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A Characterization of Complex Manifolds Biholomorphic to a Circular Domain

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Dedicated to Wilhelm Stoll for his sixtieth birthday

Introduction

W. Stoll [11], developing earlier ideas of Griffiths and King [5], introduced in Value Distribution Theory the concept of parabolic exhaustion. This notion has recently found important applications in the classification of domains in $\mathbb{C}^m$ and in general of complex manifolds. A *strictly parabolic exhaustion* on a non compact complex manifold M of dimension m is a C^∞ exhaustion function $\tau: M \to [0, \Delta^2), 0 < \Delta = \sup \sqrt{\tau} \leqq \infty$, such that $d d^c \tau > 0$ on M and $d d^c \log \tau \geqq 0$, $(d d^c \log \tau)^m \equiv 0$ on $M - \tau^{-1}(0)$. Using the information that τ yields and most noteworthy the existence of a Monge-Ampère foliation associated to τ (cfr. [1]), Stoll in [12] shows that there exists a biholomorphic map $h: \mathbb{B}(\Delta) \to M$, where $\mathbb{B}(\Delta) = \{Z \in \mathbb{C}^m \mid \|Z\| < \Delta\}$, such that $\tau \circ h(Z) = \|Z\|^2$ (cfr. also [4] and [14] for other proofs). The properties of the exhaustion τ on $\tau^{-1}(0)$ play a decisive role in the classification. In fact, if we relax the differentiability assumptions on $\tau^{-1}(0)$, even assuming that this set reduces to a single point (which is always the case if τ is strictly parabolic), the range of possibilities for the underlying manifold is much wider. If $G \Subset \mathbb{C}^m$ is a smooth, strictly pseudoconvex, complete circular domain and σ is its Minkowski functional squared, then σ is strictly parabolic on $G - \{0\}$ but it is smooth on the origin if and only if G is biholomorphic to the ball. On the other hand the space of biholomorphic classes of such domains is infinite dimensional (cfr. [10]). In this class of examples the Monge-Ampère foliation associated to the exhaustion is always holomorphic. This is also quite an exceptional occurrence. Let $D \Subset \mathbb{C}^m$ be a smooth, strictly convex domain and $p \in D$ be any point. There exists an exhaustion $\tau_p: \bar{D} \to [0, 1]$ such that $\tau_p^{-1}(0) = \{p\}$, $\tau | \partial D \equiv 1$ and that τ_p is strictly parabolic on $D - \{p\}$ (cfr. [6] and [9]). For a generic strictly convex domain the Monge-Ampère foliation associated to τ_p is *not* holomorphic (cfr. [9]).

In this paper we study complex manifolds M admitting an exhaustion $\tau: M \to [0, \Delta^2)$ which is strictly parabolic on $M - \tau^{-1}(0)$ and has a prescribed behavior on $\tau^{-1}(0)$. We require that there exists $p \in \tau^{-1}(0)$ such that, if $\mathbf{P}: \tilde{M} \to M$ is the blow up of M at p, then $\tau \circ \mathbf{P} \in C^\infty(\tilde{M})$ and such that, with respect to coordinates centered at p, we have $C\|Z\|^2 \leqq \tau(Z) \leqq K\|Z\|^2$ for some positive constants C, K (cfr. Sect. 1 for detailed definitions). It is easy to see that this definition encompasses the above mentioned examples. If M is a complex manifold of dimension m equipped with such an exhaustion τ, our main result can be formulated as follows (cfr. Theorem 4.4 for the precise statement):

Theorem. *If $\Delta=\infty$, then M is biholomorphic to $\mathbb{C}^m$. If $\Delta<\infty$ and the Monge-Ampère foliation associated to τ is holomorphic, then M is biholomorphic to a bounded, strictly pseudoconvex, complete circular domain in $\mathbb{C}^m$.*

Clearly, since the assumptions are biholomorphically invariant, our Theorem characterizes, up to biholomorphic maps, $\mathbb{C}^m$ and the bounded, strictly pseudoconvex, complete circular domains in $\mathbb{C}^m$.

The methods used in this paper are a combination of Stoll's techniques ([12]) and of ideas developed in the special case of strictly convex domains ([9]). The simplifications to Stoll's original proof due to Burns [4] and Wong [14] cannot be used here because they are based on the consideration of the exponential map of the metric $dd^c\tau$ at the center point $\tau^{-1}(0)$ whereas in our situation the metric is not defined on $\tau^{-1}(0)$.

Here, once realized that, as in the case of strictly parabolic manifolds, the center $\tau^{-1}(0)$ reduces to a single point, our first step is to construct from scratch a "generalized" exponential map at the center point (Sect. 2). This map is in general only homeomorphic and in fact, it is smooth at the origin if and only if it is biholomorphic (which is the case if and only if τ is strictly parabolic on all M). The map is constructed by integrating the radial vector field $Y=\operatorname{grad}_g\sqrt{\tau}$, where g is the Riemannian metric associated to $dd^c\tau$, and by giving a precise description of its flow. The technical difficulty that one encounters is that, while it is very important to understand the flow of Y in a neighborhood of the center point, Y is not even defined at the center.

Following Stoll's original proof for the strictly parabolic manifold case, we overcome the problem by borrowing a notion from differential topology. Let $p \in M$ be the center point and D be a small ball around p. Then $M'=M-D$ is diffeomorphic to $M-\{p\}$. The orientable manifold N gotten by glueing together two copies of M' along the boundaries is called the double of M at p. Moreover there exists a differentiable map $r: N \to M$ such that r restricted to a copy of M' in N is diffeomorphic onto $M-\{p\}$ and such that $r^{-1}(p)$ is diffeomorphic to a sphere (cfr. Sect. 2 for the precise definition). The differentiability assumptions on τ guarantee that the vector field Y lifts, via r, to a smooth vector field $\tilde{Y}$ defined on *all* N (cfr. Lemma 2.3). Integrating the vector field $\tilde{Y}$ on N and pushing forward its flow to M, we get the flow of Y together with all the necessary asymptotic informations at the center (cfr. Theorem 2.6).

Next, in order to give a parametrization of the leaves of the Monge-Ampère foliation, we consider the vector field $Z=\sqrt{\tau}\, JY$, where J is the almost complex stucture of M. Since Y and Z are both tangent to the Monge-Ampère foliation, by suitably composing their flow, we get, in the case $\Delta=\sup\sqrt{\tau}<\infty$, a holomorphic covering map from a half plane onto each leaf of the foliation of $M-\tau^{-1}(0)$ (cfr. Lemma 3.2). Then, observing that the flow of Z is periodic (cfr. Proposition 3.3), we are able to parametrize each leaf by a punctured disk in such a way that the parametrization depends smoothly on the changing of the leaf (cfr. Theorem 3.4). This will also show that our generalized exponential map transforms every disk through the origin into the closure of a leaf of the Monge-Ampère foliation. In analogy with our previous work [9], in Sect. 4 we define a "model" (G,σ) for (M,τ) where G is a circular domain and σ, up to a factor, is its Minkowski functional squared. A homeomorphism $h: G\to M$ is constructed such that $\tau\circ h=\sigma$. We show that h is biholomorphic if and only if the Monge-Ampère foliation association to τ is holomorphic. If $\Delta=\sup\sqrt{\tau}=\infty$, the proof follows the same line and, moreover, we can use the fact that, because of a theorem of Burns [4], the Monge-Ampère foliation is always holomorphic. In this case the manifold M is biholomorphic to $\mathbb{C}^m$ and τ pulls back to the Minkowski functional squared of some circular domain.

Complex manifolds carrying an exhaustion which is strictly parabolic outside its zero set and whose associated Monge-Ampère foliation is holomorphic, have been classified also by Wong [15]. His approach is based on the study of the differential geometry of the level sets of the exhaustion. His results do not overlap with ours but are clearly related.

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0. Notations

Let M be a complex manifold and let $T(M)$, $T^c(M)$ denote respectively the *tangent bundle* and the *complexified tangent bundle* of M. Then $T(M)$ is a subbundle of $T^c(M)$ and we have the splitting $T^c(M)=T^+(M)\oplus T^-(M)$ where $T^+(M)$ denote respectively the *holomorphic* and the *antiholomorphic tangent bundle* of M. The projections $\pi_+: T^c(M)\to T^+(M)$ and $\pi_-: T^c(M)\to T^-(M)$ restrict to $\mathbb{R}$-bundle isomorphisms $\pi_+: T(M)\to T^+(M)$ and $\pi_-: T(M)\to T(M)$. To the complex structure of M is canonically associated an *almost complex structure* $J: T^c(M)\to T^c(M)$ which is a bundle isomorphism such that $J^2=-\mathrm{Id}$ and such that $J_{|T^+(M)}$ is the multiplication by i and $J_{|T^-(M)}$ is the multiplication by $-i$. By an *hermitian metric* ℓ on M we intend here a C^∞ function $\ell: T^+(M)\oplus T^+(M)\to\mathbb{C}$ such that for all $p\in M$ its restriction $\ell_p: T_p^+(M)\oplus T_p^+(M)\to\mathbb{C}$ is a positive definite hermitian form. The canonical form associated to ℓ is a positive (1,1) form ω defined, for all $p\in M$ and $X,Y\in T_p^+(M)$, by $\omega_p(X,\bar{Y})=(i/2\pi)\ell_p(X,Y)$. The metric ℓ is *Kähler* if and only if $d\omega=0$. Also to ℓ is associated a Riemannian metric g defined for all $p\in M$ and $X,Y\in T(M)$, by $g_p(X,Y)=\mathrm{Re}(\ell_p(\pi_+(X),\pi_+(Y)))$.

