the assumption $f(0) < 0$ guarantees that the origin is a source. Assumption iii) plays the same role of formula (1.20). On the other hand f is positive when $|x| > \delta$, but we cannot say that the distance from the origin is decreasing as for the Van Der Pol equation. In order to get a similar result one must require the assumption i) and this will be clear in the following discussion concerning the Duffing equation. In this situation we are able to produce a winding trajectory and apply the Poincaré-Bendixson Theorem to get the existence of at least a stable limit cycle for system (1.14), that is the existence of a stable periodic solution for equation (1.13).

At this point we can observe that in general the methods used to attack this problem are basically two, and such methods were actually used for the

Van Der Pol equation, as well as in the above presented Theorems 1 and 2. We can call the first one the "method of energy", because one may consider the Liénard equation as perturbation of the Duffing equation

$$\ddot{x} + g(x) = 0 \tag{1.21}$$

which plays the role of the energy. Let us discuss in details this situation.

1.6.1 The method of energy

The Duffing equation is equivalent in both planes to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) \end{cases} \tag{1.22}$$

and it is well known that the level curves of the function

$$H(x,y) = \frac{1}{2}y^2 + G(x) \tag{1.23}$$

where $G(x) = \displaystyle\int_0^x g(x)dx$, are its solutions.

Therefore, by a straightforward calculation, the phase portrait of the Duffing equation is a global center if $\lim_{x\to\pm\infty} G(x) = +\infty$, while it is a local center if $G(x)$ is bounded (We just note that for the Van Der Pol equation treated above $H(x,y)$ was the square distance from the origin $D(x,y)$). Here we follow the elegant and concise description of Lefschetz [19]. If we consider the level curve

$$\frac{1}{2}y^2 + G(x) = K \tag{1.24}$$

in the dynamical interpretation as motion of a particle, the first term represents its kinetic energy and (1.17) expresses the law of conservation of energy as applied to the particle.

For this reason, we may consider the level curves of the function $H(x,y)$ as energy levels.

Coming back to the Liénard system (1.15), consider a generic point $S = (x_S, y_S)$. Keeping the dynamical interpretation, we can say that this point lies on a certain level of energy. For sake of simplicity, we consider a generic point of the y-axis $S = (0, y_S)$, which lies on the level energy $\frac{1}{2}y_S^2 = K_S$.

Define $\gamma^+(S)$ as the positive semi-trajectory starting from S, and assume that $\gamma^+(S)$ moves around the origin and intersects again the y-axis in the same half-plane of S at a point $R = (0, y_R)$. Clearly, such semi-trajectory is winding if $|y_R| < |y_S|$, unwinding if $|y_R| > |y_S|$ and a cycle if $|y_R| = |y_S|$.

In terms of energy, this means that in the first case we are losing energy, in

the second one we are gaining energy and in the last one there is no increment.
Therefore, one can investigate the variation of energy, even if in most papers this was not explicitly observed. In general, the situation is the following: we start at a point $P = (x_P, y_P)$ at the time t_0 and follow $\gamma^+(P)$ for the time T until we reach the point $Q = (x_Q, y_Q)$ at the time $t_0 + T$. The variation of energy is

$$H(x_Q, y_Q) - H(x_P, y_P) = \frac{1}{2}y_Q^2 + G(x_Q) - \frac{1}{2}y_P^2 - G(x_P) = \int_{t_0}^{t_0+T} \dot{H}(x,y)\,dt \tag{1.25}$$

