PROJECTIVE GEOMETRY

b3 course 2003

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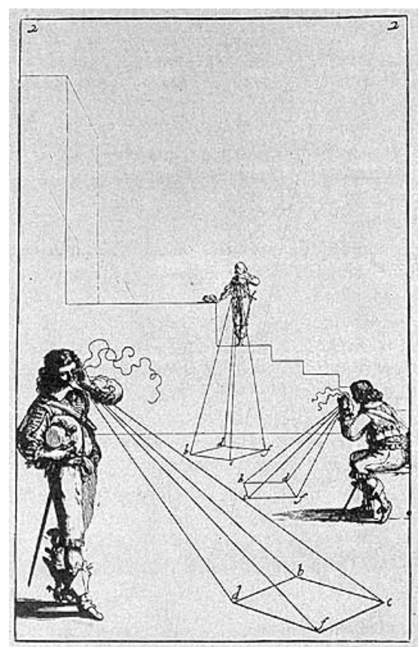
1 Introduction

This is a course on projective geometry. Probably your idea of geometry in the past has been based on triangles in the plane, Pythagoras' Theorem, or something more analytic like three-dimensional geometry using dot products and vector products. In either scenario this is usually called *Euclidean geometry* and it involves notions like distance, length, angles, areas and so forth. So what's wrong with it? Why do we need something different?

Here are a few reasons:

- Projective geometry started life over 500 years ago in the study of perspective drawing: the distance between two points on the artist's canvas does not represent the true distance between the objects they represent so that Euclidean distance is not the right concept.

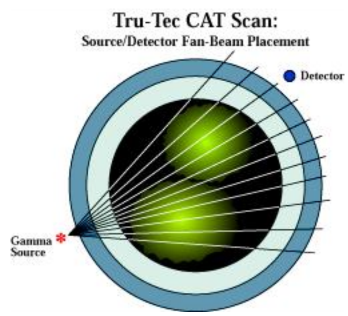


  The techniques of projective geometry, in particular homogeneous coordinates, provide the technical underpinning for perspective drawing and in particular for the modern version of the Renaissance artist, who produces the computer graphics we see every day on the web.

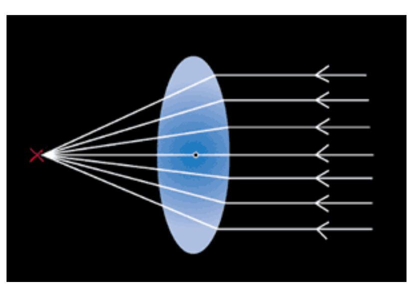
- Even in Euclidean geometry, not all questions are best attacked by using distances and angles. Problems about intersections of lines and planes, for example are not really metric. Centuries ago, projective geometry used to be called "descriptive geometry" and this imparts some of the flavour of the subject. This doesn't mean it is any less quantitative though, as we shall see.

- The Euclidean space of two or three dimensions in which we usually envisage geometry taking place has some failings. In some respects it is incomplete and asymmetric, and projective geometry can counteract that. For example, we know that through any two points in the plane there passes a unique straight line. But we can't say that any two straight lines in the plane intersect in a unique point, because we have to deal with parallel lines. Projective geometry evens things out – it adds to the Euclidean plane extra points at infinity, where parallel lines intersect. With these new points incorporated, a lot of geometrical objects become more unified. The different types of conic sections – ellipses, hyperbolas and parabolas – all become the same when we throw in the extra points.

- It may be that we are only interested in the points of good old $\mathbf{R}^2$ and $\mathbf{R}^3$ but there are always other spaces related to these which don't have the structure of a vector space – the space of lines for example. We need to have a geometrical and analytical approach to these. In the real world, it is necessary to deal with such spaces. The CT scanners used in hospitals essentially convert a series of readings from a subset of the space of straight lines in $\mathbf{R}^3$ into a density distribution.



At a simpler level, an optical device maps incoming light rays (oriented lines) to outgoing ones, so how it operates is determined by a map from the space of straight lines to itself.



> Projective geometry provides the means to describe analytically these auxiliary spaces of lines.

In a sense, the basic mathematics you will need for projective geometry is something you have already been exposed to from your linear algebra courses. Projective geometry is essentially a geometric realization of linear algebra, and its study can also help to make you understand basic concepts there better. The difference between the points of a vector space and those of its dual is less apparent than the difference between a point and a line in the plane, for example. When it comes to describing the space of lines in three-space, however, we shall need some additional linear algebra called *exterior algebra* which is essential anyway for other subjects such as differential geometry in higher dimensions and in general relativity. At this level, then, you will need to recall the basic properties of :

- vector spaces, subspaces, sums and intersections
- linear transformations
- dual spaces

After we have seen the essential features of projective geometry we shall step back and ask the question "What is geometry?" One answer given many years ago by Felix Klein was the rather abstract but highly influential statement: "Geometry is the study of invariants under the action of a group of transformations". With this point of view both Euclidean geometry and projective geometry come under one roof. But more than that, non-Euclidean geometries such as spherical or hyperbolic geometry can be treated in the same way and we finish these lectures with what was historically a driving force for the study of new types of geometry — Euclid's axioms and the parallel postulate.

