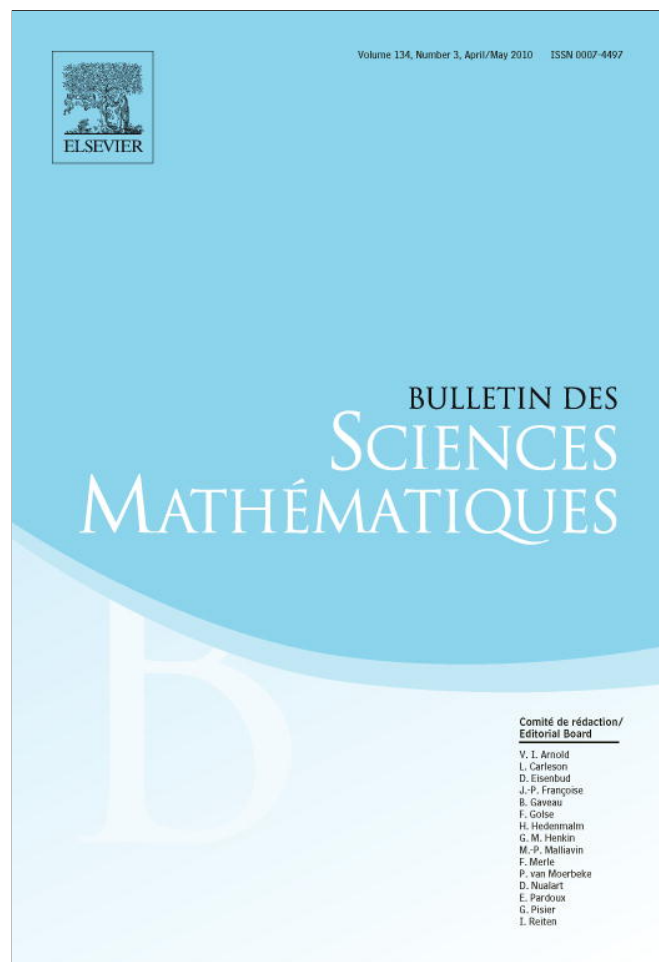


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Foliations by stationary disks of almost complex domains [☆]

G. Patrizio ^{a,*}, A. Spiro ^b^a *Dipartimento di Matematica “U. Dini”, Università di Firenze, Firenze, Italy*^b *Dipartimento di Matematica e Informatica, Università di Camerino, Camerino, Italy*

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Abstract

We study the problem of existence of stationary disks for domains in almost complex manifolds. As a consequence of our results, we prove that any almost complex domain which is a small deformation of a strictly linearly convex domain $D \subset \mathbb{C}^n$ with standard complex structure admits a singular foliation by stationary disks passing through any given internal point. Similar results are given for foliations by stationary disks through a given boundary point.

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Résumé

Nous étudions le problème de l'existence des disques stationnaires pour des domaines dans une variété presque complexe. Comme conséquence de nos résultats, nous montrons que tous les domaines presque complexes obtenu comme une petite déformation d'un domaine strictement linéairement convexe $D \subset \mathbb{C}^n$, avec la structure complexe standard, admet une foliation dans disques stationnaires passant par un point interne donné de D . Des résultats similaires sont obtenus pour foliations dans disques stationnaires dont les bords passent pour un point donné dans le bord du domaine.

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* Corresponding author.

E-mail addresses: patrizio@math.unifi.it (G. Patrizio), andrea.spiro@unicam.it (A. Spiro).

1. Introduction

Let (M, J) be an almost complex manifold and $D \subset M$ a strongly pseudoconvex domain with smooth boundary. Given a point $x_o \in D$, let us call *foliation by stationary disks of (D, x_o)* any collection of stationary disks centered at x_o and smoothly parameterized by the points of a unit sphere $S = \{v \in T_{x_o}D: \|v\| = 1\}$ for some Euclidean norm $\|\cdot\|$ on $T_{x_o}D$. By “stationary disk” we mean any J -holomorphic embedding $f: \Delta \rightarrow D$ of the unit disk $\Delta \subset \mathbb{C}$ that satisfies the definition of Coupet, Gaussier and Sukhov in [4], which naturally generalizes the usual notion of Lempert’s stationary disks for bounded domains in \mathbb{C}^n .

In case $(M, J) = (\mathbb{C}^n, J_{\text{st}})$, natural examples of foliations by stationary disks are given by the straight disks through the origin of the pseudoconvex, smoothly bounded complete circular domains D in \mathbb{C}^n . Other interesting examples are provided by the celebrated results by Lempert on Kobayashi extremal disks in strictly linearly convex domains [13–15]. In fact, an immediate consequence of those results is that for any smoothly bounded, strictly linearly convex domain $D \subset \mathbb{C}^n$ and any $x_o \in D$, the Kobayashi extremal disks of D through x_o give a foliation by stationary disks of (D, x_o) . The existence of a foliation by stationary disks is also one of the main properties of the smoothly bounded *domains of circular type*, a class of domains in \mathbb{C}^n with an exhaustion of a special kind, which naturally include all complete strictly pseudoconvex bounded circular domains, all bounded strictly linearly convex domains and, more generally, all strictly pseudoconvex domains with (singular) foliations given Kobayashi extremal disks satisfying some special regularity conditions [18,19].

In all these cases, the foliation by stationary disks $\mathcal{F}^{(x_o)}$ can be used to construct a so-called *generalized Riemann map*, i.e. a homeomorphism $\varphi: \overline{B^n} \rightarrow \overline{D}$, which is smooth on $B^n \setminus \{0\}$ and maps the straight complex lines in B^n through 0 into corresponding disks of $\mathcal{F}^{(x_o)}$. This generalized Riemann map have been often used in at least two important research areas: (a) generalizations of Fefferman’s theorem on boundary regularity of biholomorphisms between pseudoconvex domains; (b) Green’s functions with logarithmic pole for Monge–Ampère equations and plurisubharmonic exhaustions of pseudoconvex domains (see e.g. [13,14,1,24,7]).

At the best of our knowledge, the first use of foliations by stationary disks in the contest of almost complex manifolds can be found in [4]. There, the authors generalize Lempert’s notion of stationary disks in the almost complex setting and show the existence of a foliation by stationary disks of the unit ball $B^n \subset \mathbb{C}^n$, endowed with an almost complex structure J which is a sufficiently small deformation of standard complex structure J_{st} . The corresponding generalized Riemann map has been used to prove \mathcal{C}^∞ -regularity of biholomorphisms between two almost complex domains (B^n, J) and (B^n, J') of this kind, which admit \mathcal{C}^1 -extensions up to the boundary (see also [20]). Later, Gaussier and Sukhov showed in [9] that the hypothesis of \mathcal{C}^1 -extendibility can be removed and that the result holds true for any pair of smoothly bounded, strictly pseudoconvex almost complex domains, proving Fefferman’s theorem in almost complex setting in full generality (see also [5]).

Motivated by these results and possible applications on plurisubharmonic exhaustions, in this paper we determine more general situations in which the existence of foliations by stationary disks (and hence of generalized Riemann maps) is granted. Basically, we follow the approach of [4]. We first consider the differential problem that characterizes the stationary disks of an almost complex domain (D, J_o) and we explicitly determine the associated linearized operator \mathfrak{R} at a given stationary disk $f_o: \Delta \rightarrow \overline{D}$. When \mathfrak{R} is invertible, we say that ∂D is *good relatively to the pair (f_o, J_o)* . A direct application of the Implicit Function Theorem implies that if ∂D is good, then there exist stationary disks in a neighborhood of f_o also when J_o is replaced by

a sufficiently close almost complex structure J . On the basis of this observation, one has that *if an almost complex domain (D, J_o) has a foliation $\mathcal{F}^{(x_o)}$ of stationary disks through x_o , and if the boundary is good for (f_o, J_o) for any $f_o \in \mathcal{F}^{(x_o)}$, then there exists a foliation for (D, J) of stationary disks passing through x_o , also when J_o is replaced by a sufficiently close almost complex structure $J \neq J_o$.*

Secondly, by a line of arguments that goes back to Lempert and Pang ([13,17]; see also [23,22, 4,20]), we are able to prove that *any smoothly bounded, strictly linearly convex domain $D \subset \mathbb{C}^n$ has a boundary which is “good” for any of its stationary disks.* This fact and previous observation bring directly to our result, which generalizes the quoted Coupet, Gaussier and Sukhov’s theorem on the unit ball: *if a smoothly bounded, strongly pseudoconvex domain D in an almost complex manifold (M, J) is biholomorphic to a strictly linearly convex domain $\widehat{D} \subset (\mathbb{C}^n, J')$, endowed with small deformation J' of J_{st} , then there exists a foliation by stationary disks of (D, x_o) for any $x_o \in D$ (Theorem 4.1).*

This shows that the class of almost complex domains, admitting a foliation by stationary disks, is indeed much larger than the class considered in [4]. In fact, via a diffeomorphism $\varphi: \mathcal{U} \rightarrow \mathcal{V} \subset \mathbb{C}^n$ mapping \overline{D} onto $\overline{B^n}$, one obtains the existence of foliations by stationary disks on (B^n, J') also when $J' = \varphi_*(\widehat{J})$ is not a small deformation of J_{st} .

We also prove that, for any almost complex domain (D, J) as above and with J' sufficiently close to J_{st} , there exists a generalized Riemann map $\varphi: \overline{B^n} \rightarrow \overline{D}$ for any $x_o \in D$ and the function $u = (\varphi^{-1})^*(u_o): D \rightarrow]-\infty, 0[$ of $u_o(z) \stackrel{\text{def}}{=} \log(|z|)$ is a plurisubharmonic exhaustion for D . When J is integrable, u is a solution of the Monge–Ampère equation $(\partial\bar{\partial}u)^n = 0$ with boundary data $u|_{\partial D} = 0$ and logarithmic singularity at x_o . It would be interesting to know if this and other related properties have counterparts in almost complex setting.

Finally, we consider the families $\mathcal{G}^{(x_o, a)}$, formed by all stationary disks in a given almost complex domain (D, J) , passing through a given boundary point $x_o \in \partial D$ and with tangent vector v at x_o , with inner product $\langle v, \nu \rangle$ with the unit normal ν_{x_o} larger than a value $a \geq 0$. In case D is a strictly convex domain in \mathbb{C}^n , the disks in $\mathcal{G}^{(x_o, a)}$ give a (regular) foliation of a certain subdomain $D^{(x_o, a)} \subset D$ that coincide with D in case $a = 0$ [6]. We prove that *if $a > 0$, this is true also when the standard complex structure J_{st} is replaced by an almost complex structure J sufficiently close to J_{st} and we therefore have an analogue of the previous results also for what concerns foliations of conical subdomains $D^{(x_o, a)}$, $a > 0$, of almost complex domains.* A proof for the case $a = 0$ seems to be at the moment out of reach, because the family of stationary disks $\mathcal{G}^{(x_o, 0)}$ is not parameterized by a compact set, in contrast with all other considered situations.

As final remark, notice that when J is integrable, the regular foliations $\mathcal{G}^{(x_o, a)}$ determine analogues of the Riemann map and have been used in [2,3] to construct solutions to the Monge–Ampère equation $(\partial\bar{\partial}u)^n = 0$ with singularity at a given boundary point. It would be interesting to know if a similar construction can be obtained in an almost complex setting.

