Some remarks on Schwarz–Christoffel transformations from the unit disk to a regular polygon and their numerical computation

by

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Abstract. In this paper we give a description of some particular aspects of Schwarz–Christoffel transformations either from a theoretical point of view or related to numerical approach. Furthermore some interesting geometrical properties of these transforms are shown and displayed by means of a graphical interface. In particular, an approximation of the identity in the unit disc of $\mathbb{C}$ by means of a sequence of Schwarz–Christoffel transformations is exhibited.

Key words. Schwarz–Christoffel transformations, Riemann mapping theorem for regular polygons.

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0.1 Introduction to Schwarz–Christoffel transformations

In 1851, in his Dissertation Thesis, Bernhard Riemann proved that any simply connected open subset of $\mathbb{C}$ different from $\mathbb{C}$ is biholomorphically equivalent to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{C}$. This result (nowadays known as the *Riemann mapping theorem*) gives only few information about the biholomorphism involved and, indeed, its analytic expression remains completely unknown except for some special class of open sets in $\mathbb{C}$, such as the halfplanes[1] or the polygons. If P_n is a polygon of n edges in $\mathbb{C}$, the value of a biholomorphism $f : \mathbb{D} \longrightarrow P_n$ in $z \in \mathbb{D}$ can be calculated from an expression (known as *Schwarz–Christoffel transformation* in $\mathbb{D}$) which involves an integral of a product of n functions related to the "geometry of the polygon". To be more precise, if the vertices of a polygon P_n are denoted by $w_1, \ldots, w_n$ and $\beta_i \pi$ is the measure of the amplitude of the outer angle of P_n at the vertex w_i, one has (see, for instance, [15])

$$f(z) = A \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - e^{i\vartheta_k})^{-\beta_k} \, d\zeta + B, \tag{1}$$

where z_0 is a point in $\mathbb{D}$, $A \neq 0$ and B are two constants, and $e^{i\vartheta_1}, \ldots, e^{i\vartheta_n} \in \partial\mathbb{D}$, called *prevertices*, are such that $f(e^{i\vartheta_k}) = w_k$ for any $k = 1, \ldots, n$. Because of the Cayley transformation, it is easy to see that if $g : \mathbb{H} \longrightarrow P_n$ is a biholomorphism,

[1] If the halfplane is $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ then the map, called *Cayley transformation*

$$\begin{aligned} \tau_c : \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{z - i}{z + i}, \end{aligned}$$

is a biholomorphism of $\mathbb{H}$ in $\mathbb{D}$ which maps i into 0 and has as its inverse the map

$$\begin{aligned} \tau_c^{-1} : \mathbb{D} &\longrightarrow \mathbb{H} \\ w &\longmapsto i\frac{1 + w}{1 - w}. \end{aligned}$$

then

$$g(z) = A \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - x_k)^{-\beta_k}\, d\zeta + B, \tag{2}$$

where z_0 is a point in $\mathbb{H}$, $A \neq 0$ and B are two constants, and $x_1, \ldots, x_n \in \mathbb{R}$, called *prevertices*, are such that $f(x_k) = w_k$ for any $k = 1, \ldots, n$. We will define $\mathcal{SC}_n\mathbb{D}$ as the set of all Schwarz–Christoffel transformations in $\mathbb{D}$ such that $\beta_1 + \beta_2 + \ldots + \beta_n = 2$ whereas $\mathcal{SC}_n\mathbb{H}$ will be the set of all

$$z \mapsto A \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - x_k)^{-\beta_k}\, d\zeta + B \tag{3}$$