Given an exhaustion function $\tau: M\to[0,R^2)$, we shall use the following notations:

$$(0.1)\qquad \begin{aligned} M[0] &= \{x\in M \mid \tau(x)=0\}\\ M(r) &= \{x\in M \mid \tau(x)<r^2\}\\ M[r] &= \{x\in M \mid \tau(x)\leqq r^2\}\\ M\langle r\rangle &= \{x\in M \mid \tau(x)=r^2\} = M[r]-M(r)\\ M_* &= \{x\in M \mid \tau(x)\neq 0\} = M - M[0]. \end{aligned}$$

In particular $\mathbb{C}(r)$, $\mathbb{C}[r]$ will always denote respectively the open and the closed disk of radius r in $\mathbb{C}$. For $m>1$ we will denote the ball of radius r in $\mathbb{C}$ by $\mathbb{B}(r)$ and write $\mathbb{B}=\mathbb{B}(1)$. The unit sphere in $\mathbb{R}^n$ will be denoted by S^{n-1} or simply by S if the dimension is clear from the context. We will use the differential operators $d=\partial+\bar{\partial}$, $d^c=(i/4\pi)(\bar{\partial}-\partial)$ and $d\,d^c=(i/2\pi)\,\partial\,\bar{\partial}$. Finally, throughout the paper, upper indices will denote components and lower indices derivatives. Moreover, whenever possible, Einstein summation convention will be used. Also we shall use the Kroneker symbol $\delta_{\mu\nu}$.

1. Definition and Preliminaries

Let M be a non compact complex manifold of dimension m and $\tau\colon M\to[0,\Delta^2)$, $0<\Delta=\sup\sqrt{\tau}\leqq\infty$, be an exhaustion function. We say the pair (M,τ) is a *manifold of circular type* if the following assumptions are satisfied:

(1.1) $\tau\in C^0(M)\cap C^\infty(M_*)$.

(1.2) $d\,d^c\tau>0$ and $d\,d^c\log\tau\geqq 0$ on M_*.

(1.3) $(d\,d^c\log\tau)^m\equiv 0$ on M_*.

(1.4) There exists $p\in M[0]$ such that:

(i) with respect to local coordinates centered at p, if $\|\ \|$ denotes the euclidean norm, we have $C\|Z\|^2\leqq\tau(Z)\leqq K\|Z\|$ for some positive constants C, K;

(ii) if $\mathbf{P}\colon \tilde{M}\to M$ is the blow up of M at p, then $\tau\circ\mathbf{P}\in C^\infty(\tilde{M})$.

In what follows we shall only use the following consequence of (1.4):

(1.4′) There exists $p\in M[0]$ and a coordinate neighbourhood U_p, centered at p, such that:

(i) there exists positive constants C, K such that $C\|Z\|^2\leqq\tau(Z)\leqq K\|Z\|^2$ for all $Z\in U_p$;

(ii) there exists $\varepsilon_0>0$ so that $tZ\in U_p$ if $|t|<\varepsilon_0$ and $\|Z\|<2$ and such that the map $h\colon(-\varepsilon_0,\varepsilon_0)\times\mathbb{B}(2)-\{0\}\to\mathbb{R}_+$ defined by $h(t,Z)=\tau(tZ)$ is of class C^∞.

As it will turn out from Theorem 3.4, (1.4′) is in fact equivalent to (1.4) if (1.1), (1.2), (1.3) are fulfilled. We also say that $\Delta=\sup\sqrt{\tau}$ is the *radius* of (M,τ).

It is useful to bear in mind some concrete example from the very beginning. In fact, the basic motivation for this paper was to try to generalize Example 1 according to the lines suggested by Example 2 and 3.

Example 1. Stoll defines in [12] a *strictly parabolic manifold* to be a pair (M,τ) where M is a complex manifold of dimension m and $\tau\colon M\to(0,\Delta^2)$, $0<\Delta=\sup\sqrt{\tau}\leqq\infty$, is a C^∞ strictly plurisubharmonic exhaustion such that $(dd^c\log\tau)^m\equiv 0$ on M_*. He proves that, up to exhaustion-preserving biholomorphic maps, the only such pairs are the balls of radius Δ in $\mathbb{C}^m$ equipped with the exhaustion $\tau_0=\|\ \|^2$ (simplified proofs of this theorem are given by Burns [4] and Wong [14]). Clearly, strictly parabolic manifolds are manifolds of circular type. This can be also easily checked directly without refering to the classification theorem (cfr. [12], Proposition 2.2).

Example 2. Let G be a strictly pseudoconvex, bounded, smooth, complete circular domain in $\mathbb{C}^m$. The *Minkowski functional* $\rho\colon\mathbb{C}^m\to\mathbb{R}_+$ associated to G is defined by

$$\rho(Z)=\begin{cases}0 & \text{if } Z=0\\ \inf\{1/t\,|\,t>0 \text{ and } tZ\notin G\} & \text{if } Z\neq 0.\end{cases}\tag{1.5}$$

It is not hard to check that (G,ρ^2) is a manifold of circular type with radius 1 (cfr. [3] and [10]).

Example 3. Let D be a smooth, bounded, strictly convex domain in $\mathbb{C}^m$. According to [6], for any $p\in D$ there exists a surjective C^∞ map $F\colon\mathbb{C}[1]\times S^{2m-1}\to\bar{D}$, which we called stationary representation of D at p in [9,], such that for all $b\in S^{2m-1}$ $F(\square,b)\colon\mathbb{C}(1)\to D$ is a proper holomorphic embedding with $F(0,b)=p$, $F'(0,b)=\|F'(0,b)\|\,b$ and such that, if $\mathbf{K}$ denotes the Kobayashi metric of D, then $\mathbf{K}(p,b)=(\|F'(0,b)\|)^{-1}$. Also if $\lambda\in\mathbb{C}$ with $|\lambda|=1$ and $(z,b)\in\mathbb{C}[1]\times S^{2m-1}$, then $F(\lambda z,b)=F(z,\lambda b)$. An exhaustion $\tau\colon\bar{D}\to[0,1]$, called the *Lempert exhaustion* of D at p, is well defined by $\tau(F(z,b))=|z|^2$. Using results of Lempert [6] one can show that (D,τ) is a manifold of circular type with radius 1. Properties of the Lempert exhaustions and applications to the classification of strictly convex were studied in [9].

In the remainder of this section we give some preliminary results on manifold of circular type. First we have the following.

Theorem 1.1. *Let (M,τ) be a manifold of circular type. Then the set $M[0]$ consists exactly of one point which we call the center of M.*

Proof. Since no use of the smoothness of τ on $M[0]$ is made, a word by word repetition of the proof of Theorem 2.5 of [12] shows that $M[0]$ is connected. On the other hand, since τ satisfies (1.4), then $M[0]$ has an isolated point and therefore the claim follows. q.e.d.

Remark. One can show that given any smooth, bounded, strictly convex domain $D\subset\mathbb{C}^m$ and any $p\in D$, the only exhaustion σ of $\bar{D}$ such that (D,σ) is a manifold of circular type with center $\{p\}$ and radius 1 is the Lempert exhaustion at p. A proof of this fact based on the Bedford-Taylor minimum principle for the Monge-Ampère operator (cfr. [2]) is in [7]. Likewise given a strictly pseudoconvex, bounded, smooth, complete circular domain $G\subset\mathbb{C}^m$ then the unique exhaustion τ of $\bar{G}$ such that (G,σ) is a manifold of circular type with center at the origin and radius 1 is $\sigma=\rho^2$ where ρ is the Minkowski functional of G. The proof of this fact goes along the same lines of the case of strictly convex domains.