We observe that, using the Liénard system (1.14),

$$\dot{H}(x,y) = y\dot{y} + g(x)\dot{x} = -g(x)y + g(x)(y - F(x) = -g(x)F(x)$$

This well known result in the dynamical interpretation shows that when $g(x)F(x) > 0$ we are losing energy, while when $g(x)F(x) < 0$ we are gaining energy, and in order to have the existence of the limit cycle it is necessary that $g(x)F(x)$ change sign.
More precisely, considering the positive semi-trajectory $\gamma^+(P)$ reaching the point Q, the variation of energy given by the integral in (1.25) may be split in four parts as follows

$$\int_{t_0}^{t_0+T} \dot{H}(x,y)dt = \int_{\alpha}^{\beta} \frac{-g(x)F(x)}{y - F(x)}\,dx + \int_{y_1}^{y_2} F(x)\,dy + \\ + \int_{\beta}^{\alpha} \frac{-g(x)F(x)}{y - F(x)}dx + \int_{y_3}^{y_Q} F(x)\,dy$$

where $\alpha < 0$ and $\beta > 0$ are necessary in order to make a correct change of variables. Such semi-trajectory will be winding if and only if the sum of the four integrals is negative.
We observe that actually the integral $\int F(x)dy$ plays a crucial role. This because the integral $\int_{\alpha}^{\beta} \frac{-g(x)F(x)}{y - F(x)}\,dx$ may be considered arbitrarily small. In principle there are two ways for which this can be achieved. The first one is letting the difference $(\beta - \alpha$ be arbitrarily small. In virtue of the regularity of the function $\frac{-g(x)F(x)}{y - F(x)}$, the integral will be arbitrarily small. On the other hand being the 0-isocline the y-axis, the trajectory may be read as a function of y, and hence the integral $\int F(x)dy$ is well defined outside the strip $[\alpha, \beta]$. This way cannot be further developed because in general we cannot evaluate the integral unless we have strong condition on $F(x)$ on the whole line.

The second way is more efficient. The interval $[\alpha,\beta]$ is now fixed and can be large. However, the integral $\int_\alpha^\beta \frac{-g(x)F(x)}{y-F(x)}\,dx$ may be considered arbitrarily small if we take a trajectory with $|y|$ arbitrarily large. In this case we can evaluate the integral $\int F(x)dy$, provided that outside the interval $[\alpha,\beta]$, there are conditions on $F(x)$ which allow an estimation of the previous integral.

For this reason, the role of $\int F(x)dy$ was already emphatized by Lefschetz [15, p.267], who called this integral taken along a path as the energy "dissipated" by the system.

In order to work with energy, it is therefore necessary to consider trajectories arbitrarily large, and in this light we need sufficient conditions which guarantee that any positive semi-trajectory starting from a point $P=(x,y)$ with $|y|$ arbitrarily large, intersects the vertical isocline $y=F(x)$, otherwise the method fails. Such conditions were at first introduced in [36], but we are not going to discuss them. We just present the classical Dragilev Theorem because it is based on this idea, even if not explicitely stated

Theorem 3. *Dragilev [8] Assume that*

i) *f is continuous, and g is locally Lipschitz continuous; $xg(x)>0$ for all $x\neq 0$. $\lim_{x\to\pm\infty} G(x)=+\infty$;*
ii) *$xF(x)<0$ for $x\neq 0$ and $|x|$ sufficiently small.*
iii) *There exist constants N, K_1, K_2 with $K_1<K_2$ such that $F(x)\geq K_1$ for $x>N$ and $F(x)<K_2$ for $x<-N$.*

Then system (1.15) *admits at least one limit cycle.*

1.6.2 Intersection with the vertical isocline

The second method to obtain a winding trajectory is strictly related with the intersection with the vertical isocline. More precisely, if there is $\overline{P}=(0,\overline{y})$ with $\overline{y}\neq 0$ such that $\gamma^-(\overline{P})$ does not intersect the curve $y=F(x)$ and $\gamma^+(\overline{P})$ is oscillatory, clearly such trajectory is winding. Such property was called "property K" in [36] where this problem has been investigated.
We observe that, in order to get that $\gamma^+(\overline{P})$ is oscillatory, it is again necessary that $\gamma^+(\overline{P})$ intersects $y=F(x)$ for $|y|$ large enough.
This method seems more effective because no balance of energy is necessary, but actually, the requirement that $\gamma^-(\overline{P})$ does not intersects the vertical isocline is a strong condition on the structure of the system, and in general this implies that $F(x)$ dominates $G(x)$ .This is precisely the case of Theorem 1.
As it was already discussed, an example in which both methods may be used

is the Van Der Pol equation(1.16), and one can easily check that $F(x)$ is $\frac{x^3}{3} - x$ while $G(x)$ is $\frac{x^2}{2}$ and $F(x)$ actually dominates $G(x)$.

