4 Exterior algebra

4.1 Lines and 2-vectors

The time has come now to develop some new linear algebra in order to handle the space of lines in a projective space $P(V)$. In the projective plane we have seen that duality can deal with this but lines in higher dimensional spaces behave differently. From the point of view of linear algebra we are looking at 2-dimensional vector subspaces $U \subset V$.

To motivate what we shall do, consider how in Euclidean geometry we describe a 2-dimensional subspace of $\mathbf{R}^3$. We could describe it through its unit normal $\mathbf{n}$, which is also parallel to $\mathbf{u}\times\mathbf{v}$ where $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in the space and $\mathbf{u}\times\mathbf{v}$ is the vector cross product. The vector product has the following properties:

- $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$
- $(\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2)\times\mathbf{v} = \lambda_1\mathbf{u}_1\times\mathbf{v} + \lambda_2\mathbf{u}_2\times\mathbf{v}$

We shall generalize these properties to vectors in any vector space V – the difference is that the product will not be a vector in V, but will lie in another associated vector space.

Definition 12 *An alternating bilinear form on a vector space V is a map $B : V \times V \to F$ such that*

- $B(v, w) = -B(w, v)$
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$

This is the skew-symmetric version of the symmetric bilinear forms we used to define quadrics. Given a basis $\{v_1, \dots, v_n\}$, B is uniquely determined by the skew symmetric matrix $B(v_i, v_j)$. We can add alternating forms and multiply by scalars so they form a vector space, isomorphic to the space of skew-symmetric $n \times n$ matrices. This has dimension $n(n-1)/2$, spanned by the basis elements E^{ab} for $a < b$ where $E^{ab}_{ij} = 0$ if $\{a, b\} \neq \{i, j\}$ and $E^{ab}_{ab} = -E^{ab}_{ba} = 1$.

Definition 13 *The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.*

This definition is a convenience – there are other ways of defining $\Lambda^2 V$, and for most purposes it is only its characteristic properties which one needs rather than what its objects are. A lot of mathematics is like that – just think of the real numbers.

Given this space we can now define our generalization of the cross-product, called the *exterior product* or *wedge product* of two vectors.

Definition 14 *Given $u, v \in V$ the exterior product $u \wedge v \in \Lambda^2 V$ is the linear map to F which, on an alternating bilinear form B, takes the value*

$$(u \wedge v)(B) = B(u, v).$$

From this definition follows some basic properties:

- $(u \wedge v)(B) = B(u, v) = -B(v, u) = -(v \wedge u)(B)$ so that

  $$v \wedge u = -u \wedge v$$

  and in particular $u \wedge u = 0$.

- $((\lambda_1 u_1 + \lambda_2 u_2) \wedge v)(B) = B(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v)$ which implies

  $$(\lambda_1 u_1 + \lambda_2 u_2) \wedge v = \lambda_1 u_1 \wedge v + \lambda_2 u_2 \wedge v.$$

- if $\{v_1, \ldots, v_n\}$ is a basis for V then $v_i \wedge v_j$ for $i < j$ is a basis for $\Lambda^2 V$.

This last property holds because $v_i \wedge v_j(E^{ab}) = E^{ab}_{ij}$ and in facts shows that $\{v_i \wedge v_j\}$ is the dual basis to the basis $\{E^{ab}\}$.

Another important property is:

Proposition 15 *Let $u \in V$ be a non-zero vector. Then $u \wedge v = 0$ if and only if $v = \lambda u$ for some scalar λ.*

Proof: If $v = \lambda u$, then

$$u \wedge v = u \wedge (\lambda u) = \lambda(u \wedge u) = 0.$$

Conversely, if $v \neq \lambda u$, u and v are linearly independent and can be extended to a basis, but then $u \wedge v$ is a basis vector and so is non-zero. $\square$

It is the elements of $\Lambda^2 V$ of the form $u \wedge v$ which will concern us, for suppose $U \subset V$ is a 2-dimensional vector subspace, and $\{u, v\}$ is a basis of U. Then any other basis is of the form $\{au + bv, cu + dv\}$, so, using $u \wedge u = v \wedge v = 0$, we get

$$(au + bv) \wedge (cu + dv) = (ad - bc)u \wedge v$$

and since the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible $ad - bc \neq 0$. It follows that the 1-dimensional subspace of $\Lambda^2 V$ spanned by $u \wedge v$ for a basis of U is well-defined by U itself and is independent of the choice of basis. To each line in $P(V)$ we can therefore associate a *point* in $P(\Lambda^2 V)$.