Let (M,τ) be a manifold of circular type. We shall now collect some known facts which will be needed in this paper (cfr. for proofs [4] or [12] or [14]). A simple calculation shows that (1.2) and (1.3) imply that on M_*, with respect to any choice of local coordinates, the following fundamental formula holds

$$\tau_{\bar{\nu}}\tau^{\bar{\nu}\mu}\tau_{\mu}=\tau \tag{1.6}$$

where we set $(\tau^{\bar{\nu}\mu})=(\tau_{\mu\bar{\nu}})^{-1}$. A Kähler metric $\mathscr{k}$ on M_* is defined by the form $dd^c\tau>0$. A unique vector field X of type $(1,0)$, called the *complex gradient* of τ, exists which is dual to $\bar{\partial}\tau$ with respect to $\mathscr{k}$ i.e. such that:

$$\bar{\partial}\tau(\bar{W})=\mathscr{k}(X,W) \tag{1.7}$$

for all vector field W of type $(1,0)$. In local coordinates we have the following expression for X:

$$X=X^{\mu}\frac{\partial}{\partial z^{\mu}}=\tau^{\bar{\nu}\mu}\tau_{\bar{\nu}}\frac{\partial}{\partial z^{\mu}}. \tag{1.8}$$

Because of (1.6) we have also

$$\bar{\partial}\tau(\bar{X})=\mathscr{k}(X,X)=\tau. \tag{1.9}$$

From (1.2) and (1.3) it follows that the form $dd^c\log\tau$ has rank $m-1$ at every point of M_*. Then a distribution $\mathscr{A}$ of rank 1 is defined by

$$\mathscr{A}=\{W\in T^+(M_*)\,|\,dd^c\log\tau(W,\bar{V})=0 \text{ for all } V\in T^+(M_*)\}. \tag{1.10}$$

The distribution $\mathscr{A}$ is integrable and the associated foliation of M_* in Riemann surfaces is called the *Monge-Ampère foliation* associated to τ. This foliation, first introduced by Bedford and Kalka [1], has been focus of many studies recently (see for instance [4], [10], [12], [14]). Here we shall just recall few well known properties. The distribution $\mathscr{A}$ is a trivial subbundle of $T^+(M_*)$ spanned by the vector field X i.e. for every $p\in M_*$ we have

$$\mathscr{A}_p=\mathbb{C}X_p. \tag{1.11}$$

As a consequence we get that the Monge-Ampère foliation is holomorphic – i.e., $\mathscr{A}$ is a holomorphic subbundle of $T^+(M_*)$ – if and only if X is holomorphic.

A further remarkable property of the Monge-Ampère foliation is that its leaves are totally geodesic submanifold of M_* with respect the Kähler metric $\mathscr{k}$. In fact if $\mathscr{g}$ is the Riemannian metric associated to $\mathscr{k}$ and we define a vector field Y on M_* by

$$Y=\frac{1}{\sqrt{\tau}}(X+\bar{X}), \tag{1.12}$$

then we have

$$Y=\operatorname{grad}_{\mathscr{g}}\sqrt{\tau} \quad\text{and}\quad \mathscr{g}(Y,Y)=1. \tag{1.13}$$

Every integral curve of Y lies on a leaf of the Monge-Ampère foliation and it is a geodesic for the metric $\mathscr{g}$.

2. The Gradient Flow

In this section we shall give a precise description of the flow of the vector field Y defined in (1.12). We shall use the same method used by Stoll in [12] for the case of strictly parabolic manifolds. We have to recall first the concept of the double at a point of a differentiable manifold. Let N be a differentiable manifold with $\dim_{\mathbb{R}} N=n$. Let $L\subset N$ be a closed submanifold with $\dim_{\mathbb{R}} L=n-1$. Let $h: N\to\mathbb{R}$ be a function of class C^∞ with $h_{|L}=0$ and $a\in L$. We say that *h vanishes of order k* at a if there exists a neighbourhood U of a and a function $g: U\to\mathbb{R}$ with $g^{-1}(0)=L\cap U$ and $dg(x)\neq 0$ if $x\in U$, such that on a neighborhood $U_1\subset U$ of a we have $h=g^k h_1$ where $h_1: U_1\to\mathbb{R}$ is a function of class C^∞ with $h_1(a)\neq 0$. Clearly h need not vanish of any order at a. If M is another differentiable manifold, with $\dim_{\mathbb{R}} M=n$, and $f: N\to M$ is a differentiable map with $f_{|L}=b\in M$, we say that *f branches of order s* at $a\in L$ if, given a local coordinates around $a\in N$ and $b\in M$, one has that

$$D=\det\left(\frac{\partial f^i}{\partial x^j}\right)_{i,j=1,\dots,m}$$

vanishes of order $s+m-1$ at $a\in L$. This definition does not depend on the choice of local coordinates and one easily sees that $s\geqq 0$. Let $0\in M$ be any point. We say that (N,r) is the *double* of M at 0 (or connected sum of M and M at 0) (cfr. Milnor [8] and Stoll [12]) if:

(i) N is a differentiable manifold with $\dim_{\mathbb{R}} N=\dim_{\mathbb{R}} M=n$.

(ii) The map $r: N\to M$ is proper, surjective and of class C^∞.

(iii) The inverse image $r^{-1}(0)=S_0$ is a compact submanifold of N diffeomorphic to S^{m-1}.

(iv) The map r branches of order 0 at every point of S_0.

(v) There exists two open sets $N_1, N_2\subset N$ such that $N=N_1\cup N_2\cup S_0$ is a disjoint union and $r_j=r_{|N_j}: N_j\to M-\{0\}$ is a diffeomorphism.

In a neighbourhood of S_0 the double of M can be described as follows (cfr. [12], p. 113). Let U be a coordinate neighbourhood of 0 in M and let $\mathbb{B}(\varepsilon)\subset U$ be a ball of radius ε centered at 0. Let $V=r^{-1}(\mathbb{B}(\varepsilon))$. If ε is small enough, there exists a diffeomorphism $f: V\to(-\varepsilon,\varepsilon)\times S^{m-1}$ such that $f(S_0)=\{0\}\times S^{n-1}$ and $r(x)=r_0(f(x))$ for all $x\in V$ where r_0 is defined by $r_0(t,b)=t\,b$. When working near S_0, we shall identify V with $(-\varepsilon,\varepsilon)\times S^{m-1}$ so that $f=\mathrm{Id}$ and $r=r_0$ on V.

Let now (M,τ) be a manifold of circular type of dimension m and radius Δ. To describe the flow of the vector field Y defined in (1.12) we will first pull it back to a vector field $\tilde{Y}$ on the double (N,r) of M at the center $0\in M$ and study the flow of $\tilde{Y}$ on N.

Let U_0 be the coordinate neighbourhood of the center 0 of M with the properties listed in (1.4). Also the notations of Sect. 1 will be assumed. Furthermore here we write $S=S^{2m-1}$. Define $U=\{Z\in U_0 | Z=t\,b$ with $t\in(-\varepsilon,\varepsilon)$ and

$b \in S\}$. If (N, r) is the double of M at 0 then set $V = r^{-1}(U)$. We can choose ε small enough so that we can identily $V = (-\varepsilon, \varepsilon) \times S$ and so that $r(t, b) = t\,b$. We also set $V_1 = N_1 \cap V = (-\varepsilon, 0) \times S$ and $V_2 = N_2 \cap V = (0, \varepsilon) \times S$.

Lemma 2.1. *There exists a function $G\colon V - \{0\} \to \mathbb{R}_0$ of class C^∞ and constants $C_2 > C_1 > 0$, such that for all $Z \in U - \{0\}$ we have $\tau(Z) = G(Z)\|Z\|^2$ and $C_1 < G(Z) < C_2$. Furthermore there exists $c\colon V \to \mathbb{R}$, $H\colon S \to \mathbb{R}_+$, $R\colon V \to \mathbb{R}$ of class C^∞ such that $c^2(t, b) = \tau(r(t, b))$ for all $(t, b) \in V$ with $c_{|V_1} < 0$, $c_{|V_2} > 0$ and such that*

$$c(t, b) = t\sqrt{H(b)} + t^2 R(t, b). \tag{2.1}$$

Proof. The fact that a function G on $U - \{0\}$ exists with the above properties is an immediate consequence of (1.4′) part (i). Let $h\colon (-\varepsilon_0, \varepsilon_0) \times \mathbb{B}(2) - \{0\} \to \mathbb{R}_+$ be defined as in (1.4′) part (ii). Then its restriction $h\colon V \to \mathbb{R}_+$ is of class C^∞. There exists $G\colon V \to \mathbb{R}_+$ of class C^∞ and such that $\tilde{G} > 0$ on V so that $h(t, b) = \tilde{G}(t, b)\, t^2$. Then we have $\dot{h}(0, b) = 0$ and $\ddot{h}(0, b) > 0$ for all $b \in S$ where we have set $\dot{h} = \frac{\partial h}{\partial t}$. Define $H\colon S \to \mathbb{R}_+$ by

$$H(b) = \tfrac{1}{2}\ddot{h}(0, b). \tag{2.2}$$

Then there exists $T\colon V \to \mathbb{R}$ of class C^∞ such that if $(t, b) \in V$, then $0 \leqq h(t, b) = t^2 H(b) + t^3\, T(t, b)$. In particular we have that $H(b) + t\,T(t, b) > 0$ for all $(t, b) \in V$. The function $L\colon V \to \mathbb{R}$ defined by $L(t, b) = (H(b) - t\,T(t, b))^{1/2} - (H(b))^{1/2}$ is of class C^∞ and $L(0, b) = 0$ for all $b \in S$. Thus there exists $R\colon V \to \mathbb{R}$ of class C^∞ such that $L(t, b) = t\,R(t, b)$. But then $c\colon V \to \mathbb{R}$ given by

$$c(t, b) = t(H(b))^{1/2} + t^2 R(t, b) = t(H(b) + t\,T(t, b)^{1/2}$$

is the required function. q.e.d.