1.6.3 The Massera theorem

Finally we discuss the problem of uniqueness of limit cycles. Among the results we present only the classical Massera Theorem, because his prof is based on a clever and simple geometrical idea, which works in the case $g(x) = x$. Other results involve more sophisticated mathematical background, and therefore cannot take place in this short note.

Theorem 4. *(Massera, [23]) The system* (1.14) *has at most one limit cycle which is stable, and hence equation* (1.13) *has at most one non trivial periodic solution which is stable, provided that f is continuous, $g(x) = x$ and $f(x)$ is monotone decreasing for $x < 0$, monotone increasing for $x > 0$.*

We just observe that Van Der Pol equation satisfies such assumptions, and hence has an unique stable non trivial periodic solution. The Theorem of Massera improved a previous result due to Sansone [29] in which there was the additional assumption $|f(x)| < 2$. This assumption comes from the fact that Sansone was using the polar coordinates. Such strong restriction on f is clearly not satisfied in the polynomial case and hence the Massera's result is much more powerful.

We must observe that Massera was proving the uniqueness of limit cycles regardless the existence because only the monotonicity properties and the continuity were required. It is easy to prove that, in order to fulfill the necessary conditions for the existence of limit cycles, the only cases to be considered are.

– $f(x)$ has two zeros a and b, $a < 0 < b$. In this case the existence of limit cycles is granted, in virtue of Theorem 2
– $f(x)$ remains negative for $x < 0$, (or for $x > 0$), while intersects the x axis once in $x > 0$ (or for $x < 0$).

Now the existence of limit cycles is no more granted. Moreover this case does not cover the crucial polynomial case, which is still the most important. For related resuls, and a comparison of the theorems of Massera and Sansone, one can see [37, 27, 28]

Proof. A crucial step in the proof is the fact that any limit cycle of system (1.14) must be star-like, which means thar any halfray intersects the limit cycle only once. For some strange reason, this was not explicitly mentioned in the paper of Massera, but it is proved in the paper of Sansone, where for this result the additional assumption $|f(x)| < 2$ was not necessary. Roughly speaking, the key of the proof is the following: One performs a similarity transormation on the limit-cycle Γ and in this way obtain a family os simple closed curves Γ_k on the plane at both sides of Γ. This because Γ is star-like. Then one proves that the positive semi-trajectories of system (1.14) moving from the points of Γ_k stay in the interior of Γ_k for $k > 1$ and in the exterior of Γ_k for $k < 1$. Therefore we get that cannot exist any other limit-cycle other than Γ. Let us see what is going on in more details.

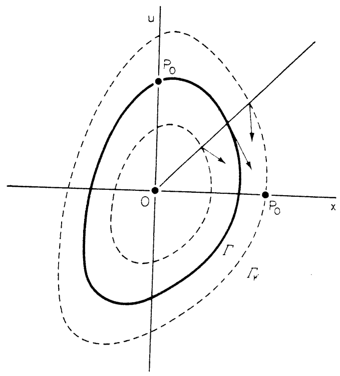

Fig. 1.11 The geometric idea of the Massera theorem

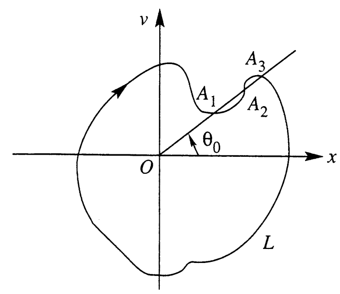

Fig. 1.12 An example of a not star-like limit cycle. Guess why Massera proof it is no more working in this case.