The problem is, not every vector in $\Lambda^2 V$ can be written as $u \wedge v$ for vectors $u, v \in V$. In general it is a linear combination of such expressions. The task, in order to describe the space of lines, is to characterize such *decomposable* 2-vectors.

Example: Consider $v_1 \wedge v_2 + v_3 \wedge v_4$ in a 4-dimensional vector space V. Suppose we can write this as

$$v_1 \wedge v_2 + v_3 \wedge v_4 = (a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4) \wedge (b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4).$$

Equating the coefficient of $v_1 \wedge v_2$ gives

$$a_1 b_2 - a_2 b_1 = 1$$

and so (a_1, b_1) is non-zero. On the other hand the coefficients of $v_1 \wedge v_3$ and $v_1 \wedge v_4$ give

$$\begin{aligned} a_1 b_3 - a_3 b_1 &= 0 \\ a_1 b_4 - a_4 b_1 &= 0 \end{aligned}$$

and since $(a_1, b_1) \neq 0$, $b_3 a_4 - a_3 b_4 = 0$. But the coefficient of $v_3 \wedge v_4$ gives $a_4 b_3 - a_3 b_4 = 1$ which is a contradiction. This 2-vector is not therefore decomposable. We shall find an easier method of seeing this by working with p-vectors and exterior products.

4.2 Higher exterior powers

Definition 15 *An alternating multilinear form of degree p on a vector space V is a map $M : V \times \ldots \times V \to F$ such that*

- $M(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_p) = -M(u_1, \ldots, u_j, \ldots, u_i, \ldots, u_p)$
- $M(\lambda_1 v_1 + \lambda_2 v_2, u_2, \ldots, u_p) = \lambda_1 M(v_1, u_2, \ldots, u_p) + \lambda_2 M(v_2, u_2, \ldots, u_p)$

Example: Let $u_1, \ldots, u_n$ be column vectors in $\mathbf{R}^n$. Then

$$M(u_1, \ldots, u_n) = \det(u_1 u_2 \ldots u_n)$$

is an alternating multilinear form of degree n.

The set of all alternating multilinear forms on V is a vector space, and M is uniquely determined by the values

$$M(v_{i_1}, v_{i_2}, \ldots, v_{i_p})$$

for a basis $\{v_1, \ldots, v_n\}$. But the alternating property allows us to change the order so long as we multiply by -1 for each transposition of variables. This means that M is uniquely determined by the values of indices for

$$i_1 < i_2 < \ldots < i_p.$$

The number of these is the number of p-element subsets of n, i.e. $\binom{n}{p}$, so this is the dimension of the space of such forms. In particular if $p > n$ this space is zero. We define analogous constructions to those above for a pair of vectors:

Definition 16 *The p-th exterior power $\Lambda^p V$ of a finite-dimensional vector space is the dual space of the vector space of alternating multilinear forms of degree p on V. Elements of $\Lambda^p V$ are called p-vectors.*

and

Definition 17 *Given $u_1, \ldots, u_p \in V$ the exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p \in \Lambda^p V$ is the linear map to F which, on an alternating multilinear form M takes the value*

$$(u_1 \wedge u_2 \wedge \ldots \wedge u_p)(M) = M(u_1, u_2, \ldots, u_p).$$

The exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ has two defining properties

- it is linear in each variable u_i separately
- interchanging two variables changes the sign of the product

- if two variables are the same the exterior product vanishes.