Lemma 2.2. *Let $W = (-\varepsilon_0, \varepsilon_0) \times (\mathbb{B}(2) - \{0\})$. There exists a map $A\colon W \to \mathbb{C}^m$ of class C^∞ so that if X is the vector field defined in (1.7), $(t, Z) \in W$ and $t \neq 0$, then*

$$X(t, Z) = t\,Z + t^2 A(t, Z). \tag{2.3}$$

Proof. Let $(t, Z) \in W$ and $t \neq 0$, then because of Lemma 2.1 we have for $\mu, \nu = 1, \ldots, m$:

$$\tau(tZ) = t^2 G(tZ)\,\|Z\|^2$$
$$\tau_{\bar{\nu}}(tZ) = t\,G(tZ)\,z^\nu + t^2\,G_{\bar{\nu}}(tZ)\,\|Z\|^2$$
$$\tau_{\mu\bar{\nu}}(tZ) = G(tZ)\,\delta_{\mu\nu} + t\,G_\mu(tZ)\,z^\nu + t\,G_{\bar{\nu}}(tZ)\,\bar{z}^\mu + t^2\,G_{\mu\bar{\nu}}(tZ)\,\|Z\|^2$$

Because of (1.4′) there exists $\tilde{G}\colon W \to (0, \infty)$ of class C^∞ such that $h(t, Z) = \tilde{G}(t, Z)\,\|tZ\|^2$ and, in fact, $\tilde{G}(t, Z) = G(tZ)$ if $(t, Z) \in W$ and $t \neq 0$. Functions $\hat{G}^\mu\colon W \to \mathbb{C}$, $\hat{G}^{\bar{\nu}}\colon W \to \mathbb{C}$ and $\hat{G}^{\mu\bar{\nu}}\colon W \to \mathbb{C}$ of class C^∞ are defined by

$$\hat{G}^\mu(t, Z) = \begin{cases} (1/t)\,\tilde{G}_\mu(t, Z) = G_\mu(tZ) & \text{if } t \neq 0 \\ \dot{\tilde{G}}_\mu(t, Z) & \text{if } t = 0 \end{cases}$$

$$\hat{G}^{\bar{\nu}}(t,Z)=\begin{cases}(1/t)\,\tilde{G}_{\bar{\nu}}(t,Z)=G_{\bar{\nu}}(tZ) & \text{if } t\neq 0\\ \dot{\tilde{G}}_{\bar{\nu}}(t,Z) & \text{if } t=0\end{cases}$$

$$\hat{G}^{\mu\bar{\nu}}(t,Z)=\begin{cases}(1/t)\,G^{\mu}_{\bar{\nu}}(t,Z)=G_{\mu\bar{\nu}}(tZ) & \text{if } t\neq 0\\ \dot{\hat{G}}^{\mu\bar{\nu}}(t,T) & \text{if } t=0\end{cases}$$

where the dot denotes the derivative with respect to t. Then we have for $(t,Z)\in W$ and $t\neq 0$

$$\tau_{\bar{\nu}}(tZ)=t\,\tilde{G}(t,Z)\,z^{\nu}+t^2\,\hat{G}^{\bar{\nu}}(t,Z)\,\|Z\|^2 \tag{2.4}$$

$$\tau_{\mu\bar{\nu}}(tZ)=\tilde{G}(t,Z)\,\delta_{\mu\nu}+t\,B^{\mu\bar{\nu}}(t,Z)$$

where $B^{\mu\bar{\nu}}\colon W\to\mathbb{C}$ are functions of class C^{∞} defined by

$$B^{\mu\bar{\nu}}(t,Z)=\hat{G}^{\mu}(t,Z)\,z^{\nu}+\hat{G}^{\nu}(t,Z)\,\bar{z}^{\mu}+t\,\hat{G}^{\mu\bar{\nu}}(t,Z)\,\|Z\|^2.$$

There exist functions $R^{\mu\bar{\nu}}\colon W\to\mathbb{C}$ of class C^{∞} so that if $(t,Z)\in W$ and $t\neq 0$ we have

$$\tau^{\bar{\nu}\mu}(tZ)=\frac{1}{\tilde{G}(t,Z)}\,\delta_{\mu\nu}+t\,R^{\bar{\nu}\mu}(t,Z)$$

where $(\tau^{\bar{\nu}\mu})=(\tau_{\mu\bar{\nu}})^{-1}$. But recalling (1.8), we get at once (2.2) where $A\colon W\to\mathbb{C}^m$ is the map of class C^{∞} defined by

$$A^{\mu}(t,Z)=\sum_{\nu=1}^{m}\frac{\|Z\|^2}{\tilde{G}(t,z)}\,\delta_{\mu\nu}\,\hat{G}^{\bar{\nu}}(t,Z)+\tilde{G}(t,Z)\,R^{\mu\bar{\nu}}(t,Z)\,z^{\nu}$$
$$+t\,\|Z\|^2\,R^{\mu\bar{\nu}}(t,Z)\,\hat{G}^{\bar{\nu}}(t,Z).\quad \text{q.e.d.}$$

In order to study vector fields on the double (N,r) of M we make the following identifications. Let $(t,b)\in V$. We identify $T_{(t,b)}(V)=\mathbb{R}\oplus T_b(S)$ where $T_b(S)$ is the tangent space of S at b. Also if $H_b(S)$ is the holomorphic tangent space to S we identify $T_{(t,b)}(V)=\mathbb{R}\oplus i\mathbb{R}b\oplus H_b(S)$. As seen above we can consider V as a neighbourhood of $S_0\subset N$ and set $r(t,b)=t\,b$ if $(t,b)\in V$. Also if $p\in V$ then we identify $T_p(U)=T_p^+(U)=T_p^-(U)=\mathbb{C}^m$ so that the projection π_+,π_- are the identity, the almost complex structure J is the multiplication by i and the conjugation is obtained by conjugation of the coordinates. We denote the hermitian product in $\mathbb{C}^m$ by $(\,|\,)$ so that the real scalar product $\langle\,|\,\rangle$ is given by $\langle Z|W\rangle=\mathrm{Re}(Z|W)$. Then we have the following formula:

$$dr^{-1}(t,b)(Z)=\left(\mathrm{Re}(Z|b),\frac{\mathrm{Im}(Z|b)}{t}\,i\,b+\frac{Z}{t}-\frac{(Z|b)}{t}\right)\in\mathbb{R}\oplus i\,\mathbb{R}\,b\oplus H_b(S) \tag{2.5}$$

Let J be the almost complex structure on M. On $N-S_0$ a complex structure is defined by pulling back by r the complex structure of M. The associated almost complex structure is denoted by $\tilde{J}$ and we have $\tilde{J}=dr^{-1}\circ J\circ dr$. Thus if $(t,b)\in V-S_0$ and $(u,i\,v\,b+w)\in\mathbb{R}\oplus i\mathbb{R}\oplus H_b(S)$

$$\tilde{J}_{t,b}(u,i\,v\,b+w)=\left(-t\,v,\frac{u}{t}\,i\,b+i\,w\right). \tag{2.6}$$

On $N-S_0$ we can define vector fields of class C^∞ $\tilde{X}$ and $\tilde{Y}$ by lifting via r the vector fields X, Y. Also define $\tilde{\tau}: N\to\mathbb{R}_+$ by $\tilde{\tau}=\tau\circ r$.

Lemma 2.3. *The function $\tilde{\tau}$ is of class C^∞ on N. There exists $c: N\to\mathbb{R}$ of class C^∞ such that $c^2=\tilde{\tau}$, $c_{|N_1}<0$ and $c_{|N_2}>0$. Furthermore $\tilde{X}$, $J\tilde{X}$ and $\tilde{Y}$ extend to vector fields of class C^∞ on N.*

Proof. We need only to check the claim on V. It is an immediate consequence of Lemma 2.1 that $\tilde{\tau}$ is of class C^∞ and that $c: N\to\mathbb{R}$ exists with the required properties. Using (2.3) and (2.5) we have for $(t,b)\in V-S_0$:

$$\begin{aligned}\tilde{X}(t,b)&=dr^{-1}(t,b)(X(r(t,b)))\\&=t(F_1(t,b), F_2(t,b)\,i\,b+F_3(t,b))\end{aligned}\tag{2.7}$$

where $F_1: V\to\mathbb{R}$, $F_2: V\to\mathbb{R}$, $F_3: V\to H_b(S)$ are of class C^∞ and defined by

$$\begin{cases}F_1(t,b)=1+t\,\mathrm{Re}(A(t,b)|b)\\F_2(t,b)=\mathrm{Im}(A(t,b)|b)\\F_3(t,b)=A(t,b)-(A(t,b)|b).\end{cases}\tag{2.8}$$

But then $\tilde{X}$ extends of class C^∞ on all V and thus on N. From (2.6) and (2.7) we have for $(t,b)\in V-S_0$

$$J\tilde{X}(t,b)=(-t^2F_2(t,b), F_1(t,b)\,i\,b+i\,tF_2(t,b))\tag{2.9}$$

and thus also $J\tilde{X}$ extends of class C^∞ on all N. Finally, since if $(t,b)\in V-S_0$ we have

$$\tilde{Y}(t,b)=\frac{1}{c(t,b)}\left(\tilde{X}(t,b)+\overline{\tilde{X}(t,b)}\right),$$

$\tilde{Y}$ extends of class C^∞ on N because of (2.1) and (2.7). q.e.d.