In the system (1.14) we consider a limit-cycle Γ and make the similarity transformation $(x, y) \to (kx, ky)$. Therefore Γ is moved in a closed curve Γ_k which lies in the interior of Γ if $k < 1$ and in the exterior of Γ if $k > 1$. Take a point $P(x, y)$ on Γ and consider its tangent vector. This is given by the slope, which in this case is

$$y'(x, y) = -f(x) - \frac{x}{y} \tag{1.26}$$

Due to the similarity transformation, the tangent vector at the point $P_k(kx, ky)$ on the closed curve Γ_k is parallel to the one at the point P. But the slope of the system in this point is

$$y'(kx, ky) = -f(kx) - \frac{kx}{ky} = -f(kx) - \frac{x}{y} \tag{1.27}$$

The monotonicity assumption on f is now used to prove that the slope at $P(x, y)$, namely

$$y'(x, y) = -f(x) - \frac{x}{y} \geq -f(kx) - \frac{x}{y} = y'(kx, ky) \tag{1.28}$$

that is the slope at $P_k(kx, ky)$, if $k < 1$. Clearly such inequality holds with $\leq$ if $k > 1$. Therefore $\gamma^+(P_k)$ enters Γ_k and lies in its interior if $k > 1$, while exits Γ_k and lies in its exterior if $k < 1$. Being P_k a generic point this shows that Γ is actually unique and stable. As already mentioned, this smart proof works only in the case $g(x) = x$ and up to now was not possible to have a similar geometrical proof in the general case.

With this last result we can say that, finally, our survival kit is complete.

1.7 Water waves. Phase plane analysis versus classical theory

We are now ready to show how phase-plane analysis can be applied also in water waves. We discuss the motion of particle trajectories in linear water wawes and follow the joint paper with A. Constantin [6] where it is proved that for a linear water wave no particle trajectory is closed, unless the free surface is flat. Each trajectory involves over a period a backward/forward movement of the particle, and the path is an elliptical arc (which degenerates on the flat bed) but with a forward drift. Notice that due to the mathematical intractability of the governing equations for water waves, the classical approach towards explaining this aspect of water waves consists in analyzing the particle motion after linearization of the governing equations. Within the linear water wave theory, the ordinary differential equations system describing the motion of the particles is nevertheless nonlinear and explicit solutions are not available. In the first approximation of this nonlinear system, all particle paths are closed. Support for this conclusion is given by the only known explicit solution with a non-flat free surface for the governing equations in water of infinite depth [13], solution for which all particle paths are circles of diameters decreasing with the distance from the free surface. But as already noted in the discussion of the predator-pray model, once that after a linearization we find closed trajectory this property cannot be inferred to the nonlinear systems. This was actually the starting point of the paper, where instead of first approximation phase plane analysis was called to "help". However we observe that the presented result can actually be viewed as a more detailed version of the classical theory for the trajectories of particles below a water wave, as it was summarized by Longuet-Higgin's [22]:

"In progressive gravity waves of very small aplitude it is well known that the orbits of the particles are either elliptical or circular. In steep waves, however, the orbits become quite distorted, as shown by the existence of a mean horizontal drift or mass-transport in irrotational waves."

More precisely, we show that within linear water wave theory, the particle paths are almost closed and the more we approach the free surface, the more pronounced the deviation from a closed orbit becomes. This is in agreement with Stokes' observation [23]:

"It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of the propagation of the waves ..."

We follow the above mentioned paper [1] and begin our analysis from the governing equations

$$\begin{cases} \dfrac{dx}{dt} & = M\cosh(ky)\,\cos(x-ct) \\ \dfrac{dy}{dt} & = M\sinh(ky)\,\sin(x-ct) \end{cases} \tag{1.29}$$

because this is the standard approach for people working in this field area. Notice that from the previous part of the paper one can see that

$$M = \frac{\epsilon\omega h_0}{\sinh(kh_0)} \tag{1.30}$$

Without actually solving the system we are interested in the principal features of its solutions. At first we consider the classical first approximation in terms of the small parameter M. We restrict our attention to a the fixed time interval $[0,T]$, where $T>0$ is the wave period.

