Vladimír Souček: On three definitions of holomorphicity in Clifford analysis.

Abstract

It is a classical fact in complex function theory that there are three equivalent definitions of a holomorphic function:

1. Cauchy-Rieman (CR) equations
2. Sums of power series (Weierstrass)
3. Differentiablity = existence of the derivative $f'(z)$.

In higher dimensions, there are analogues of these definitions but they are no more equivalent.

The first definition (CR equations) is generalized to higher dimensions as the Dirac equation (solutions are called monogenic funcgtions). It is the standard definition used in Clifford analysis. Values of monogenic functions are either in Clifford algebra, or in a primitive left ideal of the Clifford algebra. The study of properties of monogenic functions is quite advanced after several decades of effort.

Recently, the second definition (Weierstrass) was intensively studied under the name of slice monogenic functions. It is a generalization of the old Cullen's idea but it developed quickly into an independent field inside Clifford analysis with a lot of interesting properties.

In the talk, I will concentrate on higher dimensional analogues of the third definition. This case is heavily underdeveloped due to one principal obstacle. It was found many times (independently) for the quaternionic case that the notion of quaternionic differentiability leads to very restricted class of functions (a subspace of linear functions), hence it was not interesting from function theory point of view. The same is true in higher dimensions. In fact, the condition for quaternionic differentiability is an overdetermined system of equation known under the name the twistor equation. It always has a finite dimensional space of solutions.

However, there is an interesting possibility how to generalize the differentiability condition which also solves the old problem in Clifford analysis (the space of monogenic functions in not a ring!). For spinor valued version of Clifford analysis, it is clear that there is no possibility to multiply monogenic functions, because there is no invariant product on the spinor space with values in the same space. Only solution is to extend the space of values.

In the lecture, we shall discuss the following possibility. There is a nice product on the sum of all irreducible Spin representations called the Cartan

product. We can consider the sum $\mathbb{S} := \oplus_0^\infty S^n$ of all Cartan powers of the basic spinor representation S. It is a (commutative) ring. It has infinite dimension as a vector space. For functions with values in $\mathbb{S}$, the usual differentiability condition can be applied and the space of solutions is quite rich. It has infinite dimension and product of two solutions is again a solution. Hence the space of all solutions has a structure of a commutative ring. It seems to be a nice and reasonable generalization of the differentiability condition to higher dimension. This possibility for dimension equal to 4 was, in fact, already studied by N. Hitchin even in more general context (the same equations can be extended to self-dual Riemannian 4-manifolds and there is a twistor interpretation of the space of solutions). In dimension 4, it was possible also to extend the equations to the case when the equations are coupled with a fixed Yang-Mills field. Generalizations of all that from dimension 4 to higher dimensions are still to be studied (with partial results available).