The structure of the paper is as follows. In Section 2 we recall a few basic facts and the definition of stationary disks in almost complex domains. In Section 3, we consider the so-called foliations of circular type, prove their stability under small deformations of J in case of a “good boundary”. In Section 4, general conditions for a boundary “to be good” are given and are used to show that any strictly linearly convex domain has a “good” boundary. This and the results of Section 3 give our main Theorem 4.1 as immediate consequence. Section 5 is devoted to the quoted results on foliations of conical subdomains.

2. Preliminaries

2.1. Notations

Given a real manifold M and a system of coordinates $\xi = (x^i) : \mathcal{U} \subset M \rightarrow \mathbb{R}^n$, we call *associated coordinates on T^*M* the coordinates $\widehat{\xi} = (x^i, p_i)$, where for any $\alpha \in T_x^*M$ the “ p_i ” are the components of $\alpha = p_i dx^i$ in the basis (dx^i) . If (M, J) is an almost complex manifold of real dimension $2n$, we call *system of complex coordinates* any local diffeomorphism $\xi = (z^i) : \mathcal{U} \subset M \rightarrow \mathbb{C}^n$. We call them *holomorphic* whenever J is integrable and $\xi = (z^i)$ is a chart of the corresponding complex manifold structure of (M, J) . We also call *associated complex coordinates on T^*M* the complex coordinates $\widehat{\xi} = (z^i, w_j) : \pi^{-1}(\mathcal{U}) \subset T^*M \rightarrow \mathbb{C}^{2n}$, where the w_j 's are defined for any 1-form α by the expression $\alpha = w_i dz^i + \bar{w}_i d\bar{z}^i$.

For any Banach space X and $\mathcal{U} \subset \mathbb{R}^M$, $\alpha \in]0, 1[$, we denote by $C^\alpha(\mathcal{U}, X)$ the Banach space of the functions $f : \mathcal{U} \rightarrow X$ such that

$$\|f\|_\alpha \stackrel{\text{def}}{=} \sup_{\zeta \in \mathcal{U}} \|f(\zeta)\| + \sup_{\theta, \eta \in \mathcal{U}, \theta \neq \eta} \frac{\|f(\theta) - f(\eta)\|}{|\theta - \eta|^\alpha} < \infty.$$

If $\alpha = m + \beta$, for some $m \in \mathbb{N}$ and $\beta \in]0, 1[$, we denote by $C^\alpha(\mathcal{U}, X)$ the Banach space $C^\alpha(\mathcal{U}, X) = \{r \in C^m(\mathcal{U}, X) : D^v r \in C^\beta(\mathcal{U}, X), v : |v| \leq m\}$. Finally, for any $\alpha, \epsilon > 0$, we set $C^{\alpha, \epsilon}(\bar{\Delta}, \mathbb{C}^n) = C^\epsilon(\bar{\Delta}, \mathbb{C}^n) \cap C^\alpha(\Delta, \mathbb{C}^n)$ and $H^\epsilon(\bar{\Delta}, \mathbb{C}^n) = C^\epsilon(\bar{\Delta}, \mathbb{C}^n) \cap \text{Hol}(\Delta, \mathbb{C}^n)$.

2.2. Lifts of J -holomorphic disks

We recall that a C^α -map $f : M \rightarrow M'$, $1 \leq \alpha$, between two almost complex manifolds (M, J) , (M', J') is called (J, J') -holomorphic if and only if $\bar{\partial}_{J, J'} f(v) = 0$ for any $v \in TM$, where $\bar{\partial}_{J, J'} f$ is the operator

$$\bar{\partial}_{J, J'} f : TM \rightarrow TM', \quad \bar{\partial}_{J, J'} f(v) \stackrel{\text{def}}{=} f_*(J(v)) - J'(f_*(v)). \tag{2.1}$$

When $(M, J) = (\mathbb{C}^n, J_{\text{st}})$, we will shortly write $\bar{\partial}_{J'}$ for $\bar{\partial}_{J_{\text{st}}, J'}$. A J -holomorphic disk of (M, J) is a (J_{st}, J) -holomorphic map $f : \Delta \rightarrow M$ from the unit disk $\Delta \subset \mathbb{C}$ into (M, J) . Recall that $\bar{\partial}_J f = 0$ if and only if $\bar{\partial}_J f(\frac{\partial}{\partial x}|_{x+iy}) = 0$ at any $x + iy \in \Delta$ (see e.g. [11]).

If (M, J) is a complex manifold, the cotangent bundle T^*M is naturally endowed with an *integrable* complex structure \mathbb{J} , determined by the identifications of open subsets $\mathcal{U} \subset M$ with open subsets of \mathbb{C}^n and by the identifications of the sets $T^*M|_{\mathcal{U}}$ with open subsets of $\mathbb{C}^{2n} = T^*\mathbb{C}^n$. When J is not integrable, these identifications are no longer valid, but there still exists a natural almost complex structure \mathbb{J} on T^*M , which reduces to the usual one if J is integrable [10]. The main properties of \mathbb{J} are summarized in the next proposition. Here, $J_j^i = J_j^i(x)$ are the components of $J = J_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ in a system of real coordinates $\xi = (x^i)$.

Proposition 2.1. (See [10].) *For any almost complex manifold (M, J) , there exists a unique almost complex structure \mathbb{J} on T^*M with the following properties:*

- (i) *the projection $\pi : T^*M \rightarrow M$ is (\mathbb{J}, J) -holomorphic;*
- (ii) *for any (J, J') -biholomorphism $f : M \rightarrow N$ between two almost complex manifolds (M, J) and (N, J') , the induced map $\widehat{f} : T^*N \rightarrow T^*M$ is $(\mathbb{J}', \mathbb{J})$ -holomorphic;*

- (iii) if J is integrable, then also \mathbb{J} is integrable and coincides with the natural complex structure of T^*M ;
- (iv) in a system of coordinates

$$\widehat{\xi} = (x^1, \dots, x^{2n}, p_1, \dots, p_{2n}) : \pi^{-1}(U) \subset T^*M \rightarrow \mathbb{R}^{4n}, \tag{2.2}$$

associated with $\xi = (x^i)$, the tensor \mathbb{J} is of the form

$$\begin{aligned} \mathbb{J} = & J_i^a \frac{\partial}{\partial x^a} \otimes dx^i + J_i^a \frac{\partial}{\partial p_i} \otimes dp_a \\ & + \frac{1}{2} p_a (-J_{i,j}^a + J_{j,i}^a + J_\ell^a (J_{i,m}^\ell J_j^m - J_{j,m}^\ell J_i^m)) \frac{\partial}{\partial p_j} \otimes dx^i. \end{aligned} \tag{2.3}$$

The almost complex structure \mathbb{J} is called *canonical lift of J on T^*M* .

Lemma 2.2. *Let \mathbb{J} be the canonical lift on T^*M of an almost complex structure J . For any $0 \neq t \in \mathbb{R}$, the map $\varphi_t : T^*M \rightarrow T^*M$ defined by $\varphi_t(\alpha) = t \cdot \alpha$ is a \mathbb{J} -biholomorphic diffeomorphism, i.e. $\varphi_{t*} \circ \mathbb{J} = \mathbb{J} \circ \varphi_{t*}$.*

Proof. Writing φ_t in a system of coordinates (2.2), one has that $\varphi_t(x^i, p_j) = (x^i, tp_j)$. Using (2.3), the claim is then immediately checked. \square

Given a J -holomorphic disk $f : \Delta \rightarrow (M, J)$, we call *lift of f* any \mathbb{J} -holomorphic disk $\widehat{f} : \Delta \rightarrow (T^*M, \mathbb{J})$ so that $f = \pi \circ \widehat{f}$.

2.3. Stationary disks

Let $\Gamma \subset M$ be a smooth hypersurface of an almost complex manifold (M, J) . The *conormal bundle of Γ* is defined as

$$\mathcal{N} \stackrel{\text{def}}{=} \{ \alpha \in T_x^*M, x \in \Gamma : \alpha|_{T_x\Gamma} \equiv 0 \} \subset T^*M|_\Gamma. \tag{2.4}$$

In the following, we denote by $\mathcal{N}_* = \mathcal{N} \setminus \{\text{zero section}\}$ and when we mention “the conormal bundle” we will always mean \mathcal{N}_* .

The CR structure of Γ is defined as the pair (\mathcal{D}, J) given by the distribution

$$\mathcal{D} = \bigcup_{x \in \Gamma} \mathcal{D}_x \subset T\Gamma, \quad \mathcal{D}_x \stackrel{\text{def}}{=} \{ v \in T_x\Gamma : J(v) \in T_x\Gamma \} \tag{2.5}$$

endowed with the family $J = \{J_x\}$ of complex structures $J_x \stackrel{\text{def}}{=} J|_{\mathcal{D}_x}$. A *defining 1-form* for \mathcal{D} is a 1-form on Γ so that $\ker \vartheta|_x = \mathcal{D}_x$ for any $x \in \Gamma$. The *Levi form at x* is the quadratic form $\mathcal{L}_x : \mathcal{D}_x \rightarrow \mathbb{R}$ defined by $\mathcal{L}_x(v) \stackrel{\text{def}}{=} -d\vartheta_x(v, Jv)$ for any $v \in \mathcal{D}_x$ and (up to a scalar factor) it is independent on the choice of ϑ . This last property follows immediately from the fact that for any vector field $X^{(v)} \in \mathcal{D}$ so that $X_x^{(v)} = v$ one has

$$\mathcal{L}_x(v) = -d\vartheta_x(X^{(v)}, JX^{(v)}) = \vartheta_x([X^{(v)}, JX^{(v)}]). \tag{2.6}$$

An *oriented* hypersurface $\Gamma \subset M$ is called *strongly pseudoconvex* if \mathcal{L}_x is positive definite at every $x \in \Gamma$ when determined by a defining 1-form ϑ with $\vartheta_x(Jn) > 0$ for any n pointing in the “outwards” direction. If $D \subset M$ is a bounded domain with smooth boundary ∂D , we say that

D is *strongly pseudoconvex* when ∂D , oriented so that the “outwards” directions are pointing outside D , is strongly pseudoconvex.

The following notion of “stationary disk” for domains in almost complex manifolds was considered for the first time by Coupet, Gaussier and Sukhov in [4]. It generalizes the notion of stationary disks of bounded domains in \mathbb{C}^n [13,24].

Definition 2.3. Let $D \subset M$ be a domain with smooth boundary and \mathcal{N}_* the conormal bundle of ∂D . Given $\alpha \geq 1$, $\varepsilon > 0$, a map $f : \bar{\Delta} \rightarrow M$ is called $\mathcal{C}^{\alpha,\varepsilon}$ -stationary disk of D if

- (i) $f|_{\Delta}$ is a J -holomorphic embedding and $f(\partial\Delta) \subset \partial D$;
- (ii) there exists a lift $\widehat{f} : \bar{\Delta} \rightarrow T^*M$ of f so that

$$\zeta^{-1} \cdot \widehat{f}(\zeta) \in \mathcal{N}_* \quad \text{for any } \zeta \in \partial\Delta \tag{2.7}$$

and $\widehat{\xi} \circ \widehat{f} \in \mathcal{C}^{\alpha,\varepsilon}(\bar{\Delta}, \mathbb{C}^{2n})$ for some complex coordinates $\widehat{\xi} = (z^i, w_j)$ around $\widehat{f}(\bar{\Delta})$. Here “ \cdot ” denotes the usual \mathbb{C} -action on T^*M , i.e.

$$\zeta \cdot \alpha \stackrel{\text{def}}{=} \text{Re}(\zeta)\alpha - \text{Im}(\zeta)J^*\alpha \quad \text{for any } \alpha \in T^*M, \zeta \in \mathbb{C}. \tag{2.8}$$

In the following, the values of α and ε are considered as fixed and by “stationary” we always mean “ $\mathcal{C}^{\alpha,\varepsilon}$ -stationary”. Moreover, given a stationary disk f , the maps \widehat{f} satisfying (ii) are called *stationary lifts of f* .