Remark. Using (2.8) and (2.9) we also have the following useful formula for $b\in S$:

$$J\tilde{X}(0,b)=(0,i\,b).\tag{2.10}$$

Proposition 2.4. *There exists a number $\eta>0$ and a map $\Psi:(-\eta,\Delta)\times S\to N$ of class C^∞ such that*

(i) *$\Psi(0,b)=(0,b)\in S_0=\{0\}\times S$ for all $b\in S$;*
(ii) *$\dot{\Psi}(t,b)=\tilde{Y}(\Psi(t,b))$ for all $(t,b)\in(-\eta,\Delta)\times S$;*
(iii) *The map Ψ: $[0,\Delta]\to S_0\cup N_2$ is a diffeomorphism;*
(iv) *$c(\Psi(t,b))=t$ for all $(t,b)\in[0,\Delta]\times S$;*
(v) *If $t\geqq 0$, define $\Psi_t: S\to N$ by $\Psi_t(b)=\Psi(t,b)$ and $N_t=\{c=t\}$. Then $\Psi_t: S\to N_t$ is a diffeomorphism.*

Proof. Since $c: N\to\mathbb{R}$ is a function of class C^∞ such that $c^{-1}([0,\Delta))$ is non empty, non compact and connected, $c^{-1}([0,t])$ is compact for all $t\in[0,\Delta)$, $S_0=c^{-1}(\{0\}\neq\emptyset$ and $\tilde{Y}$ is a vector field of class C^∞ on N, according to Theorem 4.1 of [12] the map Ψ with the properties (i)–(v) exists provided $dc(p,\tilde{Y}(p))=1$ for all $p\in C^{-1}([0,\Delta))=S_0\cup N_2$. To show this observe that

$$dr(p, \tilde{Y}(p)) = -Y(r(p)) \quad \text{on } N_1$$
$$dr(p, \tilde{Y}(p)) = Y(r(p)) \quad \text{on } N_2.$$

But then on N_1 using (1.13) we have

$$dc(p, \tilde{Y}(p)) = -d\sqrt{\tau}(r(p), dr(p, \tilde{Y}(p)))$$
$$= -d\sqrt{\tau}(r(p), -Y(r(p))) = 1$$

In the same way one obtains $dc(p, \tilde{Y}(p)) = 1$ on N_2 and thus by continuity the claim is true on N. q.e.d.

Lemma 2.5. *Let $r_0 > 0$ small enough so that $\Psi((-\eta, r_0) \times S) \subset V$ and let $\Psi_1: (-\eta, r_0) \times S \to \mathbb{R}$, $\Psi_2: (-\eta, r_0) \times S \to S$ such that $\Psi = (\Psi_1, \Psi_2)$. There exist $\Psi_3: (-\eta, r_0) \times S \to \mathbb{R}$ and $\Psi_4: (-\eta, r_0) \times S \to \mathbb{C}^m$ of class C^∞ such that*

$$\Psi_1(t, b) = \frac{1}{\sqrt{H(b)}} t + t^2 \Psi_3(t, b), \tag{2.11}$$

$$\Psi_2(t, b) = b + t\Psi_4(t, b). \tag{2.12}$$

Proof. Because of (i) of Theorem 2.4 there exist $\Psi_4: (-\eta, r_0) \times S \to \mathbb{C}^m$ and $\Psi_5: (-\eta, r_0) \times S \to \mathbb{R}$ of class C^∞ such that (2.12) holds and $\Psi_1(t, b) = t\Psi_5(t, b)$. If π_+ is the isomorphism defined is Sect. 0, then

$$\lim_{t \to 0} \pi_+(\tilde{Y}(t, b)) = \lim_{t \to 0} \frac{1}{c(t, b)} \tilde{X}(t, b)$$
$$= \frac{1}{\sqrt{H(b)}}(1, F_2(0, b)\, ib + F_3(0, b))$$

where we have used (2.1), (2.7) and (2.8). Thus because of (ii) of Theorem 2.4 we have $\Psi_5(0, b) = \dot{\Psi}_1(0, b) = (H(b))^{-1/2}$. But then there exists $\Psi_3: (-\eta, r_0) \times S \to \mathbb{R}$ of class C^∞ such that $\Psi_5(t, b) = (H(b))^{-1/2} + t\Psi_3(t, b)$ and thus the proof is completed. q.e.d.

We have now all the ingredients to describe in detail the flow on M of the vector field Y defined in (1.12). Here we identify $T_0(M) = \mathbb{C}^m$.

Theorem 2.6. *There exists a map of class C^∞*

$$\varphi: [0, \Delta) \times S \to M \tag{2.13}$$

with the following properties

(i) *$\varphi(0, b) = 0$ for all $b \in S$;*
(ii) *$\dot{\varphi}(0, b) = (H(b))^{-1/2} b$ for all $b \in S$;*
(iii) *$\dot{\varphi}(t, b) = Y(\varphi(t, b))$ for all $(t, b) \in (0, \Delta) \times S$;*
(iv) *$\varphi: (0, \Delta) \times S \to M_*$ is a diffeomorphism;*
(v) *$\tau(\varphi(t, b)) = t^2$ for all $(t, b) \in [0, \Delta) \times S$;*
(vi) *the map $\varphi_t: S \to M\langle t \rangle$ defined by $\varphi_t(b) = \varphi(t, b)$ is a diffeomorphism;*

(vii) $t\pi_+(\dot{\varphi}(t,b)) = X(\varphi(t,b))$ *for all* $(t,b) \in [0,\Delta) \times S$;

(viii) *The curve* $\varphi(\square, b)\colon (0,\Delta) \times S \to M_*$ *is geodesic with respect the Kähler metric defined by* $dd^c\tau > 0$ *for all* $b \in S$.

Proof. If $r\colon N \to M$ is the projection from the double, we define $\varphi = r \circ \Psi\colon [0,\Delta) \to M$. Then (i), (iii), (iv), (v), (vi) and (vii) are immediate from Proposition 2.4. Also, using the usual identification of the tangent space of M at the center 0 with $\mathbb{C}^m$, with respect to local coordinates around 0 we have recalling (2.11) and (2.12):

$$\begin{aligned}\varphi(t,b) &= r(\Psi(t,b)) \\ &= r(\Psi_1(t,b), \Psi_2(t,b)) \\ &= t(H(b))^{-1/2} b + t^2 T(t,b)\end{aligned}$$

where T is of class C^∞ and thus (ii) follows. Finally, (viii) is a consequence of the well known fact recalled in Sect. 1 that the integral curves of Y are geodesics. q.e.d.

Corollary 2.7. *The map* $E\colon \mathbb{B}(\Delta) \to M$ *defined by*

$$E(Z) = \begin{cases} 0 & \text{if } Z = 0 \\ \varphi(\|Z\|, Z/\|Z\|) & \text{if } Z \neq 0 \end{cases} \tag{2.14}$$

is a homeomorphism and its restriction $E\colon \mathbb{B}(\Delta) - \{0\} \to M_*$ *is a diffeomorphism.*

Proof. The map $E\colon \mathbb{B}(\Delta) - \{0\} \to M_*$ is clearly a diffeomorphism. Also $\tau(E(Z)) = \|Z\|^2$ and hence $\lim_{Z \to 0} E(Z) = 0$ and $\lim_{p \to 0} E^{-1}(p) = 0$ so that the proof is complete. q.e.d.

Remark. The metric ℓ defined on M_* by $dd^c\tau$ is not defined on the center 0 of M. Thus one cannot define in the usual way an exponential map at 0. The map E defined in (2.14) is a generalized exponential map at the "singular" point 0 with respect the metric ℓ. The existence of an exponential map at a singular point of a Riemannian manifold has been investigated by Stone [13] in terms of curvature conditions in a neighbourhood of the singularity. Here it follows from the behavior of the exhaustion which defines the metric. It would be interesting to understand the relationship between Stone's hypothesis and ours in this particular case. Notice also that the map E is differentiable at 0 if and only if the metric extends over 0 and this is the case if M is strictly parabolic (in which case E is biholomorphic). This will be evident from the results of the next section.

3. The Leaf Parametrization Map

In order to give a parametrization of the leaves of the Monge-Ampère foliation induced by τ we must consider also the flow of the vector field Z defined on M_* by

$$Z = i(X - \bar{X}) = J(X + \bar{X}). \tag{3.1}$$

where X is as usual the complex gradient of τ. Using (1.9) we have

$$d\tau(Z) = i\partial\tau(X) - i\bar{\partial}\tau(\bar{X}) = i\tau - i\tau = 0$$

and therefore any integral curve of Z is contained in some level set of τ which is compact. But this implies that Z is complete i.e. there exists a unique map $\sigma: \mathbb{R} \times M_* \to M_*$ such that

(3.2) $\sigma(\Box, p): \mathbb{R} \to M_*$ is an integral curve of Z such that $\sigma(0, p) = p$.