We have a useful generalization of Proposition 15:

Proposition 16 *The exterior product $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ of p vectors $u_i \in V$ vanishes if and only if the vectors are linearly dependent.*

Proof: If there exists a linear relation

$$\lambda_1 u_1 + \ldots \lambda_p u_p = 0$$

with $\lambda_i \neq 0$, then u_i is a linear combination of the other vectors

$$u_i = \sum_{j \neq i} \mu_j u_j$$

but then

$$u_1 \wedge u_2 \wedge \ldots \wedge u_p = u_1 \wedge \ldots \wedge (\sum_{j \neq i} \mu_j u_j) \wedge u_{i+1} \wedge \ldots \wedge u_p$$

and expand this out by linearity, each term has a repeated variable u_j and so vanishes.

Conversely, if $u_1, \ldots, u_p$ are linearly independent they can be extended to a basis and $u_1 \wedge u_2 \wedge \ldots \wedge u_p$ is a basis vector for $\Lambda^p V$ and is thus non-zero. □

The exterior powers $\Lambda^p V$ have natural properties with respect to linear transformations: given a linear transformation $T : V \to W$, and an alternating multilinear form M on W we can define an induced one $T^* M$ on V by

$$T^* M(v_1, \ldots, v_p) = M(Tv_1, \ldots, Tv_p)$$

and this defines a dual linear map

$$\Lambda^p T : \Lambda^p V \to \Lambda^p W$$

with the property that

$$\Lambda^p T(v_1 \wedge v_2 \wedge \ldots \wedge v_p) = Tv_1 \wedge Tv_2 \wedge \ldots \wedge Tv_p.$$

One such map is very familiar: take $p = n$, so that $\Lambda^n V$ is one-dimensional and spanned by $v_1 \wedge v_2 \wedge \ldots \wedge v_n$ for a basis $\{v_1, \ldots, v_n\}$. A linear transformation from a

1-dimensional vector space to itself is just multiplication by a scalar, so $\Lambda^n T$ is some scalar in the field. In fact it is the *determinant* of T. To see this, observe that

$$\Lambda^n T(v_1 \wedge \ldots \wedge v_n) = Tv_1 \wedge \ldots \wedge Tv_n$$

and the right hand side can be written using the matrix T_{ij} of T as

$$\sum_{i_1,\ldots,i_n} T_{i_1 1} v_{i_1} \wedge \ldots \wedge T_{i_n n} v_{i_n} = \sum_{i_1,\ldots,i_n} T_{i_1 1} \ldots T_{i_n n} v_{i_1} \wedge \ldots \wedge v_{i_n}.$$

Each of the terms vanishes if any two of $i_1, \ldots, i_n$ are equal by the property of the exterior product, so we need only consider the case where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Any permutation is a product of transpositions, and any transposition changes the sign of the exterior product, so

$$\Lambda^n T(v_1 \wedge \ldots \wedge v_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) T_{\sigma(1)1} T_{\sigma(2)2} \ldots T_{\sigma(n)n} v_1 \wedge \ldots \wedge v_n$$

which is the definition of the determinant of T_{ij}. From our point of view the determinant is naturally defined for a linear transformation $T : V \to V$, and what we just did was to see how to calculate it from the matrix of T.

We now have vector spaces $\Lambda^p V$ of dimension $\binom{n}{p}$ naturally associated to V. The space $\Lambda^1 V$ is by definition the dual space of the space of linear functions on V, so $\Lambda^1 V = V'' \cong V$ and by convention we set $\Lambda^0 V = F$. Given p vectors $v_1, \ldots, v_p \in V$ we also have a corresponding vector $v_1 \wedge v_2 \wedge \ldots \wedge v_p \in \Lambda^p V$ and the notation suggests that there should be a product so that we can remove the brackets:

$$(u_1 \wedge \ldots \wedge u_p) \wedge (v_1 \wedge \ldots v_q) = u_1 \wedge \ldots \wedge u_p \wedge v_1 \wedge \ldots v_q$$

and indeed there is. So suppose $a \in \Lambda^p V, b \in \Lambda^q V$, we want to define $a \wedge b \in \Lambda^{p+q} V$. Now for fixed vectors $u_1, \ldots, u_p \in V$,

$$M(u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q)$$

is an alternating multilinear function of $v_1, \ldots, v_q$, so if

$$b = \sum_{j_1 < \ldots < j_q} \lambda_{j_1 \ldots j_q} v_{j_1} \wedge \ldots \wedge v_{j_q}$$