Lemma 2.4.

- (i) If $D \subset M$ is a smoothly bounded, strongly pseudoconvex domain and $f : \bar{\Delta} \rightarrow M$ is a non-constant stationary disk of D , then $f(\bar{\Delta}) \subset \bar{D}$ and $f(\zeta) \in \partial D$ if and only if $\zeta \in \partial\Delta$.
- (ii) For any $t \in \mathbb{R}_*$ and any stationary lift \widehat{f} of a stationary disk $f : \bar{\Delta} \rightarrow \bar{D}$, also the map $\widehat{f}_t(\zeta) \stackrel{\text{def}}{=} (\varphi_t \circ \widehat{f})(\zeta) = t \cdot \widehat{f}(\zeta)$ is a stationary lift of f .

Proof. (i) If D is strongly pseudoconvex, it is known that there exists a defining function $\rho : \mathcal{U} \subset M \rightarrow \mathbb{R}$ for D which is J -plurisubharmonic, i.e. so that $\rho \circ f : \Delta \rightarrow \mathbb{R}$ is strictly subharmonic for any J -holomorphic disk $f : \Delta \rightarrow \mathcal{U}$ (see e.g. [5, p. 14]). Since $\rho \circ f|_{\partial\Delta} = 0$, the claim follows from the maximum principle.

(ii) It follows from the fact that \widehat{f}_t satisfies (2.7) and that the diffeomorphism φ_t is a \mathbb{J} -biholomorphism by Lemma 2.2. \square

We conclude recalling the following theorem that generalizes a well-known result by Webster to the almost complex setting [26].

Theorem 2.5. (See [21].) Let Γ be a strongly pseudoconvex hypersurface in an almost complex manifold (M, J) and $\mathcal{N}_* \subset T^*M$ its conormal bundle with the zero section excluded. Then \mathcal{N}_* is a totally real submanifold of (T^*M, \mathbb{J}) .

3. Foliations by stationary disks and deformations of almost complex structures

3.1. The Riemann–Hilbert problem for stationary disks

In this and the next sections, D is a strongly pseudoconvex domain in an almost complex manifold (M, J) with smooth boundary ∂D with conormal bundle $\mathcal{N} \subset T^*M|_{\partial D}$. We also assume that $\bar{D} \subset M$ is contained in a globally coordinatizable open subset $\mathcal{U} \subset M$ or, equivalently, that D is a domain of $M = \mathbb{R}^{2n} \simeq \mathbb{C}^n$ equipped with a non-standard complex structure J . We also assume that D has a smooth defining function $\rho : \mathcal{U} \subset M \rightarrow \mathbb{R}$ on \mathcal{U} , so that

$$D = \{x \in M : \rho(x) < 0\} \quad \text{and} \quad d\rho_x \neq 0 \quad \text{for any } x \in \Gamma = \partial D.$$

We want to study the differential problem that characterizes the lifts $\widehat{f} : \bar{\Delta} \rightarrow T^*M$ of stationary disks of D . First of all, consider the map

$$\widetilde{\rho} : \mathbb{R}_* \times T^*M|_{\mathcal{U}} \rightarrow \mathbb{R} \times T^*M|_{\mathcal{U}}, \quad \widetilde{\rho}(t, \alpha) \stackrel{\text{def}}{=} (\rho(\pi(\alpha)), \alpha - t \cdot d\rho_{\pi(\alpha)}). \quad (3.1)$$

Notice that the bundle $\mathcal{N}_* = \mathcal{N} \setminus \{\text{zero section}\}$, which is a $2n$ -dimensional submanifold of T^*M , can be identified with the level set

$$\{(t, \alpha) : t \neq 0, \widetilde{\rho}(t, \alpha) = (0_{\mathbb{R}}, 0_{T^*_{\pi(\alpha)}M})\} \subset \mathbb{R}_* \times T^*M|_{\mathcal{U}},$$

which is a $2n$ -dimensional submanifold of $\mathbb{R}_* \times T^*M$. Therefore, using a system of coordinates $\widehat{\xi} = (x^i, p_j)$ on $T^*M|_{\mathcal{U}}$, associated with coordinates $\xi = (x^i)$, we may identify $\mathbb{R}_* \times T^*M|_{\mathcal{U}}$ with an open subset $\mathcal{V} \subset \mathbb{R}^{4n+1}$ and \mathcal{N}_* with the level set in \mathcal{V} defined by

$$\mathcal{N}_* \simeq \{(t, \alpha) \in \mathcal{V} : \widetilde{\rho}^i(t, \alpha) = 0, \quad 1 \leq i \leq 2n + 1\}.$$

By a direct check of the rank of the Jacobian, one can see that the map $\widetilde{\rho} = (\widetilde{\rho}^1, \dots, \widetilde{\rho}^{2n+1})$ is a smooth defining function for \mathcal{N}_* .

We now consider the map $r : \mathbb{C} \times \mathcal{V} \subset \mathbb{C} \times \mathbb{R}^{4n+1} \rightarrow \mathbb{R}^{2n+1}$, defined by

$$r(\zeta, t, \alpha) \stackrel{\text{def}}{=} (\widetilde{\rho}^1(t, \zeta^{-1} \cdot \alpha), \dots, \widetilde{\rho}^{2n+1}(t, \zeta^{-1} \cdot \alpha)). \quad (3.2)$$

Here, the product $\zeta^{-1} \cdot \alpha$ is as in (2.8). By definition, a disk $f : \bar{\Delta} \rightarrow \bar{D} \subset \mathbb{R}^{2n}$ is stationary if and only if there exists $\widehat{f} \in (\mathcal{C}^{\alpha, \varepsilon}(\bar{\Delta}); \mathbb{C}^{2n})$ and $\lambda \in \mathcal{C}^\varepsilon(\partial \Delta; \mathbb{R})$ so that

$$\begin{cases} \bar{\partial}_{\mathbb{J}} \widehat{f}(\zeta) = 0, & \zeta \in \Delta, \\ r(\zeta, \lambda(\zeta), \widehat{f}(\zeta)) = 0, & \zeta \in \partial \Delta \end{cases} \quad (3.3)$$

where $\bar{\partial}_{\mathbb{J}} = \bar{\partial}_{J_{\text{st}}, \mathbb{J}} : \mathcal{C}^\alpha(\bar{\Delta}, \mathbb{C}^{2n}) \rightarrow \mathcal{C}^{\alpha-1}(\bar{\Delta}, \mathbb{C}^{2n})$ is the operator (2.1).

The differential problem (3.3) belongs to a class often called of *generalized Riemann–Hilbert problems* (see f.i. [16, Ch. VII]).

3.2. Stability under small deformations of the data

Consider a fixed almost complex structure $J = J_o$, a point $x_o \in D \subset \mathbb{R}^{2n}$ and a vector $v_o \in T_{x_o}D \simeq \mathbb{R}^{2n}$ and denote by $\mathcal{R}_{(J_o, x_o, v_o)} = (\mathcal{R}_1, \dots, \mathcal{R}_5)$ the operator from $\mathcal{C}^{\alpha, \varepsilon}(\bar{\Delta}, \mathbb{C}^{2n}) \times \mathcal{C}^\varepsilon(\partial \Delta, \mathbb{R}) \times \mathbb{R}_*$ into $\mathcal{C}^{\alpha-1, \varepsilon}(\bar{\Delta}, \mathbb{C}^{2n}) \times \mathcal{C}^\varepsilon(\partial \Delta, \mathbb{R}^{2n+1}) \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R}$ with components \mathcal{R}_i defined by

$$\begin{aligned}
 \mathcal{R}_1(\widehat{f}, \lambda, \mu) &\stackrel{\text{def}}{=} \bar{\partial}_{\mathbb{J}_o} \widehat{f}, & \mathcal{R}_2(\widehat{f}, \lambda, \mu) &\stackrel{\text{def}}{=} r(\zeta, \lambda(\zeta), \widehat{f}(\zeta)), \\
 \mathcal{R}_3(\widehat{f}, \lambda, \mu) &\stackrel{\text{def}}{=} \pi(\widehat{f})|_{\zeta=0} - x_o, & \mathcal{R}_4(\widehat{f}, \lambda, \mu) &\stackrel{\text{def}}{=} \pi(\widehat{f})_* \left(\frac{\partial}{\partial x} \Big|_{\zeta=0} \right) - \mu v_o, \\
 \mathcal{R}_5(\widehat{f}, \lambda, \mu) &\stackrel{\text{def}}{=} \widehat{f} \left(\pi(\widehat{f})_* \left(\frac{\partial}{\partial x} \Big|_1 \right) \right) - 1.
 \end{aligned} \tag{3.4}$$

Notice also that, by Hopf’s lemma and Lemma 2.4(ii), for any stationary disk, there exists a stationary lift satisfying $\widehat{f}(\pi(\widehat{f})_* (\frac{\partial}{\partial x}|_1)) = 1$. So, by the previous section, the existence of a stationary disk $f : \Delta \rightarrow D$ with $f(0) = x_o$ and $f_* (\frac{\partial}{\partial x}|_0) \in \mathbb{R}v_o$ is equivalent to the existence of a solution to

$$\mathcal{R}_{(J_o, x_o, v_o)}(\widehat{f}, \lambda, \mu) = 0. \tag{3.5}$$

Let $(\widehat{f}_o, \lambda_o, \mu_o)$ be solution of (3.5) and $\mathfrak{R}_{(J_o, x_o, v_o; \widehat{f}_o, \lambda_o, \mu_o)} \stackrel{\text{def}}{=} \dot{\mathcal{R}}_{(J_o, x_o, v_o)}|_{(\widehat{f}_o, \lambda_o, \mu_o)}$ the linearized operator at $(\widehat{f}_o, \lambda_o, \mu_o)$ determined by $\mathcal{R}_{(J_o, x_o, v_o)}$. Now, by the Implicit Function Theorem (see e.g. [12]), when $\mathfrak{R} = \mathfrak{R}_{(J_o, x_o, v_o; \widehat{f}_o, \lambda_o, \mu_o)}$ is invertible, there exists a solution to the problem $\mathcal{R}_{(J_t, x_t, v_t)}(\widehat{f}, \lambda, \mu) = 0$ for any smooth deformation (J_t, x_t, v_t) of (J_o, x_o, v_o) for t sufficiently small t and $\dim_{\mathbb{R}} \ker \mathfrak{R}_{(J_o, x_o, v_o; \widehat{f}_o, \lambda_o, \mu_o)}$ is equal to the dimension of the solutions space. This motivates the following:

Definition 3.1. Let $f_o : \bar{\Delta} \rightarrow \bar{D}$ be a stationary disk of (D, J_o) with $x_o = f_o(0)$ and $v_o = (\widehat{f}_o)_* (\frac{\partial}{\partial x}|_{\zeta=0})$. We call ∂D a good boundary for (J_o, f_o) if there is a lift \widehat{f}_o of f_o and a function λ_o so that $(\widehat{f}_o, \lambda_o, 1)$ is a solution to (3.5) and the linearized operator $\mathfrak{R} = \mathfrak{R}_{(J_o, x_o, v_o; \widehat{f}_o, \lambda_o, 1)}$ is invertible.