(3.3) For all $y \in \mathbb{R}$ the map $\sigma(y, \Box): M_* \to M_*$ is a diffeomorphism.

(3.4) For all $y_1, y_2 \in \mathbb{R}$ and $p \in M_*$ we have $\sigma(y_1 + y_2, p) = (y_1, \sigma(y_2, p))$.

Define a map $\alpha: [0, \Delta) \times M_* \to M$ by

(3.5) $$\alpha(t, p) = \varphi(t, \varphi^{-1}_{\sqrt{\tau(p)}}(p)).$$

Clearly α is continous and of class C^∞ on $(0, \Delta) \times M_*$.

Lemma 3.1. *For all* $t \in \mathbb{R}$, $s \in (0, \Delta)$ *and* $p \in M_*$ *we have*

(3.6) $$\sigma(t, \alpha(s, p)) = \alpha(s, \sigma(t, p))$$

Proof. Since α is such that for all $p \in M_*$ $\alpha(\Box, p): (0, \Delta) \to M_*$ is the unique integral curve of Y such that $\alpha(\sqrt{\tau(p)}, p) = p$, (3.6) is equivalent to $[Y, Z] = 0$ on M_*. Define $F = X + \bar{X}$ so that $Y = (\tau)^{-1/2} F$ and $Z = JF$. Then

$$[Y, Z] = \frac{1}{\sqrt{\tau}} [F, JF] - JF\left(\frac{1}{\sqrt{\tau}}\right) F.$$

Since $JF((\tau)^{-1/2}) = 0$ follows easily from (1.9) to show $[Y, Z] = 0$ we have only to prove $[F, JF] = 0$. If $[F, JF] = H^\mu \frac{\partial}{\partial z^\mu} + \bar{H}^\mu \frac{\partial}{\partial \bar{z}^\mu}$ then one computes $H^\mu = 2i\bar{X}^\nu X^\mu_{\bar{\nu}}$. On the other hand

$$\bar{X}^\nu X^\mu_{\bar{\nu}} = \tau_\alpha \tau^{\bar{\nu}\alpha} (\tau_{\bar{\beta}} \tau^{\bar{\beta}\mu})_{\bar{\nu}} = \tau_\alpha \tau^{\bar{\nu}\alpha} \tau_{\bar{\beta}\bar{\nu}} \tau^{\bar{\beta}\mu} + \tau_\alpha \tau^{\bar{\nu}\alpha} \tau_{\bar{\beta}} \tau^{\bar{\beta}\mu}_{\bar{\nu}}.$$

Differentiating the equality $\tau_{\sigma\bar{\gamma}} \tau^{\bar{\nu}\sigma} = \delta_{\gamma\nu}$ and (1.6) we obtain the following equalities:

$$\tau_{\sigma\bar{\gamma}} \tau^{\bar{\nu}\sigma}_{\bar{\beta}} + \tau_{\bar{\gamma}\bar{\beta}\sigma} \tau^{\bar{\nu}\sigma} = 0$$

$$\tau_{\bar{\beta}\bar{\nu}} \tau^{\bar{\nu}\alpha} \tau_\alpha + \tau_{\bar{\nu}} \tau^{\bar{\nu}\alpha}_{\bar{\beta}} \tau_\alpha = 0.$$

But then we have

$$\tau_\alpha \tau^{\bar{\nu}\alpha} \tau_{\bar{\beta}\bar{\nu}} \tau^{\bar{\beta}\mu} = -\tau_\alpha \tau^{\bar{\nu}\alpha}_{\bar{\beta}} \tau_{\bar{\nu}} \tau^{\bar{\beta}\mu} = -\tau_{\bar{\nu}} \tau^{\bar{\gamma}\alpha} \tau_{\sigma\bar{\gamma}} \tau^{\bar{\nu}\sigma}_{\bar{\beta}} \tau_\alpha \tau^{\bar{\beta}\mu}$$
$$= \tau_{\bar{\nu}} \tau^{\bar{\gamma}\alpha} \tau_{\sigma\bar{\gamma}\bar{\beta}} \tau^{\bar{\nu}\sigma} \tau_\alpha \tau^{\bar{\beta}\mu}$$

and

$$\tau_{\bar{\beta}} \tau^{\bar{\beta}\mu}_{\bar{\nu}} \tau_\alpha \tau^{\bar{\nu}\alpha} = \tau_{\bar{\beta}} \tau^{\bar{\gamma}\mu} \tau_{\sigma\bar{\gamma}} \tau^{\bar{\beta}\sigma}_{\bar{\nu}} \tau_\alpha \tau^{\bar{\nu}\alpha} = -\tau_{\bar{\beta}} \tau^{\bar{\gamma}\mu} \tau_{\sigma\bar{\gamma}\bar{\nu}} \tau^{\bar{\beta}\sigma} \tau_\alpha \tau^{\bar{\nu}\alpha}$$
$$= -\tau_{\bar{\nu}} \tau^{\bar{\beta}\mu} \tau_{\sigma\bar{\beta}\bar{\gamma}} \tau^{\bar{\nu}\sigma} \tau_\alpha \tau^{\bar{\gamma}\alpha}$$

where in the last line we exchange the indices in the following way: $\beta \to \nu$, $\gamma \to \beta$, $\nu \to \gamma$. Our computation shows that $\bar{X}^\nu X^\mu_{\bar{\nu}} = 0$ and thus $[F, JF] = 0$. q.e.d.

We need now some notations. For $R \in (0, \infty]$ define

$$\mathbb{H}(R) = \begin{cases} \{z \in \mathbb{C} \mid \operatorname{Re} z < \log R\} & \text{if } R < \infty \\ \mathbb{C} & \text{if } R = \infty. \end{cases} \tag{3.7}$$

Also fix $x_0 \in (0, \Delta)$. Write $q = \varphi_{x_0}$. A one parameter group of diffeomorphisms $\zeta: \mathbb{R} \times S \to S$ is defined by

$$\zeta(y, b) = q^{-1}(\sigma(y, q(b))). \tag{3.8}$$

Finally we define a surjective map W: $\mathbb{H}(\Delta) \times S \to M_*$ of class C^∞ by

$$W(x + iy, b) = \sigma(y, \varphi(e^x, b)) \tag{3.9}$$

From (3.5), (3.6) and (3.8) one has at once

$$W(x + iy, b) = \sigma(y, \alpha(e^x, q(b)) = \alpha(e^x, \sigma(y, q(b))) = \varphi(e^x, \zeta(y, b)). \tag{3.10}$$

Lemma 3.2. *For all $b \in S$ the map $W(\square, b)$: $\mathbb{H}(\Delta) \to M_*$ is holomorphic and $W'(z, b) = X(W(z, b))$. Furthermore if $x + iy$, $s + it \in \mathbb{H}(\Delta)$ and $b, c \in S$, then $W(x + iy, b) = W(s + it, c)$ if and only if $x = s$ and $c = \zeta(y - t, b)$.*

Proof. If we set $p = W(x + iy, b)$, then by (3.10):

$$\frac{\partial}{\partial x} W(x + iy, b) = \dot{\varphi}(e^x, \zeta(y, b))\, e^x = \sqrt{\tau(p)}\, Y(p) = X(p) + \overline{X(p)}$$

$$\frac{\partial}{\partial x} W(x + iy, b) = \dot{\sigma}(y, \varphi(e^x, b)) = Z(p) = J(X(p) + \overline{X(p)}).$$

But then $\frac{\partial W}{\partial x} + J\frac{\partial W}{\partial y} = \frac{\partial W}{\partial x} = \frac{\partial W}{\partial x} + JJ\frac{\partial W}{\partial x} = 0$ and $W(\square, b)$ is thus holomorphic. Moreover $W'(z, b) = \pi_+\left(\frac{\partial W(z, b)}{\partial x}\right) = X(W(z, b))$. Finally if $W(x + iy, b) = W(s + it, c)$ then $\varphi(e^x, \zeta(y, b)) = \varphi(e^s, \zeta(t, b))$ and therefore $x = s$ and $\zeta(y, b) = \zeta(t, c)$ which implies $c = \zeta(y - t, b)$ since ζ is a one parameter group. In the other direction the implication is immediate. q.e.d.

We need now to compute explicitly $\zeta(y, b)$ for $y \in \mathbb{R}$ and $b \in S$. To this end we shall again use the notations introduced in Sect. 2. Let (N, r) be the double of M at the center 0, $V = (-\varepsilon, \varepsilon) \times S \subset N$ and let $U = r(V)$. If Ψ is the map defined in Proposition 2.4, let $r_0 > 0$ be such that $\Psi([0, r_0) \times S) \subset V$. Then, since $\varphi = r \circ \Psi$, we have also $\varphi([0, r_0) \times S) \subset U$. It is not restrictive to assume that, if $x_0 \in (0, \Delta)$ is the number used in the definition (3.8) of ζ, then $x_0 \in (0, r_0)$.