Since y belongs to a set bounded a priori, from (1.29) we readily obtain that

$$x(t)-x_0 = O(M), \quad y(t)-y_0 = O(M), \quad t\in[0,T] \tag{1.31}$$

where $O(M)$ denotes an expression of order M. Using the mean-value theorem, we may write (1.29) on $[0,T]$ as

$$\begin{cases} \dfrac{dx}{dt} & = M\cosh(ky_0)\,\cos(kx_0-\omega t)+O(M^2) \\ \dfrac{dy}{dt} & = M\sinh(ky_0)\,\sin(kx_0-\omega t)+O(M^2) \end{cases} \tag{1.32}$$

(we just observe that actually $\omega = kc$). Neglecting terms of second order in M, we find that

$$\begin{cases} \dfrac{dx}{dt} & \approx M\cosh(ky_0)\,\cos(kx_0-\omega t) \\ \dfrac{dy}{dt} & \approx M\sinh(ky_0)\,\sin(kx_0-\omega t) \end{cases} \tag{1.33}$$

so that by integration we obtain

$$\begin{cases} \dfrac{dx}{dt} & \approx x_0 - \frac{M}{\omega}\cosh(ky_0)\,\sin(kx_0-\omega t) \\ \dfrac{dy}{dt} & \approx x_0 + \frac{M}{\omega}\sinh(ky_0)\,\cos(kx_0-\omega t) \end{cases} \tag{1.34}$$

Thus

$$\frac{[x(t)-x_0]^2}{\cosh^2(ky_0)}+\frac{[y(t)-y_0]^2}{\sinh^2(ky_0)}\approx\frac{M^2}{\omega^2} \tag{1.35}$$

which is the equation of an ellipse: to a first-order approximation the water particles move in closed elliptic orbits, the centre of the ellipse being (x_0, y_0). We some calculations, which we omit for sake of simplicity, one can see that the sense of the particle motion round its orbit is clockwise as viewed with respect to the positive x-direction (the propagation direction). By (1.34), to a first approximation a surface water particle traces an elliptical orbit with the vertical axis of length $\frac{M}{\omega}\sinh(kh_0)=\epsilon h_0$, equal to the height of the wave, while its horizontal axis is slightly larger. The same type of motion occurs for the water particles below the surface but both axes of the elliptical orbits become smaller and smaller with depth. Since $\cosh^2(r)-\sinh^2(r)=1$, all

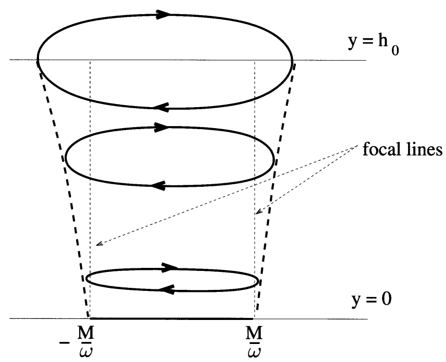

Fig. 1.13 The first approximation of a particle paths in a linear wave

the ellipses have the same distance $\frac{2M}{\omega}$ between their foci, but the lengths of their axes decrease as we go deep into the water: the ellipses are confocal if they are coaxially superimposed upon each other. Observe that the ratio of the height to its length for such an ellipse is $\tanh(ky_0)<1$ so that the flattening of the orbits becomes more pronounced as the depth increases, until at the bottom they are completely flattened into horizontal lines: at the bottom $y=0$ the vertical axis is zero but the major axis equals $\frac{M}{\omega}$ so that the ellipse degenerates into a straight line and the water particles simply oscillate backwards and forwards on the flat bed.