then

$$\sum_{j_1 < \ldots < j_q} \lambda_{j_1 \ldots j_q} M(u_1, \ldots, u_p, v_{j_1}, \ldots, v_{j_q})$$

only depends on b and not on the particular way it is written in terms of a basis $\{v_1, \ldots, v_n\}$. Similarly if

$$a = \sum_{i_1 < \ldots < i_p} \mu_{i_1 \ldots i_p} u_{i_1} \wedge \ldots \wedge u_{i_p}$$

then

$$\sum_{i_1 < \ldots < i_p} \mu_{i_1 \ldots i_p} M(u_{i_1}, \ldots, u_{i_p}, v_1, \ldots, v_q)$$

only depends on a. We can therefore unambiguously define $a \wedge b$ by its value on an alternating $p+q$-form M as

$$(a \wedge b)(M) = \sum_{i_1 < .. < i_p ; j_i, .. < j_q} \mu_{i_1 \ldots i_p} \lambda_{j_1 \ldots j_q} M(u_{i_1}, \ldots, u_{i_p}, v_{j_1}, \ldots, v_{j_q}).$$

The product just involves linearity and removing the brackets.

Example: Suppose $a = v_1 + v_2$, $b = v_1 \wedge v_3 - v_3 \wedge v_2$, with $v_1, v_2, v_3 \in V$ then

$$\begin{aligned}
a \wedge b &= (v_1 + v_2) \wedge (v_1 \wedge v_3 - v_3 \wedge v_2) \\
&= v_1 \wedge v_1 \wedge v_3 - v_1 \wedge v_3 \wedge v_2 + v_2 \wedge v_1 \wedge v_3 - v_2 \wedge v_3 \wedge v_2 \\
&= -v_1 \wedge v_3 \wedge v_2 + v_2 \wedge v_1 \wedge v_3 \\
&= v_1 \wedge v_2 \wedge v_3 - v_1 \wedge v_2 \wedge v_3 = 0
\end{aligned}$$

where we have used the basic rules that a repeated vector from V in an exterior product gives zero, and the interchange of two vectors changes the sign.

Note that

$$u_1 \wedge u_2 \wedge \ldots \wedge u_p \wedge v_1 \wedge \ldots \wedge v_q = (-1)^p v_1 \wedge u_1 \wedge u_2 \wedge \ldots \wedge u_p \wedge v_2 \wedge \ldots \wedge v_q$$

because we have to interchange v_1 with each of the p u_i's to bring it to the front, and then repeating

$$u_1 \wedge u_2 \wedge \ldots \wedge u_p \wedge v_1 \wedge \ldots \wedge v_q = (-1)^{pq} v_1 \wedge \ldots \wedge v_q \wedge u_1 \wedge u_2 \wedge \ldots \wedge u_p.$$

This extends by linearity to all $a \in \Lambda^p V, b \in \Lambda^q V$. We then have the basic properties of the exterior product;

- $a \wedge (b + c) = a \wedge b + a \wedge c$

- $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
- $a \wedge b = (-1)^{pq} b \wedge a$ if $a \in \Lambda^p V, b \in \Lambda^q V$

What we have done may seem rather formal, but it has many concrete applications. For example if $a = x \wedge y$ then $a \wedge a = x \wedge y \wedge x \wedge y = 0$ because $x \in V$ is repeated. So it is much easier to determine that $a = v_1 \wedge v_2 + v_3 \wedge v_4$ from the Exercise above is not decomposable:

$$(v_1 \wedge v_2 + v_3 \wedge v_4) \wedge (v_1 \wedge v_2 + v_3 \wedge v_4) = 2 v_1 \wedge v_2 \wedge v_3 \wedge v_4 \neq 0.$$

4.3 Decomposable 2-vectors

A line in $P(V)$ defines a point in $P(\Lambda^2 V)$ defined by a *decomposable* 2-vector

$$a = x \wedge y.$$

We need to characterize algebraically this decomposability, and the following theorem does just that:

Theorem 17 *Let $a \in \Lambda^2 V$ be a non-zero element. Then a is decomposable if and only if $a \wedge a = 0 \in \Lambda^4 V$.*

Proof: If $a = x \wedge y$ for two vectors x and y then

$$a \wedge a = x \wedge y \wedge x \wedge y = 0$$

because of the repeated factor x (or y).