2 Projective spaces

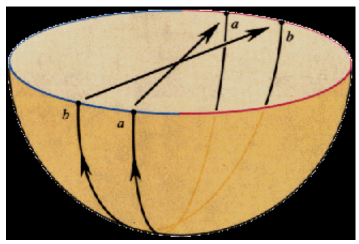
2.1 Basic definitions

Definition 1 *Let V be a vector space. The projective space $P(V)$ of V is the set of 1-dimensional vector subspaces of V.*

Definition 2 *If the vector space V has dimension $n+1$, then $P(V)$ is a projective space of dimension n. A 1-dimensional projective space is called a projective line, and a 2-dimensional one a projective plane.*

For most of the course, the field F of scalars for our vector spaces will be either the real numbers $\mathbf{R}$ or complex numbers $\mathbf{C}$. Our intuition is best served by thinking of the real case. So the projective space of $\mathbf{R}^{n+1}$ is the set of lines through the origin. Each such line intersects the unit n-sphere $S^n = \{x \in \mathbf{R}^{n+1} : \sum_i x_i^2 = 1\}$ in two points $\pm u$, so from this point of view $P(\mathbf{R}^{n+1})$ is S^n with antipodal points identified. Since each line intersects the lower hemisphere, we could equally remove the upper hemisphere and then identify opposite points on the equatorial sphere.

When $n = 1$ this is just identifying the end points of a semicircle which gives a circle, but when $n = 2$ it becomes more difficult to visualize:



If this were a course on topology, this would be a useful starting point for looking at some exotic topological spaces, but it is less so for a geometry course. Still, it does explain why we should think of $P(\mathbf{R}^{n+1})$ as n-dimensional, and so we shall write it as $P^n(\mathbf{R})$ to make this more plain.

A better approach for our purposes is the notion of a representative vector for a point of $P(V)$. Any 1-dimensional subspace of V is the set of multiples of a non-zero vector $v \in V$. We then say that v is a representative vector for the point $[v] \in P(V)$. Clearly if $\lambda \neq 0$ then λv is another representative vector so

$$[\lambda v] = [v].$$

Now suppose we choose a basis $\{v_0, \ldots, v_n\}$ for V. The vector v can be written

$$v = \sum_{i=0}^{n} x_i v_i$$

and the $n+1$-tuple $(x_0, \ldots, x_n)$ provides the coordinates of $v \in V$. If $v \neq 0$ we write the corresponding point $[v] \in P(V)$ as $[v] = [x_0, x_1, \ldots, x_n]$ and these are known as homogeneous coordinates for a point in $P(V)$. Again, for $\lambda \neq 0$

$$[\lambda x_0, \lambda x_1, \ldots, \lambda x_n] = [x_0, x_1, \ldots, x_n].$$

Homogeneous coordinates give us another point of view of projective space. Let $U_0 \subset P(V)$ be the subset of points with homogeneous coordinates $[x_0, x_1, \ldots, x_n]$

such that $x_0 \neq 0$. (Since if $\lambda \neq 0$, $x_0 \neq 0$ if and only if $\lambda x_0 \neq 0$, so this is a well-defined subset, independent of the choice of $(x_0, \dots, x_n)$). Then, in U_0,

$$[x_0, x_1, \dots, x_n] = [x_0, x_0(x_1/x_0), \dots, x_0(x_n/x_0)] = [1, (x_1/x_0), \dots, (x_n/x_0)].$$

Thus we can uniquely represent any point in U_0 by one of the form $[1, y_1, \dots, y_n]$, so

$$U_0 \cong F^n.$$

The points we have missed out are those for which $x_0 = 0$, but these are the 1-dimensional subspaces of the n-dimensional vector subspace spanned by $v_1, \dots, v_n$, which is a projective space of one lower dimension. So, when $F = \mathbf{R}$, instead of thinking of $P^n(\mathbf{R})$ as S^n with opposite points identified, we can write

$$P^n(\mathbf{R}) = \mathbf{R}^n \cup P^{n-1}(\mathbf{R}).$$

A large chunk of real projective n-space is thus our familiar $\mathbf{R}^n$.

Example: The simplest example of this is the case $n = 1$. Since a one-dimensional projective space is a single point (if $\dim V = 1$, V is the only 1-dimensional subspace) the projective line $P^1(F) = F \cup pt$. Since $[x_0, x_1]$ maps to $x_1/x_0 \in F$ we usually call this extra point $[0, 1]$ the point ∞. When $F = \mathbf{C}$, the complex numbers, the projective line is what is called in complex analysis the *extended complex plane* $\mathbf{C} \cup \{\infty\}$.

Having said that, there are many different copies of F^n inside $P^n(F)$, for we could have chosen x_i instead of x_0, or coordinates with respect to a totally different basis. Projective space should normally be thought of as a homogeneous object, without any distinguished copy of F^n inside.

2.2 Linear subspaces

Definition 3 *A linear subspace of the projective space $P(V)$ is the set of 1-dimensional vector subspaces of a vector subspace $U \subseteq V$.*

Note that a linear subspace is a projective space in its own right, the projective space $P(U)$.

Recall that a 1-dimensional projective space is called a projective line. We have the following two propositions which show that projective lines behave nicely:

Proposition 1 *Through any two distinct points in a projective space there passes a unique projective line.*

Proof: Let $P(V)$ be the projective space and $x, y \in P(V)$ distinct points. Let u, v be representative vectors. Then u, v are linearly independent for otherwise $u = \lambda v$ and

$$x = [u] = [\lambda v] = [v] = y.$$

Let $U \subseteq V$ be the 2-dimensional vector space spanned by u and v, then $P(U) \subset P(V)$ is a line containing x and y.