where $A, B \in \mathbb{C}$, $A \neq 0$, and $z_0 \in \mathbb{H}$ are constants, $x_k \in \mathbb{R}\ \forall k = 1, \ldots n$ and[2] $\beta_k \in (-1, 1)\ \forall k = 1, \ldots n$ are such that $\beta_1 + \beta_2 + \ldots + \beta_n = 2$. The set of all biholomorphisms of $\mathbb{D}$ (or $\mathbb{H}$) onto P_n will be denoted by $\mathcal{B}_n\mathbb{D}$ (or $\mathcal{B}_n\mathbb{H}$). Furthermore the biholomorphisms of $\mathcal{B}_n\mathbb{D}$ and of $\mathcal{B}_n\mathbb{H}$ turn out to be continuous in $\overline{\mathbb{D}}$ or in $\overline{\mathbb{H}} \cup \{\infty\}$ respectively: this follows from a well–known theorem due to Carathéodory (see [2] or [10]), which asserts that given a bounded open set $\Omega \subset \mathbb{C}$, a biholomorphism $f : \mathbb{D} \longrightarrow \Omega$ can be extended to a homeomorphism $\widehat{f}$ of $\overline{\mathbb{D}}$ in $\overline{\Omega}$, if and only if $\partial\Omega$ is a Jordan curve (for a complete and self–contained exposition of these aspects see [1]). On the other hand, it is not true that, given a map $f \in \mathcal{SC}_n\mathbb{H}$ (or $f \in \mathcal{SC}_n\mathbb{D}$) then necessarily $f \in \mathcal{B}_n\mathbb{H}$ (or $f \in \mathcal{B}_n\mathbb{D}$). We remind that for $g \in \mathcal{SC}_n\mathbb{H}$ to belong to $\mathcal{B}_n\mathbb{H}$ it suffices to be injective in $\mathbb{R}$ (see, for instance, [8]). But in 1983 E. Johnston (see [9]) showed that the Schwarz–Christoffel transformation g in $\mathbb{H}$ given by

$$g(z) = \int_{z_0}^{z} \prod_{k=1}^{6} (\zeta - x_k)^{-\beta_k}\, d\zeta$$

where

$$x_1 = -10, \qquad x_2 = -\frac{1}{10}, \qquad x_3 = 0,$$
$$x_4 = \frac{1}{10}, \qquad x_5 = \frac{1}{5}, \qquad x_6 = 10$$

and

$$\beta_1 = \beta_6 = -\frac{5}{7},$$
$$\beta_2 = \beta_3 = \beta_4 = \beta_5 = \frac{6}{7},$$

is not injective in $\mathbb{H}$ and so $g \notin \mathcal{B}_n\mathbb{H}$. Notice that, as observed before, the function g cannot be injective in $\mathbb{R} \cup \{\infty\}$, and then $g(\mathbb{R} \cup \{\infty\})$ is not a simple curve but it has self–intersections. The existence of such a map g can be directly obtained as an application of the following (see [9])

[2] If one admits a prevertex in ∞, then the number of factors in (3) is $n - 1$.

Theorem 1. *If $g \in \mathcal{SC}_n\mathbb{H}$ is injective in $\mathbb{H}$ then*

$$\left|\sum_{k=1}^{n} \frac{\beta_k}{z_0 - x_k}\right| < \frac{3}{\operatorname{Im} z_0}$$

for any $z_0 \in \mathbb{H}$.

Since for the map g it is not difficult to see that

$$\left|\sum_{k=1}^{6} \frac{\beta_k}{i - x_k}\right| > 3 = \frac{3}{\operatorname{Im} i},$$

Theorem 1 guarantees that g is not injective in $\mathbb{H}$ and so $g \notin \mathcal{B}_n\mathbb{H}$. In [9], Johnston proves the following

Lemma 2. *Let G be a holomorphic and injective function in $\mathbb{H}$, then*

$$\left|\frac{G''(z_0)}{G'(z_0)}\right| < \frac{3}{\operatorname{Im} z_0}$$

for any $z_0 \in \mathbb{H}$.

After that, the author claims that Theorem 1 is a consequence of the previous Lemma. This can be shown by considering that if $g \in \mathcal{SC}_n\mathbb{H}$, then, for any $z \in \mathbb{H}$, one has

$$g'(z) = \prod_{k=1}^{n} (z - x_k)^{-\beta_k},$$
$$g''(z) = -\sum_{j=1}^{n} \beta_j (z - x_k)^{-\beta_j - 1} \prod_{\substack{k=1 \\ k \neq j}}^{n} (z - x_k)^{-\beta_k}$$

so that

$$\frac{g''(z)}{g'(z)} = -\sum_{k=1}^{n} \frac{\beta_k (z - x_k)^{-\beta_k - 1}}{(z - x_k)^{-\beta_k}} = \sum_{k=1}^{n} \frac{-\beta_k}{z - x_k},$$

i.e. g satisfies the differential equation

$$g''(z) + g'(z) \cdot \sum_{k=1}^{n} \frac{\beta_k}{z - x_k} = 0. \tag{4}$$

Actually, *every* (holomorphic and locally injective in $\mathbb{H}$) solution of (4) belongs to $\mathcal{SC}_n\mathbb{H}$, how it is stated in the next Lemma.