We prove the converse by induction on the dimension of V. If $\dim V = 0, 1$ then $\Lambda^2 V = 0$, so the first case is $\dim V = 2$. In this case $\dim \Lambda^2 V = 1$ and $v_1 \wedge v_2$ is a non-zero element if v_1, v_2 is a basis for V, so any a is decomposable.

We consider the case $\dim V = 3$ separately now. Given a non-zero $a \in \Lambda^2 V$, define $A : V \to \Lambda^3 V$ by

$$A(v) = a \wedge v.$$

Since $\dim \Lambda^3 V = 1$, $\dim \ker A \geq 2$, so let u_1, u_2 be linearly independent vectors in the kernel and extend to a basis u_1, u_2, u_3 of V. We can then write

$$a = \lambda_1 u_2 \wedge u_3 + \lambda_2 u_3 \wedge u_1 + \lambda_3 u_1 \wedge u_2.$$

Now by definition $0 = a \wedge u_1 = \lambda_1 u_2 \wedge u_3 \wedge u_1$ so $\lambda_1 = 0$ and similarly $0 = a \wedge u_2$ implies $\lambda_2 = 0$. It follows that $a = \lambda_3 u_1 \wedge u_2$, which is decomposable.

Now assume inductively that the theorem is true for $\dim V \leq n-1$ and consider the case $\dim V = n$. Using a basis $v_1, \ldots, v_n$, write

$$\begin{aligned} a &= \sum_{1 \leq i < j}^{n} a_{ij} v_i \wedge v_j \\ &= (\sum_{i=1}^{n-1} a_{in} v_i) \wedge v_n + \sum_{1 \leq i < j}^{n-1} a_{ij} v_i \wedge v_j \\ &= u \wedge v_n + a' \end{aligned}$$

where $u \in U$ and $a' \in \Lambda^2 U$ and U is the $(n-1)$-dimensional space spanned by $v_1, \ldots, v_{n-1}$.

Now

$$0 = a \wedge a = (u \wedge v_n + a') \wedge (u \wedge v_n + a') = 2u \wedge a' \wedge v_n + a' \wedge a'.$$

But v_n doesn't appear in the expansion of $u \wedge a'$ or $a' \wedge a'$ so we separately obtain

$$u \wedge a' = 0, \qquad a' \wedge a' = 0.$$

By induction $a' \wedge a' = 0$ implies $a' = u_1 \wedge u_2$ and so the first equation reads

$$u \wedge u_1 \wedge u_2 = 0$$

which from Proposition 16 says that there is a linear relation

$$\lambda u + \mu_1 u_1 + \mu_2 u_2 = 0.$$

If $\lambda = 0$, then u_1 and u_2 are linearly dependent so $a' = u_1 \wedge u_2 = 0$. This means that $u = u \wedge v_n$ and is therefore decomposable. If $\lambda \neq 0$, $u = \lambda_1 u_1 + \lambda_2 u_2$, so

$$a = \lambda_1 u_1 \wedge v_n + \lambda_2 u_2 \wedge v_n + u_1 \wedge u_2$$

and this is the 3-dimensional case which is always decomposable as we showed above.

We conclude that a, in each case, is decomposable. $\square$

4.4 The Klein quadric

The first case where we can apply Theorem 17 is when $\dim V = 4$, to describe the projective lines in the 3-dimensional space $P(V)$. In this case $\dim \Lambda^4 V = 1$ with a basis vector $v_0 \wedge v_1 \wedge v_2 \wedge v_3$ if V is given the basis $v_0, \ldots, v_3$.

For $a \in \Lambda^2 V$ we write

$$a = \lambda_1 v_0 \wedge v_1 + \lambda_2 v_0 \wedge v_2 + \lambda_3 v_0 \wedge v_3 + \mu_1 v_2 \wedge v_3 + \mu_2 v_3 \wedge v_1 + \mu_3 v_1 \wedge v_2$$

and then $a \wedge a = B(a,a) v_0 \wedge v_1 \wedge v_2 \wedge v_3$ where

$$B(a,a) = 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) \tag{8}$$

This is a non-degenerate quadratic form, and so $B(a,a) = 0$ defines a nonsingular quadric $Q \subset P(\Lambda^2 V)$. Moreover, any other choice of basis rescales B by a non-zero constant and so Q is well defined in projective space.