Suppose $P(U')$ is another such line, then $u \in U'$ and $v \in U'$ and so the space spanned by u, v (namely U) is a subspace of U'. But U and U' are 2-dimensional so $U = U'$ and the line is thus unique. □

Proposition 2 *In a projective plane, two distinct projective lines intersect in a unique point.*

Proof: Let the projective plane be $P(V)$ where $\dim V = 3$. Two lines are defined by $P(U_1), P(U_2)$ where U_1, U_2 are distinct 2-dimensional subspaces of V. Now from elementary linear algebra

$$\dim V \geq \dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

so that

$$3 \geq 2 + 2 - \dim(U_1 \cap U_2)$$

and

$$\dim(U_1 \cap U_2) \geq 1.$$

But since U_1 and U_2 are 2-dimensional,

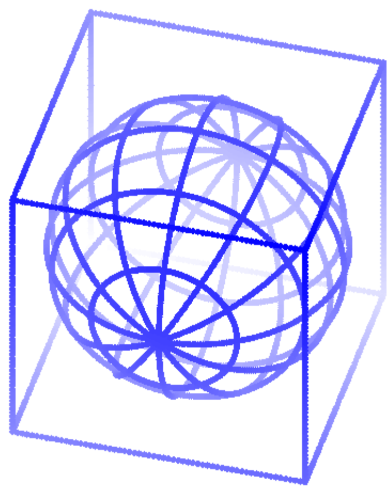
$$\dim(U_1 \cap U_2) \leq 2$$

with equality if and only if $U_1 = U_2$. As the lines are distinct, equality doesn't occur and so we have the 1-dimensional vector subspace

$$U_1 \cap U_2 \subset V$$

which is the required point of intersection in $P(V)$. □

Remark: The model of projective space as the sphere with opposite points identified illustrates this proposition, for a projective line in $P^2(\mathbf{R})$ is defines by a 2-dimensional subspace of $\mathbf{R}^3$, which intersects the unit sphere in a great circle. Two great circles intersect in two antipodal points. When we identify opposite points, we just get one intersection.



Instead of the spherical picture, let's consider instead the link between projective lines and ordinary lines in the plane, using the decomposition

$$P^2(\mathbf{R}) = \mathbf{R}^2 \cup P^1(\mathbf{R}).$$

Here we see that the real projective plane is the union of $\mathbf{R}^2$ with a projective line $P^1(\mathbf{R})$. Recall that this line is given in homogeneous coordinates by $x_0 = 0$, so it corresponds to the 2-dimensional space spanned by $(0, 1, 0)$ and $(0, 0, 1)$. Any 2-dimensional subspace of $\mathbf{R}^3$ is defined by a single equation

$$a_0x_0 + a_1x_1 + a_2x_2 = 0$$

and if a_1 and a_2 are not both zero, this intersects $U_0 \cong \mathbf{R}^2$ (the points where $x_0 \neq 0$) where

$$0 = a_0 + a_1(x_1/x_0) + a_2(x_2/x_0) = a_0 + a_1y_1 + a_2y_2$$

which is an ordinary straight line in $\mathbf{R}^2$ with coordinates y_1, y_2. The projective line has one extra point on it, where $x_0 = 0$, i.e. the point $[0, a_2, -a_1]$. Conversely, any straight line in $\mathbf{R}^2$ extends uniquely to a projective line in $P^2(\mathbf{R})$.

Two lines in $\mathbf{R}^2$ are parallel if they are of the form

$$a_0 + a_1y_1 + a_2y_2 = 0, \qquad b_0 + a_1y_1 + a_2y_2 = 0$$

but then the added point to make them projective lines is the same one: $[0, a_2, -a_1]$, so the two lines meet at a single point on the "line at infinity" $P^1(\mathbf{R})$.

2.3 Projective transformations

If V, W are vector spaces and $T : V \to W$ is a linear transformation, then a vector subspace $U \subseteq V$ gets mapped to a vector subspace $T(U) \subseteq W$. If T has a non-zero kernel, $T(U)$ may have dimension less than that of U, but if $\ker T = 0$ then $\dim T(U) = \dim U$. In particular, if U is one-dimensional, so is $T(U)$ and so T gives a well-defined map

$$\tau : P(V) \to P(W).$$

Definition 4 *A projective transformation from $P(V)$ to $P(W)$ is the map τ defined by an invertible linear transformation $T : V \to W$.*

Note that if $\lambda \neq 0$, then λT and T define the same linear transformation since

$$[(\lambda T)(v)] = [\lambda(T(v))] = [T(v)].$$

The converse is also true: suppose T and T' define the same projective transformation τ. Take a basis $\{v_0, \ldots, v_n\}$ for V, then since

$$\tau([v_i]) = [T'(v_i)] = [T(v_i)]$$

we have

$$T'(v_i) = \lambda_i T(v_i)$$

for some non-zero scalars λ_i and also

$$T'(\sum_{i=0}^{n} v_i) = \lambda T(\sum_{i=0}^{n} v_i)$$

for some non-zero λ. But then

$$\sum_{i=0}^{n} \lambda T(v_i) = \lambda T(\sum_{i=0}^{n} v_i) = T'(\sum_{i=0}^{n} v_i) = \sum_{i=0}^{n} \lambda_i T(v_i).$$

Since T is invertible, $T(v_i)$ are linearly independent, so this implies $\lambda_i = \lambda$. Then $T'(v_i) = \lambda T(v_i)$ for all basis vectors and hence for all vectors and so

$$T' = \lambda T.$$

Example: You are, in fact, already familiar with one class of projective transformations – Möbius transformations of the extended complex plane. These are just projective transformations of the complex projective line $P^1(\mathbf{C})$ to itself. We describe points in $P^1(\mathbf{C})$ by homogeneous coordinates $[z_0, z_1]$, and then a projective transformation τ is given by

$$\tau([z_0, z_1]) = ([az_0 + bz_1, cz_0 + dz_1])$$

where $ad - bc \neq 0$. This corresponds to the invertible linear transformation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is convenient to write $P^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ where the point ∞ is now the 1-dimensional space $z_1 = 0$. Then if $z_1 \neq 0$, $[z_0, z_1] = [z, 1]$ and

$$\tau([z, 1]) = [az + b, cz + d]$$

and if $cz + d \neq 0$ we can write

$$\tau([z, 1]) = [\frac{az + b}{cz + d}, 1]$$

which is the usual form of a Möbius transformation, i.e.