Lemma 3. *Let g be a locally injective and holomorphic function in $\mathbb{H}$. Then g is a solution of the differential equation (4) if and only if g belongs to $\mathcal{SC}_n\mathbb{H}$.*

Proof. It has been shown above that every $g \in \mathcal{SC}_n\mathbb{H}$ is a solution of (4). Now, let g be a locally injective and holomorphic function in $\mathbb{H}$ which satisfies (4). Since g is a locally injective function in $\mathbb{H}$, then g' never vanishes and, therefore, the logarithm ℓ of g' is defined in $\mathbb{H}$. Since (4) is satisfied, one can write

$$\ell'(z) = \frac{g''(z)}{g'(z)} = \sum_{k=1}^{n} \frac{-\beta_k}{z - x_k} \quad \text{for any } z \in \mathbb{H}.$$

For $k = 1, \ldots, n$ let ℓ_k be the logarithm of $g_k(z) = z - x_k$ in $\mathbb{H}$ in such a way that $(z - x_k)^{-\beta_k} = |z - x_k|^{-\beta_k} \cdot e^{-i\beta_k \arg \ell_k(z)}$. Given $z_0 \in \mathbb{H}$ we have

$$\ell(z) = \int_{z_0}^{z} \sum_{k=1}^{n} \frac{-\beta_k}{\zeta - x_k}\, d\zeta = \sum_{k=1}^{n} \int_{z_0}^{z} \frac{-\beta_k}{\zeta - x_k}\, d\zeta =$$

$$= -\sum_{k=1}^{n} \beta_k \int_{z_0}^{z} \frac{d\zeta}{\zeta - x_k} = -\sum_{k=1}^{n} \beta_k \left[\ell_k(z) - \ell_k(z_0)\right] =$$

$$= -\sum_{k=1}^{n} \beta_k \ell_k(z) + \sum_{k=1}^{n} \beta_k \ell_k(z_0).$$

If we define $C = \sum_{k=1}^{n} \beta_k \ell_k(z_0)$ and $A = e^C \neq 0$ we finally obtain

$$g'(z) = \exp\left(-\sum_{k=1}^{n} \beta_k \ell_k(z) + C\right) = A \exp\left(-\sum_{k=1}^{n} \beta_k \ell_k(z)\right) =$$

$$= A \prod_{k=1}^{n} \exp\left[-\beta_k \ell_k(z)\right] = A \prod_{k=1}^{n} (z - x_k)^{-\beta_k},$$

hence, after putting $B := g(z_0)$, we get

$$g(z) = A \int_{z_0}^{z} \prod_{k=1}^{n} (\zeta - x_k)^{-\beta_k} d\zeta + B$$

that is $g \in \mathcal{SC}_n\mathbb{H}$. □

Now it is very interesting to continue by considering some special cases such as the biholomorphisms f_n of $\mathbb{D}$ into the regular polygon of n edges (shortly the n–agon); its vertices can be easily described as the n–th roots of the unity, that is $w_k = e^{2\pi(k-1)i/n}$. If we assume (as it is natural to think) that the prevertices coincide with the vertices of the n–agon and that $f_n(0) = 0$, then $\beta_k = 2/n$ $\forall k = 1, \ldots n$ and

$$f_n(z) = A \int_0^z \prod_{k=1}^{n} (\zeta - e^{2\pi(k-1)i/n})^{-2/n}\, d\zeta$$

or

$$f_n(z) = A' \int_0^z (\zeta^n - 1)^{-2/n}\, d\zeta;$$

so we are left only to evaluate the constant A'. To do so, we recall that integrating over the real segment $[0,1] \subset \mathbb{R}$ one has

$$I := \int_0^1 \frac{ds}{(1-s^n)^{2/n}} = \frac{1}{n}\int_0^1 \tau^{1/n-1}(1-\tau)^{-2/n}\, d\tau. \tag{5}$$

Since the Beta function is defined, for $s, t > 0$, as

$$\mathsf{B}(s,t) = \int_0^1 \tau^{s-1}(1-\tau)^{t-1}\, d\tau,$$

and since B and the Γ function are related by

$$\mathsf{B}(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad \text{for } s, t > 0,$$

it is possible to rewrite (5) in the following form

$$I = \frac{1}{n}\mathsf{B}\left(\frac{1}{n}, 1 - \frac{2}{n}\right) = \frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{2}{n})}{n\Gamma(1-\frac{1}{n})}.$$