Proposition 3.3. *For all $y \in \mathbb{R}$ and $b \in S$ we have*

$$\zeta(y, b) = e^{iy} b. \tag{3.11}$$

Proof. Fix $b \in S$. For $t \in (0, r_0)$ and $y \in \mathbb{R}$ define $\gamma(t, y) = r^{-1}(\varphi(t, \zeta(y, b))$. Then γ is of class C^∞ on $(0, r_0) \times \mathbb{R}$ and we have

$$\gamma(t, y) = \Psi(t, \zeta(y, b)) = r^{-1}(W(\log t + iy, b)).$$

Using (2.11) and (2.12) we have

$$\begin{aligned}\gamma(t, y) &= (\Psi_1(t, \zeta(y, b)), \Psi_2(t, \zeta\, y, b))) \\ &= \left(\frac{t}{\sqrt{H(\zeta(y, b))}} + t^2 \Psi_3(t, \zeta(y, b)), \zeta(y, b) + t \Psi_4(t, \zeta(y, b))\right).\end{aligned}$$

Thus

$$\gamma_y(t, y) = \left(t \frac{\partial}{\partial y}\Big(\frac{1}{\sqrt{H(\zeta(y, b))}} + t \Psi_3(t, \zeta(y, b)), \zeta_y(y, b) + t \frac{\partial}{\partial y} \Psi_4(t, \zeta(y, b))\right).$$

The limit of $\gamma_y(t, y)$ as $t \to 0$ exists and

$$\gamma_y(0, y) = \lim_{t \to 0} \gamma_y(t, y) = (0, \zeta_y(y, b)).$$

On the other hand since

$$W_y(\log t + iy, b) = iW'(\log t + iy, b) = iX(W(\log t + iy, b)),$$

we have:

$$\begin{aligned}\gamma_y(t, b) &= dr^{-1}(W(\log t + iy, b), W_y(\log t + iy, b)) \\ &= dr^{-1}(\varphi(t, \zeta(y, b)), iX(\varphi(t, \zeta(y, b)))) \\ &= \tilde{J}\tilde{X}(\Psi(t, \zeta(y, b))).\end{aligned}$$

Thus using (2.10) we have:

$$\begin{aligned}\gamma_y(0, y) &= \lim_{t \to 0} \gamma_y(t, y) = \tilde{J}\tilde{X}(\Psi_1(0, \zeta(y, b)), \Psi_2(0, \zeta(y, b))) \\ &\tilde{J}\tilde{X}(0, \zeta(y, b)) = i\zeta(y, b).\end{aligned}$$

Comparing the two expressions that we have obtained for $\gamma_y(0, y)$ we get

$$\zeta_y(y, b) = i\zeta(y, b).$$

Integrating this equation and imposing the initial condition $\zeta(0, b) = b$ we conclude that (3.11) holds. q.e.d.

We can now state the main result of this section. As usual we shall identify the tangent space to M at the center 0 with $\mathbb{C}^m$ and we shall denote by $\| \ \|$ the euclidean norm on it.

Theorem 3.4. *There exists uniquely a surjective map, called the leaf parametrization map,*

$$F: \mathbb{C}(\Delta) \times S \to M \tag{3.12}$$

of class C^∞ and with the following properties:

(i) $F(0, b) = 0$ *for all* $b \in S$;
(ii) $\tau(F(z, b)) = |z|^2$ *for all* $b \in S$ *and* $z \in \mathbb{C}(\Delta)$;
(iii) $F(t, b) = \varphi(t, b)$ *for all* $b \in S$ *and* $t \in [0, \Delta)$;
(iv) $F(z, \lambda b) = F(\lambda z, b)$ *for all* $z \in \mathbb{C}(\Delta)$, $b \in S$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$;
(v) $F(\square, b)\colon \mathbb{C}(\Delta) \to M$ *is holomorphic, proper and injective*;
(vi) $X(F(z, b)) = zF'(z, b)$;
(vii) $F'(0, b) = \|F'(0, b)\|\, b = \dfrac{b}{\sqrt{H(b)}}$.

Proof. Since $\exp\colon \mathbb{H}(\Delta) \to \mathbb{C}(\Delta) - \{0\}$ is the universal cover and, because of Lemma 3.2 and Proposition 3.3, we have $W(z_1, b) = W(z_2, b)$ if and only if $z_2 = z_1 + 2k\pi i$, a unique map $F\colon \mathbb{C}(\Delta) - \{0\} \to M_*$ of class C^∞ is defined by $F(e^z, b) = W(z, b)$. Also $F(\square, b)\colon \mathbb{C}(\Delta) - \{0\} \to M_*$ is holomorphic and injective. Define $F(0, b) = 0$ for all $b \in S$. Then $\tau(F(0, b)) = 0$. If $z \in \mathbb{C}(\Delta) - \{0\}$, then $z = e^{x+iy}$ with $x + iy \in \mathbb{H}(\Delta)$. Thus $\tau(F(z, b)) = \tau(W(x+iy, b)) = e^{2x} = |z|^2$ and (ii) is verified. Clearly $F\colon \mathbb{C}(\Delta) \times S \to M$ so defined is continous and, by Riemann extension theorem, $F(\square, b)\colon \mathbb{C}(\Delta) \to M$ is holomorphic for all $b \in S$. Let $U \subset M$ be a coordinate neighbourhood centered at $0 \in M$. With no loss of generality we can identify it with an open subset of $\mathbb{C}^m$. Let $t_0 \in (0, \Delta)$ be such that $F(\mathbb{C}(t_0) \times S) \subset U$. Let $s \in (0, t_0)$ then if $z \in \mathbb{C}(s)$ and $b \in S$ we have

$$F(z, b) = \frac{1}{2\pi i} \int_{|w|=s} \frac{F(w, b)}{w - z}\, dw$$

and hence F is of class C^∞ on $\mathbb{C}(s) \times S$ and therefore on $\mathbb{C}(\Delta) \times S$. If $K \subset M$ is compact then $K \subset M[r]$ for some $r \in (0, \Delta)$. Then $F^{-1}(K) \subset \mathbb{C}[r] \times S$ and therefore F is proper and thus so is $F(\square, b)$ for all $b \in S$. Also (iii) follows by construction and F is surjective because so is φ. As a consequence of the definition of F, Lemma 3.2 and Proposition 3.3 one obtains (iv) immediately. Again from Lemma 3.2 one gets (vi). Finally (vii) is a consequence of (iii) and of (ii) of Theorem 2.6. q.e.d.

Remark. If $E\colon \mathbb{B}(\Delta) \to M$ is the map defined in (2.14), because of (iii) and (iv) of Theorem 3.4 we have that $E(zb) = F(z, b)$ for all $(z, b) \in \mathbb{C}(\Delta) \times S$. In other words the restriction of E on any disk through the origin is holomorphic. This proves the claim made earlier that E is biholomorphic if and only if it is of class C^∞ at the origin.

Examples. Before moving on it may clarify the matter to write down the leaf parametrization map for some explicit example. Let G be a strictly pseudoconvex, bounded, smooth, complete circular domain in $\mathbb{C}^m$ and let ρ its Minkowski functional. If $\sigma = \rho^2$ one computes easily that the complex gradient vector field X of σ is given by $X(Z) = Z^\mu \dfrac{\partial}{\partial z^\mu}$. It follows then that the map $F\colon \mathbb{C}(1) \times S \to G$, relative to the manifold of circular type (G, σ), is defined by $F(z, b) = z(\rho(b))^{-1} b$. Let $D \subset \mathbb{C}^m$ be a bounded, smooth, strictly convex domain, $p \in D$ and τ be the Lempert exhaustion at p. From the results of [9] if follows that the map F for (D, τ) is just the stationary parametrization map which defines τ. However, it is not hard to check directly that $F(\square, b)\colon \mathbb{C}(1)\to D$ is a stationary map in the sense of [6] for all $b\in S$ and thus that the map F is indeed the stationary parametrization of D.

4. The Circular Representation

Let as before (M,τ) be a manifold of circular type and Δ be its radius. We shall now define an "approximation" (G,σ) of (M,τ) where G is a circular domain and σ is, up to a factor, its Minkowski functional squared. Let $F\colon \mathbb{C}(\Delta)\to S\times M$ be the leaf parametrization map of (M,τ). As usual we identify freely $T_0(M)$ with $\mathbb{C}^m$. Define $\sigma\colon \mathbb{C}^m\to \mathbb{R}_+$ by $\sigma(0)=0$ and

$$\sigma(Z)=\left(\frac{\|Z\|}{\|F'(0,Z/\|Z\|)\|}\right)^2=H(Z/\|Z\|)\,\|Z\|^2 \tag{4.1}$$

if $Z\neq 0$ where H is defined in (2.7). One easily sees that $\sigma\in C^0(\mathbb{C}^m)\cap C^\infty(\mathbb{C}^m-\{0\})$. Also from the definition of σ and (iv) of Theorem 3.4 one has for all $Z\in\mathbb{C}^m$ and $\lambda\in\mathbb{C}$:

$$\sigma(\lambda Z)=|\lambda|^2\sigma(Z). \tag{4.2}$$

For $r\in(0,\infty]$ we define also complete circular domains $G(r)\subset\mathbb{C}^m$ by

$$G(r)=\begin{cases}\{Z\}\in\mathbb{C}^m\,|\,\sigma(Z)<r^2\} & \text{if } r<\infty\\ \mathbb{C}^m & \text{if } r=\infty.\end{cases} \tag{4.3}$$

Clearly the Minkowski functional of $G(r)$ is given by $\sqrt{\sigma}/r$.