$$z \mapsto \frac{az + b}{cz + d}.$$

The advantage of projective geometry is that the point $\infty = [1, 0]$ plays no special role. If $cz + d = 0$ we can still write

$$\tau([z, 1]) = [az + b, cz + d] = [az + b, 0] = [1, 0]$$

and if $z = \infty$ (i.e. $[z_0, z_1] = [1, 0]$) then we have

$$\tau([1, 0]) = [a, c].$$

Example: If we view the real projective plane $P^2(\mathbf{R})$ in the same way, we get some less familiar transformations. Write $P^2(\mathbf{R}) = \mathbf{R}^2 \cup P^1(\mathbf{R})$ where the projective line at infinity is $x_0 = 0$. A linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ can then be written as the matrix

$$T = \begin{pmatrix} d & b_1 & b_2 \\ c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \end{pmatrix}$$

and its action on $[1, x, y]$ can be expressed, with $\mathbf{v} = (x, y) \in \mathbf{R}^2$, as

$$\mathbf{v} \mapsto \frac{1}{\mathbf{b} \cdot \mathbf{v} + d} (A\mathbf{v} + \mathbf{c})$$

where A is the 2×2 matrix a_{ij} and $\mathbf{b}, \mathbf{c}$ the vectors $(b_1, b_2), (c_2, c_2)$. These are the 2-dimensional versions of Möbius transformations. Each one can be considered as a composition of

- an invertible linear transformation $\mathbf{v} \mapsto A\mathbf{v}$
- a translation $\mathbf{v} \mapsto \mathbf{v} + \mathbf{c}$
- an inversion $\mathbf{v} \mapsto \mathbf{v}/(\mathbf{b} \cdot \mathbf{v} + d)$

Clearly it is easier here to consider projective transformations defined by 3×3 matrices, just ordinary linear algebra.

Example: A more geometric example of a projective transformation is to take two lines $P(U), P(U')$ in a projective plane $P(V)$ and let $K \in P(V)$ be a point disjoint from both. For each point $x \in P(U)$, the unique line joining K to x intersects $P(U')$ in a unique point $X = \tau(x)$. Then

$$\tau : P(U) \to P(U')$$

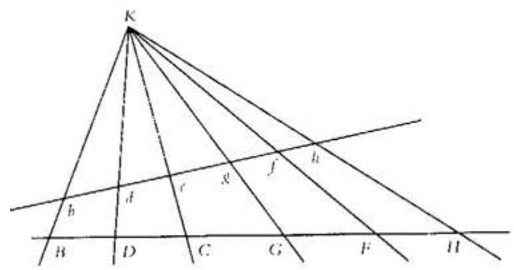
is a projective transformation.

To see why, let W be the 1-dimensional subspace of V defined by $K \in P(V)$. Then since K does not lie in $P(U')$, $W \cap U' = 0$. This means that

$$V = W \oplus U'.$$

Now take $a \in U$ as a representative vector for x. It can be expressed uniquely as $a = w + a'$, with $w \in W$ and $a' \in U'$. The projective line joining K to x is defined by the 2-dimensional vector subspace of V spanned by w and a and so $a' = a - w$ is a representative vector for $\tau(x)$. In linear algebra terms the map $a \mapsto a'$ is just the linear projection map $P : V \to U'$ restricted to U. It has zero kernel since K does not lie in $P(U)$, and hence $W \cap U = 0$. Thus $T : U \to U'$ is an isomorphism and τ is a projective transformation.

If we restrict to the points in $\mathbf{R}^2$, then this is what this *projection from K* looks like:



A linear transformation of a vector space of dimension n is determined by its value on n linearly independent vectors. A similar statement holds in projective space. The analogue of linear independence is the following

Definition 5 *Let $P(V)$ be an n-dimensional projective space, then $n+2$ points in $P(V)$ are said to be in general position if each subset of $n+1$ points has representative vectors in V which are linearly independent.*

Example: Any two distinct points in a projective line are represented by linearly independent vectors, so any three distinct points are in general position.

Theorem 3 *If $X_1, \ldots, X_{n+2}$ are in general position in $P(V)$ and $Y_1, \ldots, Y_{n+2}$ are in general position in $P(W)$, then there is a unique projective transformation $\tau : P(V) \to P(W)$ such that $\tau(X_i) = Y_i$, $1 \leq i \leq n+2$.*

Proof: First choose representative vectors $v_1, \ldots, v_{n+2} \in V$ for the points $X_1, \ldots, X_{n+2}$ in $P(V)$. By general position the first $n+1$ vectors are linearly independent, so they form a basis for V and there are scalars λ_i such that

$$v_{n+2} = \sum_{i=1}^{n+1} \lambda_i v_i \tag{1}$$

If $\lambda_i = 0$ for some i, then (1) provides a linear relation amongst a subset of $n+1$ vectors, which is not possible by the definition of general position, so we deduce that $\lambda_i \neq 0$ for all i. This means that each $\lambda_i v_i$ is also a representative vector for x_i, so (1) tells us that we could have chosen representative vectors v_i such that

$$v_{n+2} = \sum_{i=1}^{n+1} v_i \tag{2}$$

Moreover, given v_{n+2}, these v_i are unique for

$$\sum_{i=1}^{n+1} v_i = \sum_{i=1}^{n+1} \mu_i v_i$$

implies $\mu_i = 1$ since $v_1, \ldots, v_{n+1}$ are linearly independent.

[*Note: This is a very useful idea which can simplify the solution of many problems*].