Now $f_n(1) = 1$ and $f_n(1) = A' \cdot I$, so that

$$A' = \frac{1}{I} = \frac{n\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)}.$$

Since the following identity holds

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t} \quad \text{for } t > 0,$$

we have

$$A' = \frac{n}{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})} \cdot \frac{\Gamma\left(1-\frac{1}{n}\right)^2}{\Gamma\left(1-\frac{2}{n}\right)} = \frac{n \sin\frac{\pi}{n}}{\pi} \cdot \frac{\Gamma\left(1-\frac{1}{n}\right)^2}{\Gamma\left(1-\frac{2}{n}\right)}. \tag{6}$$

Finally we can write the biholomorphism f_n in the following form

$$f_n(z) = \frac{n \sin\frac{\pi}{n}}{\pi} \cdot \frac{\Gamma\left(1-\frac{1}{n}\right)^2}{\Gamma\left(1-\frac{2}{n}\right)} \int_0^z \frac{d\zeta}{(1-\zeta^n)^{2/n}}. \tag{7}$$

There is some evidence that as n increases the polygon $P_n = f_n(\mathbb{D})$ inscribed in $\partial\mathbb{D}$ is a better approximation of the disc $\mathbb{D}$ itself. Since each biholomorphism

f_n fixes 0 and 1, it is conceivable believe that the sequence of biholomorphisms $\{f_n\}_{n\geq 3}$ converges to the identity map; this is proved[3] in the following

Proposition 4. *The sequence of biholomorphisms $\{f_n\}_{n\geq 3}$ defined in (7) converges uniformly on compact sets $K \subset \mathbb{D}$ to the identity.*

Proof. Since the sequence $\{f_n\}_{n\geq 3}$ is bounded, it is, in particular, locally bounded. By Montel Theorem (see, for instance, [12]) it suffices to prove that given any subsequence $\{f_{n_k}\}_{k\geq 0}$ of $\{f_n\}_{n\geq 3}$ which converges uniformly on compact sets $K \subset \mathbb{D}$, it converges to the identity. Let $\{f_{n_k}\}_{k\geq 0}$ be a subsequence of $\{f_n\}_{n\geq 3}$ which converges uniformly on compact sets $K \subset \mathbb{D}$ to the holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C}$. Since $f_{n_k}(\mathbb{D}) \subset \mathbb{D}$ for any $k \geq 0$, one has $f(\mathbb{D}) \subset \overline{\mathbb{D}}$; but since f is an open map, $f(\mathbb{D}) \subset \mathbb{D}$. Finally, since

$$f(0) = \lim_{k\to+\infty} f_{n_k}(0) = 0,$$

$$f'(0) = \lim_{k\to+\infty} f'_{n_k}(0) = \lim_{k\to+\infty} \frac{\sin\frac{\pi}{n_k}}{\pi/n_k} \cdot \frac{\Gamma\left(1-\frac{1}{n_k}\right)^2}{\Gamma\left(1-\frac{2}{n_k}\right)} = 1,$$

the proposition follows from the Schwarz Lemma (see, for instance, [11]). $\square$

0.2 Numerical approach to Schwarz–Christoffel transformations

As seen in the previous section, even for the case of regular polygons, the full description of a Schwarz–Christoffel transformation relies upon the *choice of some parameters*, that is to say essentially the choice of the prevertices, since the constants A and B and the real numbers $\beta_1, \ldots, \beta_n$ can be obtained by geometric considerations. But even when one has all the parameters required, the evaluation of the integral in (1) still remains in general impossible so that it is normally preferred a numerical approach to the problem. This way of considering the evaluation of Schwarz–Christoffel transformations starts in the 60's, but one of the major contributions is of the end of the 70's when L. N. Trefethen wrote the package SCPACK in Fortran (see [13, 14]), which numerically solves the problem of determining the parameters and of evaluating the integral of a Schwarz–Christoffel transformation for a generic polygon by means of the *Gauss–Jacobi formulae* (see, for instance, [3]). More recently, in 1996, T. A. Driscoll has developed the algorithms proposed by G. H. Golub and J. H. Welsch (see [7] and [5]) [4] already adopted by Trefethen

[3]The proof given here of this quite intuitive proposition – a disc can be approximated by regular polygons – seems to be new or, at least, so the proof looks like to the authors, even though it is from the Ancient Greek period that many mathematicians and philosophers have developed and used this idea in several ways. In particular the Neoplatonic philosopher Nikolaus Chrypffs or Krebs (1401-1464), born in Cusa (near Treviri, now Germany) and therefore known as “Cusanus”, used this image to describe human knowledge when compared to the vastness of the Universe.