The Implicit Function Theorem and previous remarks bring immediately to the next proposition. In the statement, we denote by g a fixed Riemannian metric $g = g_{ij} dx^i \otimes dx^j$ on a neighborhood of \bar{D} and by $g^* = g_{ij} dx^i \otimes dx^j + g^{ij} dp_i \otimes dp_j$ the corresponding Riemannian metric on T^*M . We also set

$$\|J - J'\|_{\bar{D}}^{(1)} \stackrel{\text{def}}{=} \sup_{x \in \bar{D}, v \in T(T_x^*M)} \frac{\|\mathbb{J}(v) - \mathbb{J}'(v)\|_{g^*}}{\|v\|_{g^*}}, \tag{3.6}$$

where $\|\cdot\|_{g^*}$ is the norm function determined by g^* . The topology determined by the norm $\|\cdot\|_{\bar{D}}^{(1)}$ is clearly independent on the choice of g .

Proposition 3.2. Let $f_o : \bar{\Delta} \rightarrow \bar{D}$ be a stationary disk of $D \subset (M, J_o)$ with $x_o = f_o(0)$ and $v_o = f_o_* (\frac{\partial}{\partial x}|_{\zeta=0})$. If ∂D is a good boundary for (J_o, f_o) , there exist a neighborhood $\mathcal{V} \subset D$ of x_o , a neighborhood $\mathcal{W} \subset TD$ of v_o , with $\pi(\mathcal{W}) = \mathcal{V} \subset D$ and a real number $\varepsilon > 0$ so that, for any $x \in \mathcal{V}$, $v \in \mathcal{W}$ and $\|J - J_o\|_{\bar{D}}^{(1)} < \varepsilon$, there exists a unique stationary disk f of (D, J) so that

$$f(0) = x, \quad f_* \left(\frac{\partial}{\partial x} \Big|_{\zeta=0} \right) = \mu v \quad \text{for some } \mu \neq 0. \tag{3.7}$$

The disk f depends differentially on x , v and J and, given $m_o > 0$, one can choose ε , \mathcal{W} and $\mathcal{V} = \pi(\mathcal{W})$ so that $\sup_{\zeta \in \bar{\Delta}} \text{dist}_g(f(\zeta), f_o(\zeta)) < m_o$.

3.3. Foliations of circular type and their stability

3.3.1. Blow-up of an almost complex domain at one point

Let x_o be a point of the almost complex manifold (M, J) and $\xi = (z^i) : \mathcal{U} \rightarrow \mathbb{C}^n$ a system of complex coordinates with

$$\xi(x_o) = 0, \quad \xi_*(J|_{x_o}) = J_{st}|_0. \tag{3.8}$$

Consider the blow up $\pi : \tilde{\mathcal{U}} \rightarrow \xi(\mathcal{U}) \subset \mathbb{C}^n$ of $\xi(\mathcal{U})$ at 0, i.e. the submanifold of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ defined by $\tilde{\mathcal{U}} = \{(z, [w]) : z \in [w], z \in \mathcal{U}\} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$. The standard projection $\pi(z, [w]) = z$ composed with ξ^{-1} determines a diffeomorphism between $\tilde{\mathcal{U}} \setminus \pi^{-1}(0)$ and $\mathcal{U} \setminus \{0\}$ that we use to glue $\tilde{\mathcal{U}}$ with $M \setminus \{x_o\}$ and obtain a manifold \tilde{M} that we call *blow up of (M, J) at x_o* .

At a first glance, this construction seems to depend on the choice of the complex coordinates $\xi = (z^i)$. But indeed the real manifold structure of \tilde{M} depends only on the linear map $J_{x_o} : T_{x_o}M \rightarrow T_{x_o}M$. This fact is a direct consequence of the following simple lemma.

Lemma 3.3. *Consider two sets of complex coordinates $\xi = (z^i)$ and $\xi' = (z'^j)$ on \mathcal{U} satisfying $\xi(x_o) = \xi'(x_o) = 0$ and $\xi_*(J_{x_o}) = \xi'_*(J_{x_o}) = J_{st}|_0$. Then the diffeomorphism $\tilde{\varphi} = \pi^{-1} \circ (\xi' \circ \xi^{-1}) \circ \pi$ of $\tilde{\mathcal{U}} \setminus \pi^{-1}(0)$ into itself admits a unique smooth extension on $\tilde{\mathcal{U}}$. It follows that the blow up \tilde{M} , defined using the chart $\xi = (z^i)$, is naturally diffeomorphic to the one constructed using the chart $\xi' = (z'^i)$.*

Proof. By construction, the map $\varphi = \xi' \circ \xi^{-1}$ is so that $\varphi_*|_0 \circ J_{st} = J_{st} \circ \varphi_*|_0$ and hence it is of the form

$$\varphi(z) = \psi(z) + g(z) \tag{3.9}$$

where ψ is the \mathbb{C} -linear map $\psi = \varphi_*|_0 : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and $g : \mathcal{U} \rightarrow \mathcal{U}$ is an infinitesimal of the second order in $|z|$. Since

$$\tilde{\varphi}(z, [z]) = (\pi^{-1} \circ \varphi \circ \pi)(z, [z]) = (\psi(z) + g(z), [\psi(z) + g(z)]),$$

an explicit computation in coordinates shows that $\tilde{\varphi}$ extends smoothly on $\pi^{-1}(0) \subset \tilde{\mathcal{U}}$ by setting $\tilde{\varphi}(0, [v]) \stackrel{\text{def}}{=} (0, [\psi(v)])$ for any $[v] \in \mathbb{C}P^{n-1}$. \square

3.3.2. Foliations of circular type

Let D be a smoothly bounded, strongly pseudoconvex domain in (M, J) and denote by $\tilde{D} \subset \tilde{M}$ the blow up of \bar{D} at a point x_o as defined in the previous section. For any stationary disk $f : \bar{\Delta} \rightarrow \bar{D}$ with $f(0) = x_o$ and $f_*(\frac{\partial}{\partial x}|_0) = v$ there exists a unique map $\tilde{f} : \bar{\Delta} \rightarrow \tilde{D}$ so that $\pi \circ \tilde{f}(\zeta) = f(\zeta)$ for any $\zeta \neq 0$. In fact, if we identify \tilde{D} with a domain in $\tilde{\mathcal{U}} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ by means of a chart like in (3.8), the lifted map \tilde{f} is of the form

$$\tilde{f}(\zeta) = \begin{cases} (f(\zeta), [f(\zeta)]) & \text{when } \zeta \neq 0, \\ (0, [v]) & \text{when } \zeta = 0. \end{cases} \tag{3.10}$$

Since f is J -holomorphic (and hence $f_*(J_{st}|_0) = J|_0 = J_{st}|_0$), we may write

$$f(\zeta) = h(\zeta) + g(\zeta) \tag{3.11}$$

for some holomorphic disk $h : \bar{\Delta} \rightarrow \mathcal{U} \subset \mathbb{C}^n$ and a smooth map $g : \bar{\Delta} \rightarrow \mathcal{U}$ which is infinitesimal of second order in $|\zeta|$. Using this, one can check that \tilde{f} is smooth also at 0. We call \tilde{f} *the smooth lift of f at \tilde{D}* .

Definition 3.4. Let $x_o \in D$ and \tilde{D} as above and denote by $\mathcal{F}^{(x_o)}$ the family of all stationary disks of D with $f(0) = x_o$. We call $\mathcal{F}^{(x_o)}$ *foliation of circular type of the pointed domain (D, x_o)* if the following conditions are satisfied:

- (i) for any $v \in T_{x_o}D$, there exists a unique disk $f^{(v)} \in \mathcal{F}^{(x_o)}$ such that $f_*^{(v)}(\frac{\partial}{\partial x}|_0) = \mu \cdot v$ for some $0 \neq \mu \in \mathbb{R}$;
- (ii) previous an identification $(T_{x_o}D, J_{x_o}) \simeq (\mathbb{C}^n, J_{st})$, the map

$$\exp: \tilde{B}^n \subset \tilde{\mathbb{C}}^n \rightarrow \tilde{D}, \quad \exp(v, [v]) \stackrel{\text{def}}{=} \tilde{f}^{(v)}(|v|), \tag{3.12}$$

between the blow up at 0 of $B^n \subset \mathbb{C}^n$ and the blow up of D at x_o is smooth with a smooth extension up to the boundary, which induces a diffeomorphism between the boundaries $\exp|_{\partial \tilde{B}^n}: \partial \tilde{B}^n \rightarrow \partial \tilde{D}$.

If $\mathcal{F}^{(x_o)}$ is a foliation of circular type, we call x_o *center of the foliation* and D a *domain of circular type w.r.t. to J* .

3.3.3. Stability under small deformations of foliations of circular type

Proposition 3.5. *Let D be of circular type w.r.t. to J_o and with center x_o . If ∂D is a good boundary for (J_o, f_o) for any stationary disk $f_o \in \mathcal{F}^{(x_o)}$, then there exists $\varepsilon > 0$ and an open neighborhood $\mathcal{U} \subset D$ of x_o so that for any J with $\|J - J_o\|_{\tilde{D}}^{(1)} < \varepsilon$ and any $x \in \mathcal{U}$, the point x is center of a foliation of circular type of D w.r.t. the almost complex structure J .*

Proof. Using a system of coordinates $\xi = (x^i)$ on a neighborhood \mathcal{W} of x_o , let us identify \mathcal{W} with an open subset of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ and its tangent space with $T\mathcal{W} \simeq \mathcal{W} \times \mathbb{R}^{2n} \subset \mathbb{R}^{4n}$. Pick also the same Euclidean inner product $\langle \cdot, \cdot \rangle$ on all tangent spaces in $T\mathcal{W} \simeq \mathcal{W} \times \mathbb{R}^{2n}$. By definitions, for any $v_o \in S_{x_o}^{2n-1} = \{v \in T_{x_o}M: \langle v, v \rangle = 1\}$, there is a unique stationary disk $f \in \mathcal{F}^{(x_o)}$ with $f_*(\frac{\partial}{\partial x}|_0) = \mu \cdot v_o$ for some $\mu \neq 0$.