Now do the same for the points $Y_1, \ldots Y_{n+2}$ in $P(W)$ and choose representative vectors such that

$$w_{n+2} = \sum_{i=1}^{n+1} w_i \tag{3}$$

Since $v_1, \ldots, v_{n+1}$ are linearly independent, they form a basis for V so there is a unique linear transformation $T : V \to W$ such that $Tv_i = w_i$ for $1 \le i \le n+1$. Since $w_1, \ldots, w_{n+1}$ are linearly independent, T is invertible. Furthermore, from (2) and (3)

$$Tv_{n+2} = \sum_{i=1}^{n+1} Tv_i = \sum_{i=1}^{n+1} w_i = w_{n+2}$$

and so T defines a projective transformation τ such that $\tau(X_i) = Y_i$ for all $n+2$ vectors v_i.

To show uniqueness, suppose T' defines another projective transformation τ' with the same property. Then $T'v_i = \mu_i w_i$ and

$$\mu_{n+2} w_{n+2} = T'v_{n+2} = \sum_{i=1}^{n+1} T'v_i = \sum_{i=1}^{n+1} \mu_i w_i.$$

But by the uniqueness of the representation (3), we must have $\mu_i/\mu_{n+2} = 1$, so that $T'v_i = \mu_{n+2}Tv_i$ and $\tau' = \tau$. □

Examples:

1. In $P^1(\mathbf{C})$ take the three distinct points $[0,1], [1,1], [1,0]$ and any other three distinct points X_1, X_2, X_3. Then there is a unique projective transformation taking X_1, X_2, X_3 to $[0,1], [1,1], [1,0]$. In the language of complex analysis, we can say that there is a unique Möbius transformation taking any three distinct points to $0, 1, \infty$.

2. In any projective line we could take the three points $[0,1], [1,1], [1,0]$ and then for X_1, X_2, X_3 any permutation of these. Now projective transformations of a space

to itself form a group under composition, so we see that the group of projective transformations of a line to itself always contains a copy of the symmetric group S_3. In fact if we take the scalars to be the field $\mathbf{Z}_2$ with two elements 0 and 1, the *only* points on the projective line are $[0,1],[1,1],[1,0]$, and S_3 is the full group of projective transformations.

As an example of the use of the notion of general position, here is a classical theorem called Desargues' theorem. In fact, Desargues (1591-1661) is generally regarded as the founder of projective geometry. The proof we give here uses the method of choosing representative vectors above.

Theorem 4 *(Desargues) Let A, B, C, A', B', C' be distinct points in a projective space $P(V)$ such that the lines AA', BB' CC' are distinct and concurrent. Then the three points of intersection $AB \cap A'B'$, $BC \cap B'C'$, $CA \cap C'A'$ are collinear.*

Proof: Let P be the common point of intersection of the three lines AA', BB', CC'. Since P, A, A' lie on a projective line and are distinct, they are in general position, so as in (2) we choose representative vectors p, a, a' such that

$$p = a + a'.$$

These are vectors in a 2-dimensional subspace of V. Similarly we have representative vectors b, b' for B, B' and c, c' for C, C' with

$$p = b + b' \qquad p = c + c'.$$

It follows that $a + a' = b + b'$ and so

$$a - b = b' - a' = c''$$

and similarly

$$b - c = c' - b' = a'' \qquad c - a = a' - c' = b''.$$

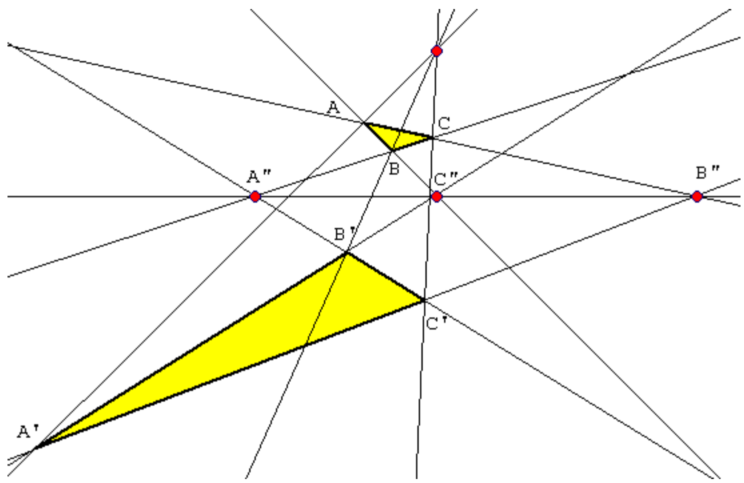
But then

$$c'' + a'' + b'' = a - b + b - c + c - a = 0$$

and a'', b'', c'' are linearly independent and lie in a 2-dimensional subspace of V. Hence the points A'', B'', C'' in $P(V)$ represented by a'', b'', c'' are collinear.

Now since $c'' = a - b$, c'' lies in the 2-dimensional space spanned by a and b, so C'' lies on the line AB. Since c'' also equals $b' - a'$, C'' lies on the line $A'B'$ and so c'' represents the point $AB \cap A'B'$. Repeating for B'' and A'' we see that these are the three required collinear points. $\square$

Desargues' theorem is a theorem in projective space which we just proved by linear algebra – linear independence of vectors. However, if we take the projective space $P(V)$ to be the real projective plane $P^2(\mathbf{R})$ and then just look at that part of the data which lives in $\mathbf{R}^2$, we get a theorem about perspective triangles in the plane:



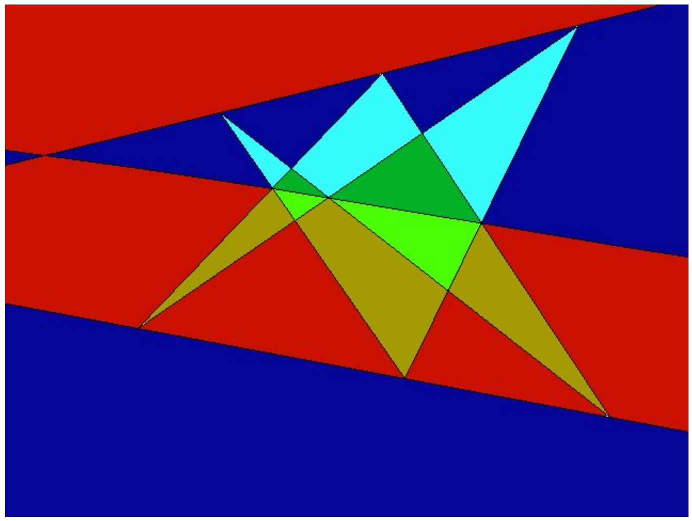
Here is an example of the use of projective geometry – a "higher form of geometry" to prove simply a theorem in $\mathbf{R}^2$ which is less accessible by other means. Another theorem in the plane for which these methods give a simple proof is Pappus' theorem. Pappus of Alexandria (290-350) was thinking again of plane Euclidean geometry, but his theorem makes sense in the projective plane since it only discusses collinearity and not questions about angles and lengths. It means that we can transform the given configuration by a projective transformation to a form which reduces the proof to simple linear algebra calculation:

Theorem 5 *(Pappus) Let A, B, C and A', B', C' be two pairs of collinear triples of distinct points in a projective plane. Then the three points $BC' \cap B'C, CA' \cap C'A, AB' \cap A'B$ are collinear.*

Proof: Without loss of generality, we can assume that A, B, C', B' are in general position. If not, then two of the three required points coincide, so the conclusion is trivial. By Theorem 3, we can then assume that

$$A = [1,0,0], \quad B = [0,1,0], \quad C' = [0,0,1], \quad B' = [1,1,1].$$

The line AB is defined by the 2-dimensional subspace $\{(x_0, x_1, x_2) \in F^3 : x_2 = 0\}$, so the point C, which lies on this line, is of the form $C = [1, c, 0]$ and $c \neq 0$ since $A \neq C$. Similarly the line $B'C'$ is $x_0 = x_1$, so $A' = [1, 1, a]$ with $a \neq 1$.



The line BC' is defined by $x_0 = 0$ and $B'C$ is defined by the span of $(1, 1, 1)$ and $(1, c, 0)$, so the point $BC' \cap B'C$ is represented by the linear combination of $(1, 1, 1)$ and $(1, c, 0)$ for which $x_0 = 0$, i.e.

$$(1, 1, 1) - (1, c, 0) = (0, 1 - c, 1).$$

The line $C'A$ is given by $x_1 = 0$, so similarly $CA' \cap C'A$ is represented by

$$(1, c, 0) - c(1, 1, a) = (1 - c, 0, -ca).$$

Finally AB' is given by $x_1 = x_2$, so $AB' \cap A'B$ is

$$(1, 1, a) + (a - 1)(0, 1, 0) = (1, a, a).$$

But then

$$(c - 1)(1, a, a) + (1 - c, 0, -ca) + a(0, 1 - c, 1) = 0.$$

Thus the three vectors span a 2-dimensional subspace and so the three points lie on a projective line. □

2.4 Duality

Projective geometry gives, as we shall see, a more concrete realization of the linear algebra notion of duality. But first let's recall what dual spaces are all about. Here are the essential points:

- Given a finite-dimensional vector space V over a field F, the dual space V' is the vector space of linear transformations $f : V \to F$.

- If $v_1, \ldots, v_n$ is a basis for V, there is a *dual basis* $f_1, \ldots f_n$ of V' characterized by the property $f_i(v_j) = 1$ if $i = j$ and $f_i(v_j) = 0$ otherwise.

- If $T : V \to W$ is a linear transformation, there is a natural linear transformation $T' : W' \to V'$ defined by $T'f(v) = f(Tv)$.

Although a vector space V and its dual V' have the same dimension there is no natural way of associating a point in one with a point in the other. We can do so however with vector subspaces:

Definition 6 *Let $U \subseteq V$ be a vector subspace. The annihilator $U^o \subset V'$ is defined by $U^o = \{f \in V' : f(u) = 0$ for all $u \in U\}$.*

The annihilator is clearly a vector subspace of V' since $f(u) = 0$ implies $\lambda f(u) = 0$ and if also $g(u) = 0$ then $(f + g)(u) = f(u) + g(u) = 0$. Furthermore, if $U_1 \subseteq U_2$ and $f(u) = 0$ for all $u \in U_2$, then in particular $f(u) = 0$ for all $u \in U_1$, so that

$$U_2^o \subseteq U_1^o.$$

We also have:

Proposition 6 $\qquad \dim U + \dim U^o = \dim V.$

Proof: Let $u_1, \ldots, u_m$ be a basis for U and extend to a basis $u_1, \ldots, u_m, v_1, \ldots, v_{n-m}$ of V. Let $f_1, \ldots, f_n$ be the dual basis. Then for $m+1 \leq i \leq n$, $f_i(u_j) = 0$ so $f_i \in U^o$. Conversely if $f \in U^o$, write

$$f = \sum_{i=1}^{n} c_i f_i$$

Then $0 = f(u_i) = c_i$, and so f is a linear combination of f_i for $m + 1 \leq i \leq n$. Thus $f_{m+1}, \ldots, f_n$ is a basis for U^o and

$$\dim U + \dim U^o = m + n - m = n = \dim V.$$

$\square$

If we take the dual of the dual we get a vector space V'', but this is naturally isomorphic to V itself. To see this, define $S : V \to V''$ by

$$Sv(f) = f(v).$$

This is clearly linear in v, and $\ker S$ is the set of vectors such that $f(v) = 0$ for all f, which is zero, since we could extend $v = v_1$ to a basis, and $f_1(v_1) \neq 0$. Since $\dim V = \dim V'$, S is an isomorphism. Under this transformation, for each vector subspace $U \subseteq V$, $S(U) = U^{oo}$. This follows since if $u \in U$, and $f \in U^0$

$$Su(f) = f(u) = 0$$

so $S(U) \subseteq U^{oo}$. But from (6) the dimensions are the same, so we have equality.