[4]For polygons having an irregular shape, this algorithm may lead to a loss of accuracy, mainly due to the *crowding* of the prevertices. This phenomenon can easily appear even with low values of n, because it depends on polygon “elongation”. T. A. Driscoll and S. A. Vavasis have described

and implemented them for MATLAB[5]; the toolbox he obtained is the *Schwarz–Christoffel toolbox* (briefly "SC toolbox") [6] that can be freely downloaded from the site

`http://www.math.udel.edu/˜driscoll/software`

The SC toolbox offers also some graphical *functions*; one of them allows the insertion an orthogonal grid in $\mathbb{D}$ or in $\mathbb{H}$ and displays its image through a Schwarz–Christoffel transformation. These graphical functions are integrated in a graphical interface or GUI[7] (generated by the *script file* `scgui.m`) which accepts the definition of the vertices of a polygon simply by clicking on a mouse. As examples of these functions, in figure 1 the images of $\mathbb{D}$ through f_n (as introduced in Proposition 4) are shown for $n = 5, 6, 7, 8$. These images can be easily obtained by evaluating in $5, 6, 7$ and 8 the *function* `nagon`, defined by the following *script*[8] of MATLAB

```
function nagon(n)
% definition of inner angles
b = zeros(n,1);
b(1:n,1) = -2/n;
% definition of vertices
w = zeros(n,1);
for j=1:n
   w(j,1) = exp((2*pi*i*(j))/n);
end;
% definition of prevertices
z = w;
% definition of the constant
C = (n/pi)*sin(pi/n)*gamma(1-1/n)^2/gamma(1-2/n);
dplot(w, b, z, C)
```

in [6] a method to avoid the crowding of the prevertices in case of elongated polygons; this method (which is implemented in the SC Toolbox) is based on Möbius transformations and on existence (and computability) of a particular triangulation for the polygons.

[5] MATLAB is a trade mark registered by The MathWorks, Inc.

[6] To determine the parameters involved, the SC toolbox applies the package NESOLVE (written by R. Behrens), which is an implementation of the Newton–Broyden method with line search (see [4]) and which solves the non linear system which the integral conditions are reduced to.

[7] Acronym of *Graphical User Interface*.

[8] Note that in this script (and in the other which is described later on in this paper) we do not let the SC Toolbox calculate the multiplicative constant of the transformation, but we assign it explicitly, using the expression found in (6); this is a rare case in which the constant can be written down without referring directly to the complex integral which define the SC transformation.

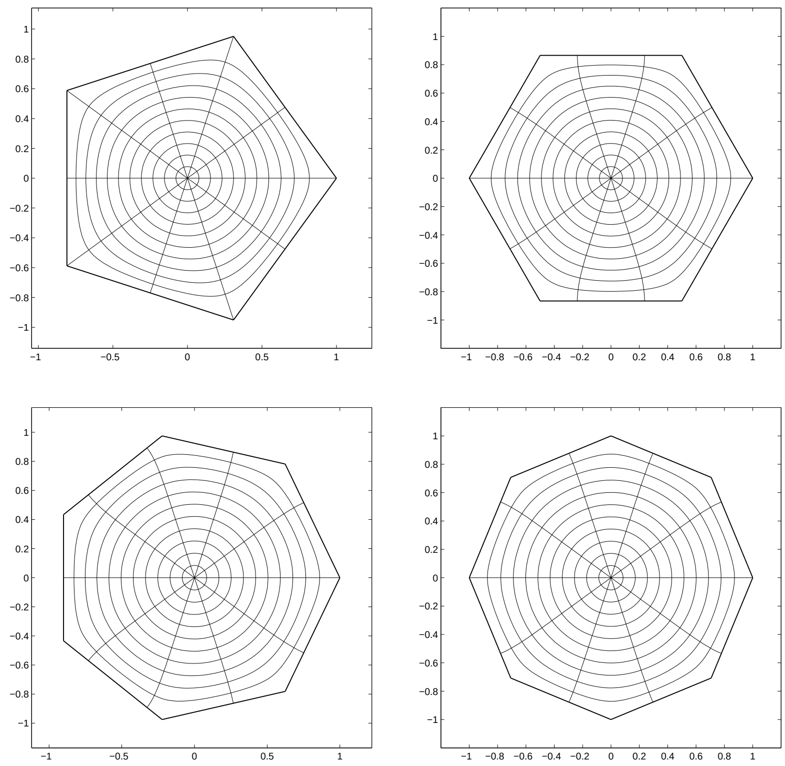

Figure 1.