We see then that a line $\ell \subset P(V)$ defines a decomposable 2-vector $a = x \wedge y$, unique up to a scalar and since $a \wedge a = 0$, it defines a point $L \in Q \subset P(\Lambda^2 V)$. Conversely, Theorem 17 tells us that every point in Q is represented by a decomposable 2-vector. Hence

Proposition 18 *There is a one-to-one correspondence $\ell \leftrightarrow L$ between lines ℓ in a 3-dimensional projective space $P(V)$ and points L in the 4-dimensional quadric $Q \subset P(\Lambda^2 V)$.*

It was Felix Klein (1849–1925), building on the work of his supervisor Julius Plücker, who first described this in detail and Q is usually called the *Klein quadric*. The equation of the quadric in the form (8) shows that there are linear subspaces inside it of maximal dimension 2 whatever the field. The linear subspaces all relate to intersection properties of lines in $P(V)$. For example:

Proposition 19 *Two lines $\ell_1, \ell_2 \subset P(V)$ intersect if and only if the line joining the two corresponding points $L_1, L_2 \in Q$ lies entirely in Q.*

Proof: Let $U_1, U_2 \subset V$ be the two-dimensional subspaces of V defined by ℓ_1, ℓ_2. Suppose the lines intersect in X, with representative vector $u \in V$. Then extend to bases $\{u, u_1\}$ for U_1 and $\{u, u_2\}$ for U_2. The line in $P(\Lambda^2 V)$ joining L_1 and L_2 is then $P(W)$ where W is spanned by $u \wedge u_1$ and $u \wedge u_2$.

Any 2-vector in W is thus of the form

$$\lambda_1 u \wedge u_1 + \lambda_2 u \wedge u_2 = u \wedge (\lambda_1 u_1 + \lambda_2 u_2)$$

which is decomposable and so represents a point in Q.

Conversely, if the lines do not intersect, $U_1 \cap U_2 = \{0\}$ so $V = U_1 \oplus U_2$. In this case choose bases $\{u_1, v_1\}$ of U_1 and $\{u_2, v_2\}$ of U_2. Then $\{u_1, v_1, u_2, v_2\}$ is a basis of V and in particular $u_1 \wedge v_1 \wedge u_2 \wedge v_2 \neq 0$. A point on the line joining L_1, L_2 is now represented by $a = \lambda_1 u_1 \wedge v_1 + \lambda_2 u_2 \wedge v_2$ so that

$$a \wedge a = 2\lambda_1 \lambda_2 u_1 \wedge v_1 \wedge u_2 \wedge v_2$$

which vanishes only if λ_1 or λ_2 are zero. Thus the line only meets Q in the points L_1 and L_2. □

Now fix a point $X \in P(V)$ and look at the set of lines passing through this point:

Proposition 20 *The set of lines $\ell \subset P(V)$ passing through a fixed point $X \in P(V)$ corresponds to the set of points $L \in Q$ which lie in a fixed plane contained in Q.*

Proof: Let x be a representative vector for X. The line $P(U)$ passes through X if and only if $x \in U$, so $P(U)$ is represented in the Klein quadric by a 2-vector of the form

$$x \wedge u.$$

Extend x to a basis $\{x, v_1, v_2, v_3\}$ of V, then any decomposable 2-vector of the form $x \wedge y$ can be written as

$$x \wedge (\mu x + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) = \lambda_1 x \wedge v_1 + \lambda_2 x \wedge v_2 + \lambda_3 x \wedge v_3.$$

Thus any line passing through X is represented by a 2-vector in the 3-dimensional space of decomposables spanned by $x \wedge v_1, x \wedge v_2, x \wedge v_3$, which is a projective plane in Q. Conversely any point in this plane defines a line in $P(V)$ through X. □

A plane in Q defined by a point $X \in P(V)$ like this is called an *α-plane.* There are other planes in Q:

Proposition 21 *Let $P(W) \subset P(V)$ be a plane. The set of lines $\ell \subset P(W)$ corresponds to the set of points $L \in Q$ which lie in a fixed plane contained in Q.*

A plane of this type contained in Q is called a *β-plane*.