So far with the classical first approximation approach. But once again we stress the fact that after a linearization we find closed trajectory this property cannot be inferred to the nonlinear systems. And in this problem we observe that actually we made two linearizations, the first one in order to be able towrite the governing equations and the second in order to recognize the principal features of its solutions. Hence the obtained result, even if in accordance with the picture of the motion below a deep water wave provided by Gerstner's explicit solution in water of infinite depth [21], where all particles move on circles, cannot be accepted. But in any case the above presented figure was present in several text-books and also (but not any longer...) in Wikipedia. It is time to study the exact solution to (1.29), and see how phase plane analysis can make the job.

To study the exact solution to (1.29) it is convenient to re-write it in a moving frame with scaled independent variables: the transformation

$$X = k(x - ct), \qquad Y = kY \tag{1.36}$$

maps system (1.29) in

$$\begin{cases} \dfrac{dX}{dt} = kM \cosh(Y)\, \cos(X) - kc \\ \dfrac{dY}{dt} = kM \sinh(Y)\, \sin(X) \end{cases} \tag{1.37}$$

Notice that in view of (1.30),

$$kM = k\frac{\varepsilon\omega h_0}{\sinh(kh_0)} = \varepsilon\omega\frac{kh_0}{\sinh(kh_0)} < kc = \omega \tag{1.38}$$

since $s > \sinh(s)$ for $s > 0$ while $\varepsilon < 1$ within the framework of linear theory.

We just observe that the autonomous system (1.37) meets the standard regularity assumptions forthe uniqueness of the Cauchy problem so that its trajectories do not intersect.

Denote

$$\begin{aligned} A(X,Y) &= kM \cosh(Y) \cos(X) - kc, \\ B(X,Y) &= kM \sinh(Y) \sin(X) \end{aligned} \tag{1.39}$$

Since both functions A and B are periodic of period 2π in the X-variable, it suffices to restrict our investigation of (1.37) to the strip

$$-\pi \le X \le \pi \qquad \infty \le Y < \infty \tag{1.40}$$

Notice that the trajectories of (1.37) have mirror symmetry with respect to the Y-axis and the X-axis. Indeed, $A(X,Y)$ is an even function in both X and Y , while $B(X,Y)$ is an odd function of both X and Y . For this reason any trajectory of the system (1.37), viewed as a curve in the phase plane (X,Y), is symmetric with respect to both axes. Notice that the physically relevant cases for (1.37) are for $Y \ge 0$ so that from now on we will only analyze the corresponding half-strip.

As usual, in order to understand what it is going on, we must study the 0-isocline and the ∞-isocline. The 0-isocline is given by the lines $X = 0$, $X = -\pi$, and $X = \pi$, and by the segment of the X-axis where $-\pi \le X \le \pi$. On the other hand, the ∞-isocline is given by the curve $(X, \alpha(X))$ with $X \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\alpha(X) \in [Y^\star, +\infty)$, where

$$Y^\star = \cosh^{-1}\left(\frac{c}{M}\right) \tag{1.41}$$

and $\alpha : \left[0, \frac{\pi}{2}\right) \to [Y^\star, +\infty)$ is the inverse of the function

$$Y \to \arccos\left(\frac{c}{M\cosh(Y)}\right)$$

defined on $[Y, +\infty)$. Now we extend α by mirror symmetry to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Notice that that for $Y > Y^\star$ we have $\dfrac{c}{M\cosh(Y)} \le 1$ so that α is well defined. The smooth function α is even, it assumes its infimum $Y^\star$ at $X = 0$, and satisfies