In general, for the *function* `dplot` with the parameters

$$\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n,$$
$$\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n,$$
$$\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n \quad \text{and} \quad C \in \mathbb{C},$$

the SC toolbox evaluates the function

$$f(z) = C \int_0^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{z_k}\right)^{b_k} \, d\zeta;$$

since, for $k = 1, \ldots, n$, the function

$$p_k(\zeta) = 1 - \frac{\zeta}{z_k}$$

never vanishes in $\mathbb{D}$ and $|p_k(\zeta) - 1| < 1$ if $\zeta \in \mathbb{D}$, we can define

$$\left(1 - \frac{\zeta}{z_k}\right)^{b_k} = \left|1 - \frac{\zeta}{z_k}\right|^{b_k} e^{b_k i \varphi(\zeta)},$$

where $\varphi(\zeta) = \arg\left(1 - \frac{\zeta}{z_k}\right) \in (-\pi, \pi)$. The *function* `dplot` automatically plots the closed curves σ_k which are obtained as images, through f_n, of the circles

$$\gamma_k(t) = \frac{k}{11} e^{it}, \quad t \in [0, 2\pi], \quad k = 1, \dots, 10. \tag{8}$$

It also plots the arcs ρ_k which are obtained as images, through f_n, of the radii in $\mathbb{D}$ defined as follows

$$r_k(t) = te^{\frac{k-1}{5}\pi i}, \quad t \in [0, 1], \quad k = 1, \dots, 10. \tag{9}$$

Other examples of application of the graphical functions are figure 2 and figure 3. It is important to remark that in the first case, for the assigned triangle of vertices $0, c > 0, w \in \mathbb{H}$ [9] and inner angles of measure $\alpha_i \pi$, the biholomorphism with prevertices $0, 1, \infty$ can be written explicitly, namely

$$g(z) = \frac{c\pi}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \sin \pi\alpha_3} \int_0^z \zeta^{\alpha_1 - 1}(1 - \zeta)^{\alpha_2 - 1} \, d\zeta, \tag{10}$$

whereas for the second case this is far for being possible (see [1] for details), even though only one vertex is added!

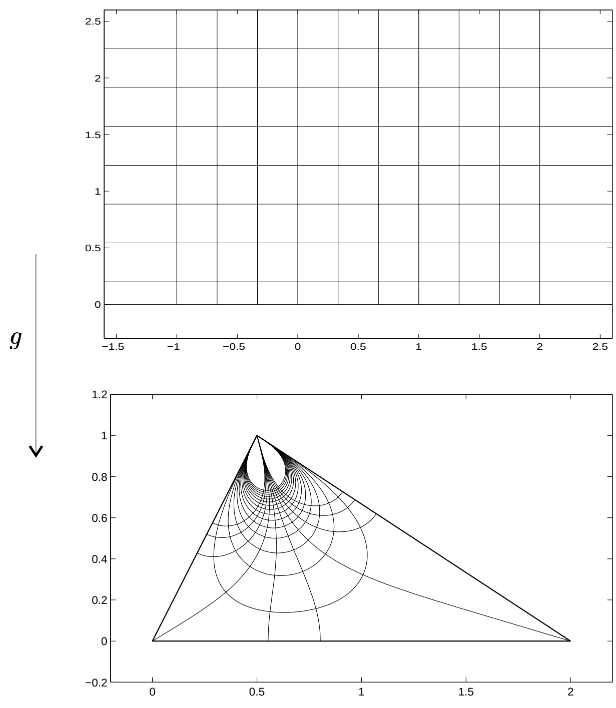

Figure 2.

[9] In figure 2 we have $c = 2$ and $w = \frac{1}{2} + i$.