By Proposition 3.2, there exist a neighborhood $\mathcal{U}^{(v_o)}$ of x_o , a neighborhood $\mathcal{V}^{(v_o)} \subset S^{2n-1}$ and $\varepsilon^{(v_o)} > 0$, so that, for any $y \in \mathcal{U}^{(v_o)}$, $v \in \mathcal{V}^{(v_o)} \subset T_yM \simeq T_{x_o}M \simeq \mathbb{R}^{2n}$ and J with $\|J - J_o\|_{\tilde{D}}^{(1)} < \varepsilon^{(v_o)}$, there exists a unique disk \tilde{f} , which is stationary for D w.r.t. J , passing through y and with $\tilde{f}_*(\frac{\partial}{\partial x}|_0)$ parallel to v . By compactness of S^{2n-1} , there exists a finite number of vectors $v_1, \dots, v_N \in S^{2n-1}$ so that the corresponding open sets $\mathcal{V}^{(v_i)} \subset S^{2n-1}$ give an open covering of S^{2n-1} . We conclude that, for any point $y \in \tilde{\mathcal{U}} = \bigcap_{i=1}^N \mathcal{U}^{(v_i)}$, $\|J - J_o\|_{\tilde{D}}^{(1)} < \min_i \varepsilon^{(v_i)}$ and $v \in T_yM$, there exists a unique disk passing through y , which is stationary w.r.t. J and with $\tilde{f}_*(\frac{\partial}{\partial x}|_0)$ parallel to $\frac{v}{|v|} \in S^{2n-1}$. In particular, the disks in $\mathcal{F}^{(y)}$, $y \in \tilde{\mathcal{U}}$, satisfy Definition 3.4(i).

Consider now the map $\exp: \tilde{B}^n \rightarrow \tilde{D}$ in (3.12). By Proposition 3.2, it is smooth and depends smoothly on y and J . Moreover, if $J = J_o$ and $y = x_o$, it is a diffeomorphism between manifolds with boundaries. Hence, there exists $\mathcal{U} \subset \tilde{\mathcal{U}}$ and $\varepsilon < \min_i \varepsilon^{(v_i)}$ so that \exp_* is invertible at all points of \tilde{B}^n whenever $y \in \mathcal{U}$ and $\|J - J_o\|_{\tilde{D}}^{(1)} < \varepsilon$. In these cases, \exp is a local homeomorphism from the compact set \tilde{B}^n to \tilde{D} and hence is a covering map of \tilde{D} . Being \tilde{B}^n simply connected, it is a diffeomorphism, i.e. also (ii) of Definition 3.4 holds true. \square

4. Conditions that force a boundary to be “good”

In this section we are going to prove a result (Theorem 4.6), which provides a condition for the existence of foliations by stationary disks of a pointed domain (D, x_o) endowed with a small deformation of the standard complex structure. An immediate consequence of this and of the contents of Section 3 is represented by the following theorem.

Theorem 4.1. *Let $D \subset M$ be a smoothly bounded, strongly pseudoconvex domain in an almost complex manifold (M, J_o) . If there is a local diffeomorphism $\varphi: \mathcal{U} \subset M \rightarrow \mathbb{C}^n$, so that $\widehat{D} = \varphi(D)$ is a strictly linearly convex domain $\widehat{D} \subset \mathbb{C}^n$ and $\varphi_* J_o$ is sufficiently close to J_{st} in \mathcal{C}^1 -norm, then D is a domain of circular type w.r.t. J and any point is a center.*

Roughly speaking, this shows that if one defines a suitable topology on the set of almost complex domains admitting foliations of circular type, such space contains a whole open neighborhood of the class of strictly linearly convex domains of \mathbb{C}^n .

4.1. The linearized operator $\mathfrak{R} = \mathfrak{R}_{(J_o, x_o, v_o; \widehat{f}_o, \lambda_o, \mu_o)}$

First of all, we want to determine an explicit expression for the tangent map $\mathfrak{R} = (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5)$ at $(\widehat{f}_o, \lambda_o, \mu_o)$ of the operator (3.4). For this, recall that, being $\widehat{f}_o: \overline{\Delta} \rightarrow T^*M$ a \mathbb{J} -holomorphic disk, one can always find a system of complex coordinates (z^i) on a neighborhood \mathcal{W} of $\widehat{f}_o(\overline{\Delta})$, in such a way that, identifying \mathcal{W} with an open subset of \mathbb{C}^{2n} , one has $\mathbb{J}|_{\mathcal{W}} = J_{st}|_{\mathcal{W}}$ at any $z \in f(\Delta)$. Moreover, by Hopf’s lemma and being the defining function ρ strongly plurisubharmonic, we have that $d\rho(f_{o*}(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y})) = d\rho(\text{Re}(z^1 \frac{\partial}{\partial z^1})) \neq 0$ at all points of $f(\partial\Delta)$. In these coordinates, the tangent map of $\mathcal{R}_1 = \widehat{\partial}_{\mathbb{J}_o}$ at \widehat{f}_o is

$$\mathfrak{R}_1(\widehat{h}) = \frac{\partial \widehat{h}}{\partial \bar{\zeta}} + \frac{1}{2i} D(\mathbb{J} - J_{st})_{\widehat{f}_o} \cdot \widehat{h}, \tag{4.1}$$

where $D(\mathbb{J} - J_{st})_{\widehat{f}_o}$ is the real differential of the matrix valued function $\zeta \mapsto (\mathbb{J} - J_{st})_{\widehat{f}_o(\zeta)}$. In matrix notation, $D(\mathbb{J} - J_{st})_{\widehat{f}_o} \cdot \widehat{h}$ can be written as

$$(D(\mathbb{J} - J_{st})_{\widehat{f}_o} \cdot \widehat{h})_{\zeta} = A(\zeta) \cdot \widehat{h}(\zeta) + B(\zeta) \cdot \widetilde{h}(\zeta),$$

for some $A, B: \overline{\Delta} \rightarrow M_{n \times n}(\mathbb{C})$ and \mathfrak{R}_1 assumes the form

$$\mathfrak{R}_1(\widehat{h}) = \frac{\partial \widehat{h}}{\partial \bar{\zeta}} + A \cdot \widehat{h} + B \cdot \widetilde{h}. \tag{4.2}$$

Consider now the tangent map \mathfrak{R}_2 . By previous remarks, the defining function $\widetilde{\rho} = (\widetilde{\rho}^1, \dots, \widetilde{\rho}^{2n+1})$ in (3.1) is locally equivalent to

$$\widehat{\varrho}(t, \alpha) = \left(\varrho(\alpha), t - \frac{\alpha(\text{Re}(z^1 \frac{\partial}{\partial z^1}))}{d\rho(\text{Re}(z^1 \frac{\partial}{\partial z^1}))|_{\pi(\alpha)}} \right)$$

where $\varrho: \mathcal{W} \subset T^*M|_{\mathcal{U}} \rightarrow \mathbb{R}^{2n}$ is the defining function for \mathcal{N}_* obtained by replacing $t = \frac{\alpha(\text{Re}(z^1 \frac{\partial}{\partial z^1}))}{d\rho(\text{Re}(z^1 \frac{\partial}{\partial z^1}))|_{\pi(\alpha)}}$ in all places of $\widetilde{\rho}$. If we set

$$\widehat{r}(\zeta, t, \alpha) \stackrel{\text{def}}{=} \left(\varrho(\zeta, \alpha), t - \frac{\alpha(\text{Re}(z^1 \frac{\partial}{\partial z^1}))}{d\rho(\text{Re}(z^1 \frac{\partial}{\partial z^1}))|_{\pi(\alpha)}} \right), \quad \text{with } \varrho(\zeta, \alpha) \stackrel{\text{def}}{=} \varrho(\zeta^{-1} \cdot \alpha) \tag{4.3}$$

we see that \mathfrak{R}_2 is equivalent to the tangent map of the operator

$$\mathcal{R}_2(\widehat{f}|_{\partial\Delta}, \lambda) = \left(\varrho(\cdot, \widehat{f}(\cdot)), \lambda - \frac{\widehat{f}(\cdot)(\operatorname{Re}(z^1 \frac{\partial}{\partial z^1}))}{d\rho(\operatorname{Re}(z^1 \frac{\partial}{\partial z^1}))|_{\pi(\widehat{f}(\cdot))}} \right)$$

and hence of the form

$$\mathfrak{R}_2(\widehat{h}, \tau) = (2 \operatorname{Re}(G \cdot \widehat{h}|_{\partial\Delta}), \tau - g(\widehat{h})) \tag{4.4}$$

where g is obtained by linearization of the map $\widehat{f} \mapsto \frac{\widehat{f}(\cdot)(\operatorname{Re}(z^1 \frac{\partial}{\partial z^1}))}{d\rho(\operatorname{Re}(z^1 \frac{\partial}{\partial z^1}))|_{\pi(\widehat{f}(\cdot))}}$ and G is the matrix valued map on $\partial\Delta$ defined by

$$G(\zeta) = \begin{pmatrix} \frac{\partial \varrho^1}{\partial z^1}(\zeta, \widehat{f}(\zeta)) & \cdots & \frac{\partial \varrho^1}{\partial z^{2n}}(\zeta, \widehat{f}(\zeta)) \\ \vdots & \ddots & \vdots \\ \frac{\partial \varrho^{2n}}{\partial z^1}(\zeta, \widehat{f}(\zeta)) & \cdots & \frac{\partial \varrho^{2n}}{\partial z^{2n}}(\zeta, \widehat{f}(\zeta)) \end{pmatrix}, \quad \zeta \in \partial\Delta. \tag{4.5}$$

By Theorem 2.5, \mathcal{N}_* is totally real w.r.t. \mathbb{J} and hence, by our choice of the coordinates, it is totally real also w.r.t. J_{st} on a neighborhood of $\widehat{f}(\partial\Delta)$. This implies that

$$\det(G(\zeta)) \neq 0, \quad \text{for any } \zeta \in \partial\Delta. \tag{4.6}$$

Finally, the maps $\mathfrak{R}_3, \mathfrak{R}_4$ and \mathfrak{R}_5 are easily seen to be (here $h \stackrel{\text{def}}{=} \pi \circ \widehat{h}$)

$$\mathfrak{R}_3(\widehat{h}) = h(0), \quad \mathfrak{R}_4(\widehat{h}, \sigma) = \frac{\partial h}{\partial x} \Big|_{\zeta=0} - \sigma v_o, \quad \mathfrak{R}_5(\widehat{h}) = \widehat{f}_o \left(\frac{\partial h}{\partial x} \Big|_1 \right) + \widehat{h} \left(\frac{\partial f_o}{\partial x} \Big|_1 \right).$$

4.2. The operator $R_{A,B,G} = (\mathfrak{R}_1, \mathfrak{R}_2)$

Consider the operator

$$R_{A,B,G} = (\mathfrak{R}_1, \mathfrak{R}_2) = \left(\frac{\partial \widehat{h}}{\partial \zeta} + A \cdot \widehat{h} + B \cdot \bar{\widehat{h}}, 2 \operatorname{Re}(G \cdot \widehat{h}) \right),$$

which is a well-known Fredholm operator related with the generalized Riemann–Hilbert problems. In the next theorem, we recall some information that will be used in the sequel (see e.g. Thms. 3.2.5, 3.3.1 in [27]).

Theorem 4.2. *If G satisfies (4.6), the operator $R_{A,B,G}$ is Fredholm with index $\nu = 2n - \frac{1}{i\pi} \int_{\partial\Delta} d \arg(\det(G))$ and hence it is surjective if and only if*

$$\dim \ker R_{A,B,G} = 2n - \frac{1}{i\pi} \int_{\partial\Delta} d \arg(\det(G)). \tag{4.7}$$

Next, we need to recall a lemma due to Globevnik and some of its direct consequences, which give a way to establish the surjectivity of $R_{A,B,G}$ in case of integrable complex structures. But in order to state them, we first need to recall the definition of “canonical system” (see e.g. [8]). In what follows, for any holomorphic function $g: \mathcal{U} \subset \mathbb{C} \rightarrow \mathbb{C}^N$ on a neighborhood of ∞ and with at most one pole at ∞ , we call *order of (zero of) g* the integer k such that $g = \frac{1}{z^k} g_0$ for some g_0 which is holomorphic at ∞ and with $g_0(\infty) \neq 0$.