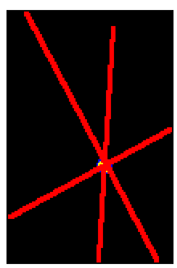
Proof: We just use duality here: if $U \subset V$ is 2-dimensional, then its annihilator $U^0 \subset V'$ is $4-2=2$-dimensional, so there is a one-to-one correspondence between lines in $P(V)$ and lines in $P(V')$. A point in Q therefore defines a line in either the projective space or its dual. Now the dual of the set of lines passing through a point is the set of lines lying in a (hyper)-plane. So applying Proposition 20 to $P(V')$ gives the result. $\square$

In fact there are no more planes:

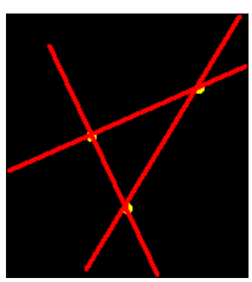
Proposition 22 *Any plane in the Klein quadric Q is either an α-plane or a β-plane.*

Proof: Take a plane in Q and three non-collinear points L_1, L_2, L_3 on it. We get three lines ℓ_1, ℓ_2, ℓ_3 in $P(V)$. Since the line joining L_1 to L_2 lies in the plane and hence in Q, it follows from Proposition 19 that each pair of ℓ_1, ℓ_2, ℓ_3 intersect. There are two possibilities:

- the three lines are concurrent:



- the three lines meet in three distinct points:



In the first case the three lines pass through a single point and so L_1, L_2, L_3 lie in an α-plane. But this must be the original plane since the three representative vectors for L_1, L_2, L_3 are linearly independent as the points are not collinear.

In the second case, if u_1, u_2, u_3 are representative vectors for the three points of intersection of ℓ_1, ℓ_2, ℓ_3, then L_1, L_2, L_3 are represented by $u_2 \wedge u_3, u_3 \wedge u_1, u_1 \wedge u_2$. A general point on the plane is then given by

$$\lambda_1 u_2 \wedge u_3 + \lambda_2 u_3 \wedge u_1 + \lambda_1 u_3 \wedge u_1$$

which is a general element of $\Lambda^2 U$ where U is spanned by u_1, u_2, u_3. Thus ℓ_1, ℓ_2, ℓ_3 all lie in the plane $P(U) \subset P(V)$. $\square$

The existence of these two families of linear subspaces of maximal dimension is characteristic of even-dimensional quadrics – it is the generalization of the two families of lines we saw on the "cooling tower" quadric surface. In the case of the Klein quadric, two different α-planes intersect in a point, since there is a unique line joining two points. Similarly (and by duality) two β planes meet in a point. An α-plane and a β plane in general have empty intersection – if X is a point and π a plane with $X \notin \pi$, there is no line in π which passes through X. If $X \in \pi$, then the intersection is a line.

4.5 Exercises

1. If $a \in \Lambda^p V$ and p is odd, show that $a \wedge a = 0$.

2. Calculate $a \wedge b$ in the following cases:

- $a = b = v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_1$
- $a = v_1 \wedge v_2 + v_3 \wedge v_1, \quad b = v_2 \wedge v_3 \wedge v_4$
- $a = v_1 + v_2 + v_3, \quad b = v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_1$.

[v_1, v_2, v_3, v_4 *are linearly independent*]

3. Which of the following 2-vectors is decomposable?

- $v_1 \wedge v_2 + v_2 \wedge v_3$
- $v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4$
- $v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4 + v_4 \wedge v_1$.

[v_1, v_2, v_3, v_4 *are linearly independent*]

4. If $\dim V = n$ shown that every $a \in \Lambda^{n-1}V$ is decomposable.

5. Let $\ell \subset P(V)$ be a line and m another such that the corresponding point $M \in Q$ lies on the polar hyperplane to $L \in Q$. Show that ℓ and m intersect.