$$\lim_{X \to \pm\infty} \alpha(X) = +\infty \tag{1.42}$$

Once that we know the 0-isocline and the ∞-isocline, we can say that in the half-strip considered there is an unique singular point $Q(0, Y^\star)$. Moreover we are able to determine the sign of the two components of the vector field given by system (1.37). For $X \in (\frac{\pi}{2}, \pi)$ the function $A(X, Y)$ is negative while $B(X, Y)$ is positive. For $X \in (0, \frac{\pi}{2})$ the function $A(X, Y)$ is negative below the graph of $X \to \alpha(X)$ and positive above it, while $B(X, Y)$ is positive here. Using the symmetry with respect to the Y-axis, we obtain the corresponding signs for $X \in (-\pi, 0)$. At this point, with some more work and using some standard phase-plane argument, it is possible to see that in the phase portrait of system (1.37) if we take a point $P(\pi, Y)$ and consider $\gamma^+(P)$ there are two possibilities, depending from the value of Y. $\gamma^+(P)$ goes to $+\infty$ for $Y > \beta$, while $\gamma^+(P)$ reaches the Y-axis for $Y < \beta$. Clearly it is not possible to determine such β, but this is not important in this framework, while we can say that if we consider $P_\beta(\pi, \beta)$, actually $\gamma^+(P_\beta)$ tends to the singular point Q and therefore it is a separatrix, that is, a phase curve that marks the boundary between phase curves with different properties. Now for $0 < X < \pi$ we consider the points $R(X, Y)$ above such separatrix. With a similar argument we can see that there are points R such that $\gamma^-(R)$ intertsects the Y-axis and points R such that $\gamma^-(R)$ intersects the line $X = \pi$. Therefore we get the existence of a second separatrix which in this case starts from the singular point Q. Using the mirror symmetry we have other two separatrices and we can argue that the singular point is actually a saddle point. A different approach, in order to study the property of Q, is presented in the paper, considering the Hamiltonian structure of (1.37) and using more sophisticated tools as Morse theory. But the qualitative approach was the first one used to attack the problem, and without any doubt deserves a place in this note. The phase portrait of (3.6) is thus complete and appears in Figure. Knowing

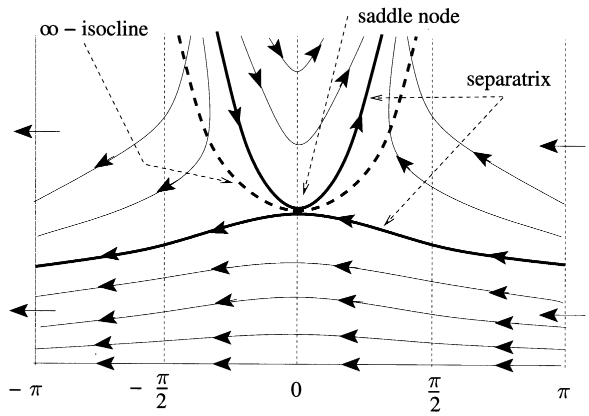

Fig. 1.14 Phase portrait in the moving frame.

the phase curves $(X(t), Y(t))$ of (1.37), the particle trajectories in the linear wave are simply given by

$$x(t) = \frac{X(t)}{k} + ct, \quad y(t) = \frac{Y(t)}{k} \tag{1.43}$$

From the above phase plane analysis of (1.37) we deduce that in order tohave a physically realistic solution it is necessary that

$$kh_0(1+\epsilon) \leq Y^\star \cosh^{-1}(\frac{\sinh(kh_0)}{\varepsilon kh_0}) \tag{1.44}$$

Indeed, (1.44) is necessary to prevent the existence of particle paths that grow indefinitely with time. Relation (1.37) can be interpreted as follows: given the average water depth $h_0 > 0$, in order to represent a realistic model of water waves of small amplitude $\varepsilon > 0$ with wavenumber $k > 0$,it is sufficient that

$$\varepsilon \leq \frac{\tanh(kh_0)}{kh_0} \tag{1.45}$$

since the function $s \to \frac{\tanh(s)}{s}$ is decreasing for $s > 0$. Such inequality gives a quantitative meaning to the statement "$\varepsilon < 1$" small. From now on we will assume his validity.

But how the phase-plane analysis plays a crucial role for the solution of this problem?

The answer to this question is given by the following Lemma, which his proved in the paper.