Definition 4.3. Given $A \in C^\epsilon(\partial\Delta, GL(N, \mathbb{C}))$, with $\epsilon \in]0, 1[$, consider the problem consisting of finding a continuous map $\Psi^+ : \bar{\Delta} \rightarrow \mathbb{C}^N$, holomorphic on Δ , and a continuous map $\Psi^- : \mathbb{C} \setminus \Delta \rightarrow \mathbb{C}^N$, holomorphic on $\mathbb{C} \setminus \bar{\Delta}$ and with at most a pole at ∞ , so that

$$\Psi^+(\zeta) = A(\zeta) \cdot \Psi^-(\zeta), \quad \zeta \in \partial\Delta. \tag{4.8}$$

A *canonical system* of A is any collection of solutions $\Phi_j = (\Phi_j^+, \Phi_j^-)$, $1 \leq j \leq N$, of the problem (4.8) so that

- (i) $\Phi^+(\zeta) = [\Phi_1^+(\zeta), \dots, \Phi_N^+(\zeta)]$ is in $GL(N, \mathbb{C})$ for any $\zeta \in \bar{\Delta}$;
- (ii) $\Phi^-(\zeta) = [\Phi_1^-(\zeta), \dots, \Phi_N^-(\zeta)]$ is in $GL(N, \mathbb{C})$ for any $\zeta \in \mathbb{C} \setminus \Delta$;
- (iii) the order k of $\det \Phi^-$ at ∞ is equal to the sum of the orders k_j of the columns Φ_j^- .

If $\{\Phi_j = (\Phi_j^+, \Phi_j^-)\}$ is a canonical system of A , the orders k_j of the Φ_j^- 's are called *partial indices* of A . The sum $k = \sum k_j$ is called *total index* of A .

An important fact is that, up to reordering, *the partial indices and the total index depend only on A and not on the considered canonical system*. We may now recall the following lemma by Globevnik, which can be considered as a corollary of N.P. Vekua's factorization theorem [25].

Lemma 4.4. (See [8, Lemma 5.1].) Let $L \in C^\epsilon(\partial\Delta, GL(N, \mathbb{C}))$, with $\epsilon \in]0, 1[$. Then there is a map $\Theta : \bar{\Delta} \rightarrow GL(N, \mathbb{C})$ in $H^\epsilon(\bar{\Delta}, \mathbb{C}^{N^2})$, such that

$$L(\zeta) \cdot \overline{L(\zeta)^{-1}} = \Theta(\zeta) \cdot \Lambda(\zeta) \cdot \overline{\Theta}^{-1}(\zeta) \quad \text{with } \Lambda(\zeta) = \begin{pmatrix} \zeta^{k_1} & 0 & \dots & 0 \\ 0 & \zeta^{k_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \zeta^{k_N} \end{pmatrix} \tag{4.9}$$

for any $\zeta \in \partial\Delta$, where k_1, \dots, k_N are the partial indices of $A(\cdot) \stackrel{\text{def}}{=} L(\cdot) \cdot \overline{L(\cdot)^{-1}}|_{\partial\Delta}$.

The integers k_i of the previous lemma and the sum $k = \sum_{i=1}^N k_i$ are the same for all maps $L' = M|_{\partial\Delta} \cdot L$ with $M : \bar{\Delta} \rightarrow GL(N, \mathbb{C})$ in $H^\epsilon(\bar{\Delta}, \mathbb{C}^{N^2})$. They are called *partial indices* and *total index* of L , respectively.

Consider now the map $G(\zeta)$ in (4.5) and let Θ^G be a map that gives a decomposition (4.9) for $L(\zeta) = G^{-1}(\zeta)$. We set

$$A^G \stackrel{\text{def}}{=} (\Theta^G)^{-1} \cdot A \cdot \Theta^G, \quad B^G \stackrel{\text{def}}{=} (\Theta^G)^{-1} \cdot B \cdot \overline{\Theta^G}. \tag{4.10}$$

It is immediate to realize that the linear map $\widehat{h} \mapsto \widetilde{h} = (\Theta^G)^{-1} \cdot \widehat{h}$ is an isomorphism between $\ker R_{A,B,G}$ the space of solutions of the problem

$$\begin{cases} \bar{\partial}\widetilde{h} + A^G \cdot \widetilde{h} + B^G \cdot \overline{\widetilde{h}} = 0, & \zeta \in \Delta, \\ \widetilde{h}^i(\zeta) = \zeta^{k_i} \overline{\widetilde{h}^i(\zeta)}, & 1 \leq i \leq 2n, \quad \zeta \in \partial\Delta \end{cases} \tag{4.11}$$

where the k_i are the partial indices of $L = G^{-1}$.

Lemma 4.5. The operator $R_{A,B,G}$ is surjective if and only if $\dim \ker R_{A,B,G} = 2n + k$, with $k = \sum_{i=1}^{2n} k_i$. Moreover, when $A = B = 0$, $R_{0,0,G}$ is surjective if and only if $k_i \geq -1$ for any $1 \leq i \leq 2n$.

Proof. The first claim follows from Theorem 4.2 and from

$$\begin{aligned} \dim \ker R_{A,B,G} &= 2n - \frac{1}{i\pi} \int_{\partial\Delta} d \arg(\det(G)) \\ &= 2n + \frac{1}{2\pi i} \int_{\partial\Delta} d \arg(\det(G^{-1} \cdot \bar{G})) \\ &= 2n + \frac{1}{\pi i} \int_{\partial\Delta} d \arg(\det(\Theta^G)) + \sum_{i=1}^{2n} \frac{1}{2\pi i} \int_{\partial\Delta} d \arg(\zeta^{k_i}) \\ &= 2n + k \end{aligned}$$

where we used the fact that $\det(\Theta^G)$ is holomorphic and never zero in Δ .

Assume now that $A = B = 0$ and recall that the elements of $\ker R_{0,0,G}$ are in natural correspondence with the elements $\tilde{h} = (\tilde{h}^1, \dots, \tilde{h}^{2n}) \in H^e(\Delta, \mathbb{C}^{2n})$ that solve (4.11) and hence of the form $\tilde{h}^i(\zeta) = \sum_{\ell \geq 0} a_\ell^i \zeta^\ell$ with coefficients $a_\ell^i \in \mathbb{C}$ so that the boundary conditions are satisfied, i.e.

$$\begin{cases} a_\ell^i = 0 & \text{when } \ell \geq \max\{k_i + 1, 0\}, \\ a_\ell^i = \overline{a_{-\ell+k_i}^i} & \text{when } k_i \geq 0 \text{ and } 0 \leq \ell \leq k_i. \end{cases} \tag{4.12}$$

From this, a simple check shows that $\dim \ker R_{0,0,G} = \sum_{k_i \geq 0} (k_i + 1)$. Since

$$\begin{aligned} 2n + k &= 2n - \sum_{k_i \leq -1} (|k_i| - 1) - (\#\{k_i \leq -1\}) + \sum_{k_i \geq 0} k_i \\ &= (\#\{k_i \geq 0\}) + \sum_{k_i \geq 0} k_i - \sum_{k_i \leq -1} (|k_i| - 1) = \sum_{k_i \geq 0} (k_i + 1) - \sum_{k_i \leq -1} (|k_i| - 1) \end{aligned}$$

it follows that $\dim \ker R_{0,0,G} = 2n + k$ if and only if $\sum_{k_i \leq -1} (|k_i| - 1) = 0$, i.e. $k_i \geq -1$ for any $1 \leq i \leq 2n$. \square

4.3. The operator $\mathfrak{R} = (R_{0,0,G}, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5)$ for convex domains in \mathbb{C}^n

Theorem 4.6. *Let D be a domain in (\mathbb{C}^n, J_{st}) , with smooth boundary and let $f_o : \bar{\Delta} \rightarrow \bar{D}$ a stationary disk D . If there is a neighborhood \mathcal{U} of $f_o(\bar{D})$ where $\mathcal{U} \cap \bar{D}$ is strictly linearly convex, then ∂D is good for (J_{st}, f_o) .*

Proof. We first need the following:

Lemma 4.7. *(See [17, Prop. 2.36, Thm. 2.45].) Let $f_o : \bar{\Delta} \rightarrow \bar{D}$ as above. Then there exists a system of complex coordinates (z^i) and a defining function ρ for ∂D on a neighborhood \mathcal{V} of $f(\bar{\Delta})$, such that $f_o(\zeta) = (\zeta, 0, \dots, 0)$ and*

$$\rho = -1 + |z^1|^2 + \sum_{\alpha, \beta=2}^n \delta_{\alpha\beta} z^\alpha \bar{z}^\beta + \operatorname{Re} \left(\sum_{\alpha, \beta=1}^n B_{\alpha\beta} z^\alpha z^\beta \right) + r(z^1, \dots, z^n) \tag{4.13}$$

with r smooth function so that $|r(z)| \leq c|z|^3$ for some $c > 0$ for all $z \in \mathcal{V}$.

Secondly, we need the following lemma, from which the theorem will follow almost immediately. There, we denote by (z^i) the coordinates in previous lemma and by (z^i, w_i) the associated complex coordinates for $T^*\mathbb{C}^n$ (see Section 2.1).

Lemma 4.8. *Let $\mathfrak{R} = (R_{0,0,G}, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5)$ be the linear operator defined in Section 4.1 using the coordinates (z^i, w_j) . Then:*

- (i) *The partial indices of G^{-1} are $k_1 = 2, k_2 = 0$ and $k_j = 1$ for all $j \geq 2$. In particular, $R_{0,0,G}$ is surjective and $\dim \ker R_{0,0,G} = 4n + 1$.*
- (ii) *The restrictions of $\mathfrak{R}_3, \mathfrak{R}_4$ on $\ker R_{0,0,G}$ are surjective.*