Thus to any vector space V we can naturally associate another vector space of the same dimension V', and to any projective space $P(V)$ we can associate another one $P(V')$. Our first task is to understand what a point of $P(V')$ means in terms of the original projective space $P(V)$.

From the linear algebra definition of dual, a point of $P(V')$ has a non-zero representative vector $f \in V'$. Since $f \neq 0$, it defines a surjective linear map

$$f : V \to F$$

and so

$$\dim \ker f = \dim V - \dim F = \dim V - 1.$$

If $\lambda \neq 0$, then $\dim \ker \lambda f = \dim \ker f$ so the point $[f] \in P(V')$ defines unambiguously a vector subspace $U \subset V$ of dimension one less than that of V, and a corresponding linear subspace $P(U)$ of $P(V)$.

Definition 7 *A hyperplane in a projective space $P(V)$ is a linear subspace $P(U)$ of dimension* $\dim P(V) - 1$ *(or* codimension *one).*

Conversely, a hyperplane defines a vector subspace $U \subset V$ of dimension $\dim V - 1$, and so we have a 1-dimensional quotient space V/U and a surjective linear map

$$\pi : V \to V/U$$

defined by $\pi(v) = v + U$. If $\nu \in V/U$ is a non-zero vector then

$$\pi(v) = f(v)\nu$$

for some linear map $f : V \to F$, and then $U = \ker f$. A different choice of ν changes f to λf, so the hyperplane $P(U)$ naturally defines a point $[f] \in P(V')$. Hence,

Proposition 7 *The points of the dual projective space $P(V')$ of a projective space $P(V)$ are in natural one-to-one correspondence with the hyperplanes in $P(V)$.*

The surprise here is that the space of hyperplanes should have the structure of a projective space. In particular there are linear subspaces of $P(V')$ and they demand an interpretation. From the point of view of linear algebra, this is straightforward: to each $m+1$-dimensional vector subspace $U \subseteq V$ of the $n+1$-dimensional vector space V we associate the $n-m$-dimensional annihilator $U^o \subseteq V'$. Conversely, given $W \subseteq V'$, take $W^o \subset V''$ then $W^o = S(U)$ for some U and since $S(U) = U^{oo}$, it follows that

$$W = U^o.$$

Thus taking the annihilator defines a one-to-one correspondence between vector subspaces of V and vector subspaces of V'. We just need to give this a geometrical interpretation.

Proposition 8 *A linear subspace $P(W) \subseteq P(V')$ of dimension m in a dual projective space $P(V')$ of dimension n consists of the hyperplanes in $P(V)$ which contain a fixed linear subspace $P(U) \subseteq P(V)$ of dimension $n-m-1$.*

Proof: As we saw above, $W = U^o$ for some vector subspace $U \subseteq V$, so $f \in W$ is a linear map $f : V \to F$ such that $f(U) = 0$. This means that $U \subset \ker f$ so the hyperplane defined by f contains $P(U)$. $\square$

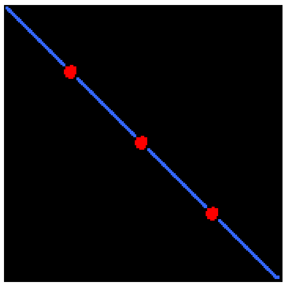
A special case of this is a hyperplane in $P(V')$. This consists of the hyperplanes in $P(V)$ which pass through a fixed point $X \in P(V)$, and this describes geometrically the projective transformation defined by S

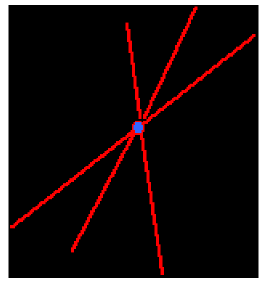
$$P(V) \cong P(V'').$$

All these features are somewhat clearer in low dimensions. A hyperplane in a projective line is just a point, so there is a natural isomorphism $P(V) \cong P(V')$ here and duality gives nothing new. In a projective plane however, a hyperplane is a line, so $P(V')$ is the space of lines in $P(V)$. The space of lines passing through a point $X \in P(V)$ constitutes a line X^o in $P(V')$. Given two points X, Y there is a unique line joining them. So there must be a unique point in $P(V')$ which lies on the two lines X^o, Y^o. Duality therefore shows that Proposition 2 is just the same as Proposition 1, if we apply the latter to the dual projective plane $P(V')$.

Here is another example of dual configurations:

three collinear points   *three concurrent lines*





In general, any result of projective geometry when applied to the dual plane $P(V')$ can be reinterpreted in $P(V)$ in a different form. In principle then, we get two theorems for the price of one. As an example take Desargues' Theorem, at least in the way we formulated it in (4). Instead of applying it to the projective plane $P(V)$, apply it to $P(V')$. The theorem is still true, but it says something different in $P(V)$. For example, our starting point in $P(V')$ consists of seven points, which now become seven lines in $P(V)$. So here is the dual of Desargues' theorem:

Theorem 9 *(Desargues) Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be distinct lines in a projective plane $P(V)$ such that the points $\alpha \cap \alpha'$, $\beta \cap \beta'$, $\gamma \cap \gamma'$ are distinct and collinear. Then the lines joining $\alpha \cap \beta$, $\alpha' \cap \beta'$ and $\beta \cap \gamma$, $\beta' \cap \gamma'$ and $\gamma \cap \alpha, \gamma' \cap \alpha'$ are concurrent.*

Here the dual theorem starts with three points lying on a line and ends with three lines meeting in a point – looked at the right way, we have the *converse* of Desargues' Theorem.