Lemma 1. *Given $Y_0 \in [0, \beta)$, let $\theta = \theta(Y_0) > 0$ be the time needed for the solution$(X(t), Y(t))$ of* (1.37) *with initial data (π, Y_0) to intersect the line $X = -\pi$. Then the phase curve $(X(t), Y(t))$ corresponds via* (1.43) *to a periodic solution of* (1.29) *if and only if $\theta = \frac{2\pi}{ck}$.*

In view of the Lemma, the statement of non existence of periodic solutions of (1.29), and therefore the non existence of closed orbits for the water particles follows once we show that

$$\theta(Y_0) > \frac{2\pi}{ck} \quad Y_0 \in [0, \beta) \tag{1.46}$$

This inequality can be checked by direct computation for $Y_0 = 0$, because in this case the trajectory lies on the X-axis and it is determined by the solution of the differential equation

$$\frac{dX}{dt} = kM\cos(X) - kc \tag{1.47}$$

with initial data $X(0) = \pi$. We omit such direct computation whis in any case is in the paper.

Consider now the case when $Y_0 \in (0, \beta)$ since for $Y_0 = \beta$, $\gamma^+(P_\beta)$ is actually the separatrix and therefore never reaches $X = -\pi$ in finite time. If we follow $\gamma^+(P)$ we easly see that such positive semi-trajectory intersects the line $X = \frac{\pi}{2}$ at some point $(\frac{\pi}{2}, Y_1)$ with $Y_1 \in (Y_0, Y^\star)$ and then reaches the Y-axis. Notice that $\gamma^+(P)$ lies below the line $Y = Y_1$ for $X \in (\frac{\pi}{2}, \pi]$ whereas

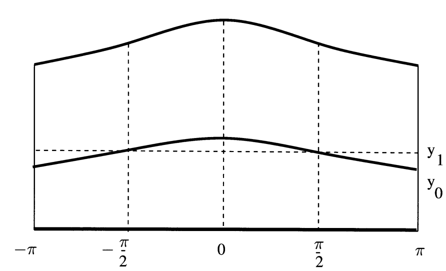

Fig. 1.15 Particle paths in the moving frame.

for $X \in [0, \frac{\pi}{2})$ it lies above it. We work only for $X > 0$ just because using the mirror simmetry we have a similar result in $[-\pi, 0]$. This is the key result because now in virtue of the second equation of system one can argue that $\theta(Y_0)$ is more than the time necessary if we move on the line $Y = Y_1$ from $X = \pi$ to $X = -\pi$. But arguing as before this time can be determined by direct computation on the line $Y = Y_1$ because in this case it is determined by the solution of the differential equation

$$\frac{dX}{dt} = kM \cosh(Y_1)\cos(X) - kc \tag{1.48}$$

with initial data $X(0) = \pi$. And also in this case, with the direct computation what we omit for sake of simplicity, one can check that this time is actually more than $\frac{2\pi}{ck}$.

With the help of phase plane analysis we proved that in time θ the particle traces a loop that fails to close-up: there is a small forward drift,and in the paper it is also shown, in virtue of the above mentioned direct computations, that this forward drift is minimal on the flat bed. We finish this note just

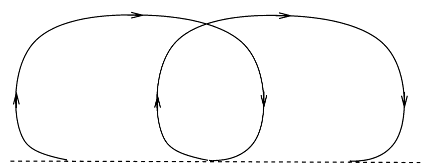

Fig. 1.16 Particle trajectory above the flat bed.

recalling that after this paper other results in water waves theory were proved using phase plane analysis, see for instance Constantin, Ehrnström and V. [4], Ehrnström and V.[9], Ehrnström, Esher and V. [10], Henry [15, 16] and Matioc [24]. These results agree with sperimental results in Umeyama [31], with numerical results in Nachbin and Ribeiro-Junior[25] and with results from nonlinear equations without approximation in Constantin [2, 3], Henry [14] and in Okamoto and Shåji [26]. And, as far as we know, more people is now attacking this kind of problems using this classical technique.

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