Having in mind the problem of describing the biholomorphism of $\mathbb{D}$ into a regular polygon, we have slightly modified the GUI created by Driscoll by developing the *script* described at the beginning of the section; in this way the new *script*, denominated `nagongui.m`, and available at the site

`http://www.math.unifi.it/~vlacci`

generates the graphical interface shown in figure 4 (and run in Windows™ environment). When the SC toolbox is properly installed, the GUI may be started by entering `nagongui` in the *command window* of MATLAB.

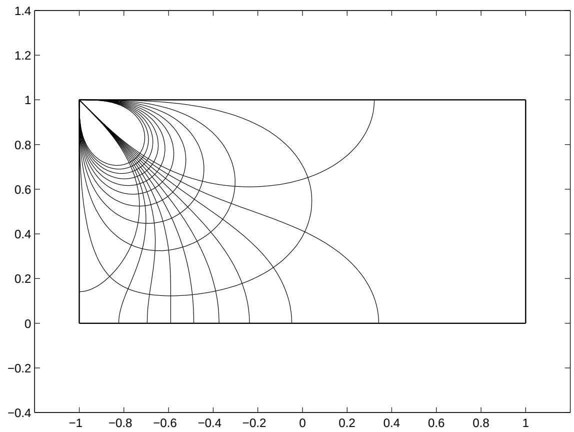

Figure 3.

Hence, after inserting a positive number n in N. of vertices (3 is by default) and clicking Trace polygon, the program displays $\partial\mathbb{D}$ on the left and a regular polygon P_n of n edges on the right. An orthogonal grid in $\mathbb{D}$ can be inserted by clicking Trace grid and its image in P_n (through the transformation f_n defined in (7)) is displayed on the right; in figure 4 an example for $n = 5$ is exhibited.

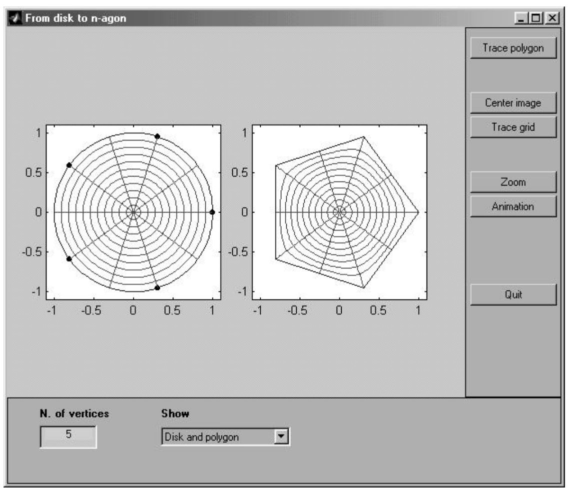

Figure 4.

The biholomorphism f_n is such that $f_n(0) = 0$ for $n \geq 3$. After clicking Center image and using the mouse, it is possible to move $f_n(0)$ to any point w_0 in P_n. The program automatically determines a biholomorphism $f : \mathbb{D} \longrightarrow P_n$ such that $f(0) = w_0$ by considering $f := f_n \circ \varphi$ where φ is an automorphism of $\mathbb{D}$ such that $\varphi(0) = f_n^{-1}(w_0)$. In this way the problem of determining the prevertices is essentially skipped (see figure 5).

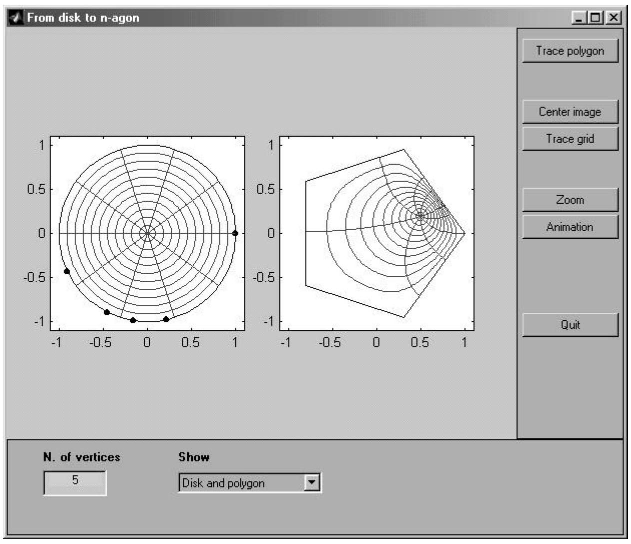

Figure 5.