Now look at Pappus' theorem. Instead of two triples of collinear points, the dual statement of the theorem gives two triples of concurrent lines α, β, γ passing through A and α', β', γ' passing through A'. Define B on α to be $\alpha \cap \gamma'$ and C to be $\alpha \cap \beta'$. Define B' on α' to be $\alpha' \cap \beta$ and C' to be $\alpha' \cap \gamma$.

The dual of Pappus says that the lines joining $\{\beta \cap \gamma', \beta' \cap \gamma\}, \{\gamma \cap \alpha', \gamma' \cap \alpha\}, \{\alpha \cap \beta', \alpha' \cap \beta\}$ are concurrent at a point P. By definition of B, B', C, C', the last two are $\{BC', B'C\}$, which therefore intersect in P. Now A lies on β and by definition so does B' so β is the line AB'. Similarly $A'B$ is the line γ'. Likewise A lies on γ and by definition so does C' so AC' is γ and $A'C$ is β'.

Thus the intersection of $\{AB', A'B\}$ is $\beta \cap \gamma'$ and the intersection of $\{AC', A'C\}$ is $\beta' \cap \gamma$. The dual of Pappus' theorem says that the line joining these passes through

P, which is the intersection of $\{BC', B'C\}$. These three points are thus collinear and this is precisely Pappus' theorem itself.

Finally, we can use duality to understand something very down-to-earth – the space of straight lines in $\mathbf{R}^2$. When we viewed the projective plane $P^2(\mathbf{R})$ as $\mathbf{R}^2 \cup P^1(\mathbf{R})$ we saw that a projective line not equal to the line at infinity $P^1(\mathbf{R})$ intersected $\mathbf{R}^2$ in an ordinary straight line. Since we now know that the lines in $P^2(\mathbf{R})$ are in one-to-one correspondence with another projective plane – the dual plane – we see that we only have to remove a single point from the dual plane, the point giving the line at infinity, to obtain the space of lines in $\mathbf{R}^2$. So in the sphere model, we remove the north and south poles and identify antipodal points.

Concretely parametrize the sphere in the usual way:

$$x_1 = \sin\theta \sin\phi, \qquad x_2 = \sin\theta\cos\phi, \qquad x_3 = \cos\theta$$

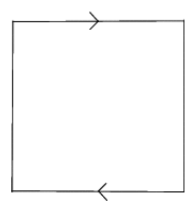
then with the poles removed the range of values is $0 < \theta < \pi$, $0 \leq \phi < 2\pi$. The antipodal map is

$$\theta \mapsto \pi - \theta, \qquad \phi \mapsto \phi + \pi.$$

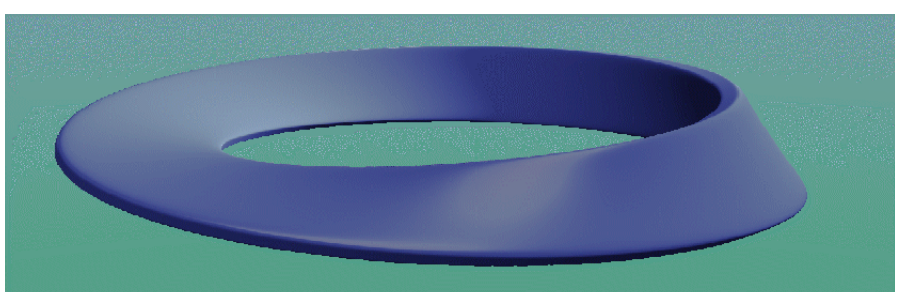
We can therefore identify the space of lines in $\mathbf{R}^2$ as the pairs

$$(\theta, \phi) \in (0, \pi) \times [0, \pi]$$

where we identify $(\theta, 0)$ with $(\pi - \theta, \pi)$:



and this is the *Möbius band*.



2.5 Exercises

1. Let U_1, U_2 and U_3 be the 2-dimensional vector subspaces of $\mathbf{R}^3$ defined by

$$x_0 = 0, \qquad x_0 + x_1 + x_2 = 0, \quad 3x_0 - 4x_1 + 5x_2 = 0$$

respectively. Find the vertices of the "triangle" in $P^2(\mathbf{R})$ whose sides are the projective lines $P(U_1), P(U_2), P(U_3)$.

2. Let U_1, U_2 be vector subspaces of V. Show that the linear subspace

$$P(U_1 + U_2) \subseteq P(V)$$

is the set of points obtained by joining each $X \in P(U_1)$ and $Y \in P(U_2)$ by a projective line.

3. Prove that three skew (i.e. non-intersecting) lines in $P^3(\mathbf{R})$ have an infinite number of transversals (i.e. lines meeting all three).

4. Find the projective transformation $\tau : P^1(\mathbf{R}) \to P^1(\mathbf{R})$ for which

$$\tau[0,1] = [1,0], \qquad \tau[1,0] = [1,1], \qquad \tau[1,1] = [0,1]$$

and show that $\tau^3 = \mathrm{i}d$.

5. Let $T : V \to V$ be an invertible transformation. Show that if $v \in V$ is an eigenvector of T, then $[v] \in P(V)$ is a fixed point of the projective transformation τ defined by T. Prove that any projective transformation of $P^2(\mathbf{R})$ has a fixed point.

6. Let V be a 3-dimensional vector space with basis v_1, v_2, v_3 and let $A, B, C \in P(V)$ be expressed in homogeneous coordinates relative to this basis by

$$A = [2,1,0], \qquad B = [0,1,1], \qquad C = [-1,1,2].$$

Find the coordinates with respect to the dual basis of the three points in the dual space $P(V')$ which represent the lines AB, BC and CA.