Proof. (i) If ρ is the defining function (4.13), the components of the function $\varrho(\zeta, \alpha)$, defined in (4.3), are (up to multiplication by a nowhere vanishing smooth function)

$$\begin{aligned} \varrho^1 &= -1 + |z^1|^2 + \sum_{\alpha, \beta=2}^n \delta_{\alpha\beta} z^\alpha \bar{z}^\beta + \operatorname{Re} \left(\sum_{\alpha, \beta=1}^n B_{\alpha\beta} z^\alpha \bar{z}^\beta \right) + O(|z|^3), \\ \varrho^2 &= i \{ 2|z^1|^2 (\zeta^{-1} w_1 - \bar{\zeta}^{-1} \bar{w}_1) - (z^1 \zeta^{-1} w_1 + \bar{z}^1 \bar{\zeta}^{-1} \bar{w}_1) (z^1 - \bar{z}^1) \} + O(|z|^2), \\ \varrho^{2\alpha-1} &= 2|z^1|^2 (\zeta^{-1} w_\alpha + \bar{\zeta}^{-1} \bar{w}_\alpha) \\ &\quad - (z^1 \zeta^{-1} w_1 + \bar{z}^1 \bar{\zeta}^{-1} \bar{w}_1) \{ (\delta_{\alpha\beta} \bar{z}^\beta + B_{\alpha\beta} z^b) + (\delta_{\alpha\beta} z^\beta + \bar{B}_{\alpha\beta} \bar{z}^\beta) \} + O(|z|^2), \\ \varrho^{2\alpha} &= i \{ 2|z^1|^2 (\zeta^{-1} w_\alpha - \bar{\zeta}^{-1} \bar{w}_\alpha) - \\ &\quad - (z^1 \zeta^{-1} w_1 + \bar{z}^1 \bar{\zeta}^{-1} \bar{w}_1) \{ (\delta_{\alpha\beta} \bar{z}^\beta + B_{\alpha\beta} z^b) - (\delta_{\alpha\beta} z^\beta + \bar{B}_{\alpha\beta} \bar{z}^\beta) \} \} + O(|z|^2) \end{aligned}$$

with $2 \leq \alpha \leq n$. Hence, the matrix (4.5) is (up to reordering of columns)

$$\begin{aligned} G(\zeta) &= \begin{pmatrix} G_1(\zeta) & 0 \\ 0 & G_2(\zeta) \end{pmatrix} \quad \text{with} \\ G_1(\zeta) &= \left(\begin{array}{cc} \frac{\partial \varrho^1}{\partial z^1} & \frac{\partial \varrho^1}{\partial w_1} \\ \frac{\partial \varrho^2}{\partial z^1} & \frac{\partial \varrho^2}{\partial w_1} \end{array} \right) \Big|_{\widehat{f}_o(\partial\Delta)} = \begin{pmatrix} \zeta^{-1} & 0 \\ i(\zeta^2 + 1)\zeta^{-2} & i(\zeta^2 + 1)\zeta^{-1} \end{pmatrix}, \\ G_2(\zeta) &= \left(\begin{array}{cc} \frac{\partial \varrho^{2\alpha-1}}{\partial z^\beta} & \frac{\partial \varrho^{2\alpha-1}}{\partial w_\gamma} \\ \frac{\partial \varrho^{2\alpha}}{\partial z^\beta} & \frac{\partial \varrho^{2\alpha}}{\partial w_\gamma} \end{array} \right) \Big|_{\widehat{f}_o(\partial\Delta)} = \begin{pmatrix} -2B_{\alpha\beta} + \delta_{\alpha\beta} & 2\zeta^{-1} \\ -2iB_{\alpha\beta} z^b + i\delta_{\alpha\beta} z^\beta & i2\zeta^{-1} \end{pmatrix}. \end{aligned}$$

Under the assumption that the real Hessian $H(\rho)_{ij}$ is positive definite at all points of $f_o(\partial\Delta)$, the partial indices of the matrix $G_2^{-1}(\zeta)$ are known to be all equal to 1. A complete proof of this can be found in [22], Lemma 3.2, being $G_2(\zeta)$ equal to the lower right block of the matrix in (3.10) of [22].

For what concerns the block $G_1^{-1}(\zeta)$, notice that for any $\zeta \in \partial\Delta$ one has that $A(\zeta) = G_1^{-1}(\zeta) \cdot \overline{G_1(\zeta)} = \begin{pmatrix} \zeta^2 & 0 \\ -2\zeta & -1 \end{pmatrix}$ and hence A admits the columns of

$$\Phi^+(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi^-(\zeta) = \begin{pmatrix} \frac{1}{\zeta^2} & 0 \\ \frac{2}{\zeta} & -1 \end{pmatrix}$$

as canonical system. Hence, by Lemma 4.5, we conclude that $k_1 = 2$ and $k_2 = 0$, since these are the orders of the columns of Φ^- .

(ii) Let $I_{x_o} \subset T_{x_o} D$ be the indicatrix of the Kobayashi metric of D at x_o . By Thm. 4.8 in [17], there exist a neighborhood $\mathcal{W} \subset D$ of x_o and a neighborhood $\mathcal{W}' \subset \partial I_{x_o}$ of v_o , so that for any $x \in \mathcal{W}$, $v \in \mathcal{W}'$ there exist exactly two stationary disks $f^{(x,v_o)}, f^{(x_o,v)} : \bar{\Delta} \rightarrow \bar{D}$ satisfying

$$f^{(x,v_o)}(0) = x, \quad f_*^{(x,v_o)}\left(\frac{\partial}{\partial x}\Big|_0\right) = v_o, \quad f^{(x_o,v)}(0) = x_o, \quad f_*^{(x_o,v)}\left(\frac{\partial}{\partial x}\Big|_0\right) = v.$$

For each of them, there is a unique stationary lift $\widehat{f}^{(x,v_o)}$ satisfying certain normalizing conditions (i.e. so that $\zeta \cdot \widehat{f}^{(x,v_o)}(\zeta)$ is the so-called *dual map*—see [17, Def. 2.10]). These lifts depend smoothly on the coordinates of the point x and the vector v and for any curves $\gamma_t \in D$ and $\gamma'_t \in T_{x_o} D$ with $\gamma_0 = x_o$ and $\gamma'_0 = v_o$, the 1-parameter families of stationary lifts $\widehat{f}_t \stackrel{\text{def}}{=} \widehat{f}^{(\gamma_t, v_o)}$ and $\widehat{f}'_t \stackrel{\text{def}}{=} \widehat{f}^{(x_o, \gamma'_t)}$ are so that $\widehat{h}(\zeta) = \frac{d\widehat{f}_t(\zeta)}{dt}\Big|_{t=0}$ and $\widehat{h}'(\zeta) = \frac{d\widehat{f}'_t(\zeta)}{dt}\Big|_{t=0}$ are in $\ker R_{0,0,G}$. Moreover, by construction,

$$\mathfrak{R}_3(\widehat{h}) = \pi(\widehat{h}(0)) = \dot{\gamma}_0 \in \mathbb{C}^n, \quad \mathfrak{R}_4(\widehat{h}, \sigma) = \frac{\partial(\pi \circ \widehat{h})}{\partial x}\Big|_0 - \sigma v_o = \dot{\gamma}'_0 - \sigma v_o \in \mathbb{C}^n.$$

Since v_o is transversal to ∂I_{x_o} , by the arbitrariness of γ_t and $\gamma'_t \in \partial I_{x_o}$ it follows that $\mathfrak{R}_3|_{\ker R_{0,0,G}}$ and $\mathfrak{R}_4|_{\ker R_{0,0,G}}$ are both surjective. \square

By the previous lemma, $\dim \ker R_{0,0,G} \cap \ker \mathfrak{R}_3 \cap \ker \mathfrak{R}_4 = 1$. So, in order to conclude, we only need to check that $\mathfrak{R}_5|_{\ker R_{0,0,G} \cap \ker \mathfrak{R}_3 \cap \ker \mathfrak{R}_4}$ is surjective onto \mathbb{R} or, equivalently, that there is $0 \neq \widehat{h} \in \ker R_{0,0,G} \cap \ker \mathfrak{R}_3 \cap \ker \mathfrak{R}_4$ so that $\mathfrak{R}_5(\widehat{h}) = \widehat{h}_1(1, 0, \dots, 0) \neq 0$. But an element of this kind is given by $\widehat{h}(\zeta) = \frac{d(\varphi_t(\widehat{f}_o(\zeta)))}{dt}\Big|_0 = (\zeta, 1, 0, \dots, 0)$, where we denote by φ_t the diffeomorphism considered in Lemma 2.2, and the proof is concluded. \square

Remark 4.9. Lemma 4.8(i) corrects and generalizes a computation in [4], where, by a minor mistake, the partial indices of G^{-1} in case $D = B^n$ are claimed to be all equal to 1.

5. Other non-singular foliations by stationary disks

5.1. Foliations of horospherical type

As before, (M, J) is an almost complex manifold of dimension $2n$. Let $x_o \in M$ and consider a Riemannian metric $\langle \cdot, \cdot \rangle$ on a neighborhood \mathcal{U} so that $\langle \cdot, \cdot \rangle|_{x_o}$ is J -Hermitian. For instance, if \mathcal{U} is identified with an open subset of \mathbb{C}^n so that $J|_{x_o} = J_{\text{st}}|_{x_o}$, we may assume that $\langle \cdot, \cdot \rangle$ is the standard Hermitian metric of \mathbb{C}^n . Denote also by ∇ the Levi-Civita connection of $\langle \cdot, \cdot \rangle$.

Definition 5.1. Let $f : \bar{\Delta} \rightarrow M$ be a J -holomorphic disk, which is \mathcal{C}^1 up to the boundary and with $v_o = f_*\left(\frac{\partial}{\partial x}\Big|_1\right) \neq 0$. We call *parameter of tangency at $x_o = f(1)$* the real number

$$p(f; x_o) \stackrel{\text{def}}{=} \left\langle \nabla_{v_o} \left(f_* \left(\frac{\partial}{\partial x} \right) \right), J v_o \right\rangle. \tag{5.1}$$

This number depends on the first order jet of $\langle \cdot, \cdot \rangle$ at x_o , but if two J -holomorphic disks f, h are so that

$$x_o = f(1) = h(1), \quad v_o = f_* \left(\frac{\partial}{\partial x} \Big|_1 \right) = h_* \left(\frac{\partial}{\partial x} \Big|_1 \right),$$

then their parameters of tangency are the same for a choice of $\langle \cdot, \cdot \rangle$ if and only if they are the same for any other choice of the metric. In fact, if we consider a new metric $\langle \cdot, \cdot \rangle'$ with Levi-Civita connection ∇' , then $S = \nabla' - \nabla$ is a tensor field of type $(1, 2)$ so that

$$\left(\nabla_{v_o} \left(f_* \left(\frac{\partial}{\partial x} \right) \right) - \nabla_{v_o} \left(h_* \left(\frac{\partial}{\partial x} \right) \right) \right) \Big|_1 = S(v_o, v_o - v_o) = 0.$$

Moreover, a simple computation shows that any disk $h = f \circ \varphi$ where $\varphi \in \text{Aut}(\Delta)$ with $\varphi(1) = 1$, $\varphi'(1) = 1$, satisfies

$$\nabla_{v_o} \left(h_* \left(\frac{\partial}{\partial x} \right) \right) = \lambda J f_* \left(\frac{\partial}{\partial x} \Big|_1 \right) = \lambda J v_o \quad \text{for some } \lambda \in \mathbb{R}.$$

Therefore, for any given $\tilde{\lambda}$, one can choose φ so that $p(f \circ \varphi; x_o) = \tilde{\lambda}$. Moreover, $p(f \circ \varphi; x_o) = p(f; x_o)$ if and only if $\varphi = \text{Id}_\Delta$ and $f = h$.

Consider now a bounded, strictly convex domain in $(\mathbb{C}^n, J_{\text{st}})$ with smooth boundary and let $x_o \in \partial D$ and ν the outward unit normal to ∂D in x_o . By [6, Thm. 2], for any $v_o \in T_{x_o}M$ so that $\langle \nu, v_o \rangle > 0$ and for any $\lambda \in \mathbb{R}$, there exists a unique stationary disk $f^{(v_o, \lambda)} : \bar{\Delta} \rightarrow \bar{D}$ so that

$$f^{(v_o, \lambda)}(1) = x_o, \quad f_* \left(\frac{\partial}{\partial x} \Big|_1 \right) = v_o, \quad p(f; x_o) = \lambda. \tag{5.2}$$

If we denote by $H_{x_o} = \{v \in T_{x_o}\mathbb{C}^n : \langle \nu, v \rangle > 0\} \subset T_{x_o}\mathbb{C}^n$, we have that the exponential map $\Phi^{(D, x_o)} : H_{x_o} \times (\Delta \setminus \{1\}) \rightarrow \bar{D} \setminus \{x_o\}$ defined by $\Phi^{(D, x_o)}(v; \zeta) = f^{(v, 0)}(\zeta)$ is a diffeomorphism.