To see the grid more in detail, one can click on Zoom and select the option Polygon only below the label Show.
The graphical interface `nagongui` gives also the possibility of showing several images in sequence; this is achieved by clicking on Animation. Let $g : \mathbb{D} \longrightarrow P_n$ be the biholomorphism currently shown in the GUI, that is the map which the GUI is using to calculate the image in P_n of the orthogonal grid in $\mathbb{D}$. The option Animation shows ten pictures in sequence of the map

$$\begin{aligned} G : \mathbb{D} \times [0,1] &\longrightarrow \mathbb{D} \\ (z,s) &\longmapsto s \cdot g(z) + (1-s) \cdot z, \end{aligned}$$

which is a *homotopy* between g and the identity; more precisely, each picture graphically represents the image of a map

$$\begin{aligned} G_j : \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{j}{9} \cdot g(z) + \left(1 - \frac{j}{9}\right) \cdot z, \end{aligned}$$

where $j = 0, \dots, 9$; moreover, in the same picture, the GUI traces the orthogonal grid which is obtained as image, through a map G_j, of the curves γ_k and r_k defined in (8) and (9).

If, in particular, one chooses $g = f_n$, then the homotopy G fixes the origin, that is $G(0, s) = 0$ for all $s \in [0, 1]$; in this case the animation shows the disk $\mathbb{D}$ which "deflates" itself till it coincides with a regular n–agon (see the example frame in figure 6).

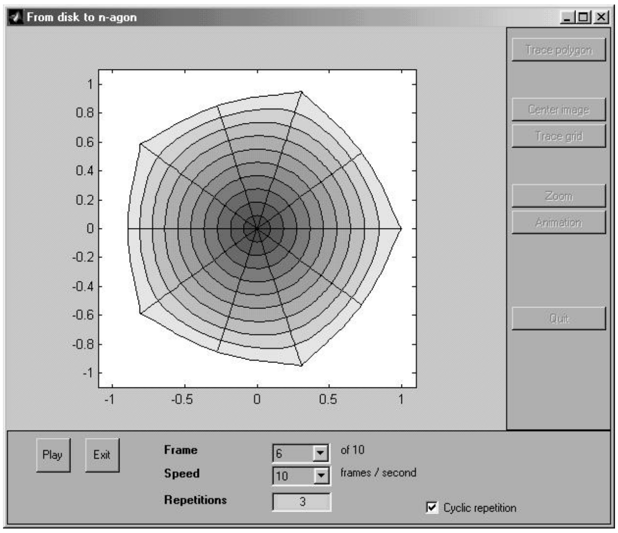

Figure 6.

The movement displayed is less and less perceptible as n increases, because of the convergence of the sequence $\{f_n\}_{n\geq 3}$ to the identity.
On the other hand, if the biholomorphism $g : \mathbb{D} \longrightarrow P_n$ chosen is such that $g(0) = w_0 \in P_n \setminus \{0\}$, then the resulting animation is more sophisticated (see figure 7), because neither all the vertices of the polygon nor the origin is fixed; in particular, the animation shows quite clearly the "movement" of the origin towards the point w_0.

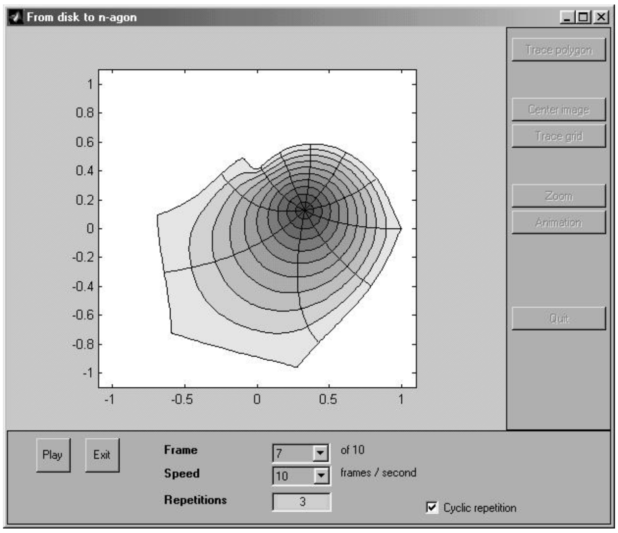

Figure 7.

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