We now consider the following definition. As before, D is a smoothly bounded, strictly pseudoconvex domain in the almost complex manifold (M, J) and, for any given $x_o \in \partial D$, we denote by ν the outward unit normal to ∂D in x_o w.r.t. some Riemannian metric $\langle \cdot, \cdot \rangle$, which is J -Hermitian at x_o . Finally, for any real number $a > 0$, we denote by $\mathcal{C}^{(a)}$ the open cone

$$\mathcal{C}^{(a)} = \{v \in T_{x_o}M : \langle \nu, v \rangle > a\} \subset T_{x_o}M.$$

Definition 5.2. For any $x_o \in \partial D$ and $a > 0$, let $\mathcal{G}^{(x_o)}$ be the family of stationary disks $f : \bar{\Delta} \rightarrow \bar{D}$ with $f(1) = x_o$ and by $\mathcal{G}^{(x_o, a)} \subset \mathcal{G}^{(x_o)}$ the subfamily of disks with $f_* \left(\frac{\partial}{\partial x} \Big|_1 \right) \in \mathcal{C}^{(a)}$. Denote also by $D^{(x_o, a)} \subset D$ the union of all images of the disks in $\mathcal{G}^{(x_o, a)}$.

We say that $\mathcal{G}^{(x_o)}$ is a foliation of horospherical type for D (resp. $\mathcal{G}^{(x_o, a)}$ is a good foliation for $D^{(x_o, a)}$) if the following conditions are satisfied:

- (i) for any $v \in T_{x_o}M$ so that $\langle \nu, v \rangle > 0$ (resp. for any $v \in \mathcal{C}^{(a)}$) and for any $\lambda \in \mathbb{R}$ there exists a unique $f^{(v, \lambda)} \in \mathcal{G}^{(x_o)}$ so that

$$f_*^{(v, \lambda)} \left(\frac{\partial}{\partial x} \Big|_1 \right) = v, \quad p(f; x_o) = \lambda; \tag{5.3}$$

- (ii) the map $\exp : \bar{B}^n \setminus \{y_o\} \rightarrow \bar{D} \setminus \{x_o\}$, $y_o \stackrel{\text{def}}{=} (1, 0, \dots, 0)$, defined by

$$\exp(\Phi^{(B^n, y_o)}(v, \zeta)) \stackrel{\text{def}}{=} f^{(v, 0)}(\zeta) \tag{5.4}$$

is a diffeomorphism on B^n (resp. on $B^n(y_o, a)$), extends smoothly at all points of the closure, different from y_o , and induces a homeomorphism between the closures of the two domains.

If $\mathcal{G}^{(x_o, a)}$ with $a > 0$ is a good foliation for $D^{(x_o, a)}$, we say that $D^{(x_o, a)} \subset D$ is a good conical subdomain with vertex in x_o . If $\mathcal{G}^{(x_o)}$ is a foliation of horospherical type, we say that x_o is a center at infinity for D and D is of horospherical type.

By the results in [6], any strictly convex domain D in $(\mathbb{C}^n, J_{\text{st}})$ is a domain of horospherical type with center at infinity at any point of the boundary.

5.1.1. Stability of foliations of horospherical type

In analogy with Section 3.2, let us consider the nonlinear operator $\mathcal{R}'_{(J_o, x_o, v_o, v)} = (\mathcal{R}'_1, \dots, \mathcal{R}'_6)$ from $\mathcal{C}^{\alpha, \varepsilon}(\bar{\Delta}; \mathbb{C}^{2n}) \times \mathcal{C}^\varepsilon(\partial\Delta; \mathbb{R})$ into $\mathcal{C}^{\alpha-1, \varepsilon}(\bar{\Delta}; \mathbb{C}^{2n}) \times \mathcal{C}^\varepsilon(\partial\Delta; \mathbb{R}^{2n+1}) \times \partial D \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$, where

$$\begin{aligned} \mathcal{R}'_1(\widehat{f}, \lambda) &= \bar{\partial}_{\mathbb{J}_o} \widehat{f}, & \mathcal{R}'_2(\widehat{f}, \lambda) &= r(\zeta, \lambda(\zeta), \widehat{f}(\zeta)), \\ \mathcal{R}'_3(\widehat{f}, \lambda) &= \pi(\widehat{f})|_{\zeta=1} - x_o, & \mathcal{R}'_4(\widehat{f}, \lambda) &= \pi(\widehat{f})_* \left(\frac{\partial}{\partial x} \Big|_{\zeta=1} \right) - v_o, \\ \mathcal{R}'_5(\widehat{f}, \lambda) &= p(\pi(\widehat{f}); x_o) - v, \\ \mathcal{R}'_6(\widehat{f}, \lambda) &= \widehat{f} \left(\pi(\widehat{f})_* \left(\frac{\partial}{\partial x} \Big|_1 \right) \right) - 1. \end{aligned}$$

Given a stationary disk $f_o: \bar{\Delta} \rightarrow \bar{D}$ with $x_o = f(1)$, $v_o = f_{o*}(\frac{\partial}{\partial x}|_{\zeta=0})$ and $p(f_o; x_o) = v_o$, we say that ∂D is a horospherically good boundary for (J_o, f_o) if f_o admits a lift \widehat{f}_o so that $(\widehat{f}_o, \lambda_o)$ is a solution of $\mathcal{R}'_{(J_o, x_o, v_o, v_o)}(\widehat{f}, \lambda) = 0$ and the tangent operator \mathfrak{R}' of $\mathcal{R}'_{(J_o, x_o, v_o, v_o)}$ at (\widehat{f}_o, λ) is invertible.

Again, by the Implicit Function Theorem, if ∂D is a horospherically good boundary for (J_o, f_o) , there is a neighborhood $\mathcal{V} \subset \partial D$ of x_o , a neighborhood $\mathcal{W} \subset TD$ of v_o , with $\pi(\mathcal{W}) = \mathcal{V}$ and a real number $\varepsilon > 0$ so that, for any $x \in \mathcal{V}$, $v \in \mathcal{W}$, $|v - v_o| < \varepsilon$ and $\|J - J_o\|_D^{(1)} < \varepsilon$, there exists a unique disk f in D with

$$f(1) = x, \quad f_* \left(\frac{\partial}{\partial x} \Big|_{\zeta=1} \right) = v \quad \text{and} \quad p(f; x) = v, \tag{5.5}$$

which is stationary for D w.r.t. the almost complex structure J . The dependence of f on x , v , v and J is differentiable and, given $m_o > 0$ and a metric g , one can choose ε , \mathcal{W} and $\mathcal{V} = \pi(\mathcal{W})$, so that $\sup_{\zeta \in \bar{\Delta}} \text{dist}_g(f(\zeta), f_o(\zeta)) < m_o$. So, in analogy with Proposition 3.2, we have:

Proposition 5.3. *Let $D^{(x_o, a)} \subset D$, $a > 0$, be a good conical subdomain w.r.t. to J_o with vertex in $x_o \in \partial D$. If ∂D is a good boundary for (J_o, f_o) for any stationary disk $f_o \in \mathcal{G}^{(x_o, a)}$, there exists $\varepsilon > 0$ and an open neighborhood $\mathcal{U} \subset \partial D$ of x_o so that for any J with $\|J - J_o\|_D^{(1)} < \varepsilon$ and any $x \in \mathcal{U}$ and $|a' - a| < \varepsilon$, the point x is vertex for a good foliation for $D^{(x, a')}$ relatively to the almost complex structure J .*

Proof. The proof can be obtained following the same steps of the proof of Prop. 6 in [4] and we give here only a sketch of it. First of all, using the Implicit Function Theorem and the compactness of $\bar{\mathcal{C}}^{(a)} \cap S^{2n-1} \subset T_{x_o}M$, one can determine \mathcal{U} and ε so that $\mathcal{G}^{(x, a')}$ satisfies (i) for Definition 5.2 for any almost complex structure such that $\|J - J_o\|_D^{(1)} < \varepsilon$ and for any $x \in \mathcal{U}$, $|a' - a| < \varepsilon$. Using the Implicit Function Theorem once again, one can also assume that for all these J , x and a , the map “exp”, defined in (ii) of that definition, is a local diffeomorphism at all points. It remains to check that \mathcal{U} and ε can be chosen so that “exp” is also injective. From this and a possible further restriction of \mathcal{U} and ε , we obtain that “exp” is a diffeomorphism and satisfies all other requirements of (ii). To prove injectivity, one may argue by contradiction as

in Step 2 of the proof of Prop. 6 in [4]. In fact, if one assumes that “exp” is never injective for any choice of \mathcal{U} and ε , one can construct sequences of complex structures J_j , of vertices x_j and of pairs $y_j \neq y'_j \in B^n$, so that $J_j \rightarrow J_o$, $x_j \rightarrow x_o$ and corresponding exponential maps $\exp^{(j)}$ are so that $\exp^{(j)}(y_j) = \exp^{(j)}(y'_j)$ for all j . Using compactness and Implicit Function Theorem, one can select a subsequence $z_{j_m} = \exp^{(j)}(y_{j_m})$, with $y_{j_m} \rightarrow y_o$, $y'_{j_m} \rightarrow y'_o$ with $y_o \neq y'_o$ and $z_{j_m} \rightarrow z_o = \exp(y_o) = \exp(y'_o) \in \overline{D^{(x_o, a)}}$, contradicting the hypothesis of bijectivity of “exp” on $\overline{D^{(x_o, a)}}$. \square

We remark that the tangent operator $\mathfrak{R}' = (\mathfrak{R}'_1, \dots, \mathfrak{R}'_6)$ of $\mathcal{R}'_{(J_o, x_o, v_o, v_o)}$ at (\widehat{f}_o, λ) is so that $(\mathfrak{R}'_1, \mathfrak{R}'_2) = R_{A, B, G}$ (see Section 4.2 for definition) and hence it coincides with operator $R_{0, 0, G}$ when $D \subset \mathbb{C}^n$. If D is a strictly linearly convex domain in \mathbb{C}^n , by Lemma 4.8(i), the dimension of $\ker(\mathfrak{R}'_1, \mathfrak{R}'_2) = 4n + 1$. From this, the results in [6] and a line of argument which is essentially the same of the proofs of Lemma 4.8(i) and Theorem 4.6, one gets that $\mathfrak{R}' = (\mathfrak{R}'_1, \dots, \mathfrak{R}'_6)$ is invertible also in this case. By Proposition 5.3, the following result is obtained.

Theorem 5.4. *Let $D \subset M$ be a smoothly bounded, strongly pseudoconvex domain in an almost complex manifold (M, J_o) and $a > 0$ be any fixed positive real number. If there is a local diffeomorphism $\varphi : \mathcal{U} \subset M \rightarrow \mathbb{C}^n$, so that $\widehat{D} = \varphi(D)$ is a strictly linearly convex domain $\widehat{D} \subset \mathbb{C}^n$ and $\varphi_*(J_o)$ is sufficiently close to J_{st} in a \mathcal{C}^1 -norm, then, for any $x_o \in \partial D$, the subset $D^{(x_o, a)} \subset D$ is a good conical subdomain.*

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