Proposition 4.1. *For all $r\in(0,\infty]$ the pair $(G(r),\sigma)$ is a manifold of circular type with radius r.*

Proof. Since (1.1) and (1.4) hold by construction only (1.2) and (1.3) must be shown. But if $dd^c\sigma>0$ on $G(r)-\{0\}$ using (4.2) the other statements are obtained by simple calculations. We shall now show $dd^c\sigma>0$ on $G(r)-\{0\}$. Since $\sigma_{\mu\bar\nu}(tZ)=\sigma_{\mu\bar\nu}(Z)$ because of (4.2), it will be enough to show that for some $\varepsilon_0>0$ one has $\sigma_{\mu\bar\nu}(Z)\,V^\mu\bar V^\nu>0$ for all $Z\neq 0$ with $\|Z\|<\varepsilon_0$ and $V\in\mathbb{C}^m$ with $\|V\|=1$. In a small enough neighbourhood U of $0\in M$ we can choose coordinates so that for some $\varepsilon,\varepsilon_0>0$ if $0<t<\varepsilon$ and $Z\in\mathbb{C}^m-\{0\}$ with $\|Z\|<\varepsilon_0$, then $tZ\in U$ and we have

$$\sigma(tz)=\tau(tZ)-t^3L(t,Z)$$

where L is a function of class C^∞ on $(-\varepsilon,\varepsilon)\times(\mathbb{B}(\varepsilon_0)-\{0\})$ (cfr. Lemma 2.1). Furthermore, recalling (2.4), there are functions $\tilde G$ and $B^{\mu\bar\nu}$ of class C^∞ defined on $(-\varepsilon,\varepsilon)\times(\mathbb{B}(\varepsilon_0)-\{0\})$ such that

$$\tau_{\mu\bar\nu}(tZ)=\tilde G(t,Z)\,\delta_{\mu\bar\nu}+tB^{\mu\bar\nu}(t,Z).$$

Also there is a constant $C>0$ such that $\tilde G(t,Z)>C$ for all $(t,Z)\in(-\varepsilon,\varepsilon)\times(\mathbb{B}(\varepsilon_0)-\{0\})$. Fix $Z\in\mathbb{B}(\varepsilon_0)-\{0\}$ and define

$$M = \max_{\substack{|t| \leq \varepsilon/2 \\ V \in S}} \|\sum_{\mu\nu} B^{\mu\bar{\nu}}(t, Z) V^\mu \bar{V}^\nu\|$$

$$N = \max_{\substack{|t| \leq \varepsilon/2 \\ V \in S}} \|\sum_{\mu\nu} L_{\mu\bar{\nu}}(t, Z) V^\mu \bar{V}^\nu\|.$$

Then for $0 < t < \min\{\varepsilon/2, C/(M+N)\}$ we have

$$\sigma_{\mu\bar{\nu}}(Z) V^\mu \bar{V}^\nu = \sigma_{\mu\bar{\nu}}(tZ) V^\mu \bar{V}^\nu = \tau_{\mu\bar{\nu}}(tZ) V^\mu \bar{V}^\nu - tL_{\mu\bar{\nu}}(tZ) V^\mu \bar{V}^\nu$$
$$\geqq C - t(M+N) > 0 \quad \text{q.e.d.}$$

Define $\rho: \mathbb{C}^m \to \mathbb{R}_+$ by

(4.4) $$\rho = \sqrt{\sigma}.$$

Then a map $h: G(\Delta) \to M$, called the *circular representation* of (M, τ), is defined by $h(0) = 0$ and

(4.5) $$h(Z) = F(\rho(Z), Z/\|Z\|),$$

where F is the leaf parametrization map of (M, τ) defined in (3.12), if $Z \neq 0$.

Proposition 4.2. *The circular representation has the following properties:*

(i) *$h: G(\Delta) \to M$ is a homeomorphism with $\tau \circ h = \sigma$;*

(ii) *$h: G(\Delta) - \{0\} \to M_*$ is a diffeomorphism;*

(iii) *If L is any complex line through the origin in $\mathbb{C}^m$, then $h_{|G(\Delta) \cap L}$ is a holomorphic map;*

(iv) *$h: G(\Delta) \to M$ is a biholomorphic map if and only if it is of class C^∞ at the origin.*

Proof. It is clear by construction that $\tau \circ h = \sigma$. Also if we define $Q: G(\Delta) \to \mathbb{B}(\Delta)$ by

$$Q(Z) = \frac{\rho(Z)}{\|Z\|} Z$$

then Q is a homeomorphism and $Q: G(\Delta) - \{0\} \to \mathbb{B}(\Delta) - \{0\}$ is a diffeomorphism. Also if $E: \mathbb{B}(\Delta) \to M$ is the map defined in (2.14) then $h = Q \circ E$. Thus (i) and (ii) follow from Corollary 2.7. Also since $\rho(\lambda Z) \subset |\lambda| \rho(Z)$ for all $\lambda \in \mathbb{C}$ and $Z \in \mathbb{C}^m$, we have $Q(\lambda Z) = \lambda Q(Z)$ for all $\lambda \in \mathbb{C}$ and $Z \in G(\Delta)$ and thus (iii) follows from the fact remarked at the end of Sect. 3 that E is holomorphic on each disk through the origin. Finally, since a smooth function on a complete circular domain which is holomorphic on each disk through the origin is holomorphic (cfr. [9], Lemma 8.1 for a detailed proof), (iii) implies (iv). q.e.d.

Remark. One can show in fact the the map h is differentiable at the origin (not necessarily of class C^1!) and that $dh(0) = \text{Id}$. The proof is the same as in the case of strictly convex domains (cfr. [9], Theorem 6.2).

Proposition 4.3. *The circular representation $h: G(\Delta) \to M$ is biholomorphic if and only if the Monge-Ampère foliation associated to τ is holomorphic.*

Proof. By the definition of h and (vi) of Theorem 3.4, the disks through the origin in $G(\Delta)$ are mapped by h into the leaves of the Monge-Ampère foliation induced by τ. Thus, if h is biholomorphic, the Monge-Ampère foliation on M is holomorphic. Assume now that the Monge-Ampère foliation is holomorphic. As recalled in Sect. 1 this is equivalent to the fact that the complex gradient X of τ is holomorphic on M_*. Thus X extends holomorphically on all M with $X(0)=0$. Moreover, if we regard X as a vector function with values in $\mathbb{C}^m$, we have, for instance from (2.3), that $dX(0)=\mathbf{Id}$. There exist a neighbourhood U of $0\in M$ and a holomorphic map $Q\colon U\to\mathbb{C}^m$ such that on U

$$X=\mathbf{Id}+Q. \tag{4.6}$$

Define a map $A\colon \mathbb{C}(1)\times G(\Delta)\to M$ by $A(z,Z)=h(zZ)$. Then from (4.5) and (iv) of Theorem 3.4, for $Z\neq 0$, one has $A(z,Z)=F(z\rho(Z), Z/\|Z\|)$ so that from (vi) of Theorem 3.4 we have

$$zA'(z,Z)=X(A(z,Z)). \tag{4.7}$$

For every $j\geqq 1$, there exists $A_j\colon G(\Delta)\to\mathbb{C}$ such that

$$A(z,Z)=\sum_{j=1}^{\infty} A_j(Z)\,z^j.$$

Thus from (4.6) and (4.7) we obtain

$$\begin{aligned} Q(A(z,Z)) &= zA'(z,Z)-A(z,Z)\\ &= \sum_{j=2}^{\infty}(j-1)\,A_j(Z)\,z^i \end{aligned} \tag{4.8}$$

Define $P(z,Z)=\sum_{j=1}^{\infty} A_j(Z)\,z^{j-1}$. If $Q=\sum_{k=2}^{\infty} Q_k$ is the development of Q in homogeneous vector polynomials of degree k, then there exist vector polynomials Q_{kr} such that

$$Q_k(P(z,Z))=\sum_{r=0}^{\infty} Q_{kr}(A_1(Z),\ldots,A_{r+1}(Z))\,z^r.$$

Thus

$$\begin{aligned} Q(A(z,Z)) &= \sum_{k=2}^{\infty} Q_k(P(z,Z))\,z^k\\ &= \sum_{k=2}^{\infty}\sum_{r=0}^{\infty} Q_{kr}(A_1(Z),\ldots,A_{r+1}(Z))\,z^{r+k}\\ &= \sum_{j=2}^{\infty}\Big(\sum_{j=k+r} Q_{kr}(A_1(Z),\ldots,A_{r+1}(Z))\Big)\,z^j. \end{aligned} \tag{4.9}$$

Comparing (4.8) and (4.9), we get

$$(j-1)\,A_j(Z)=\sum_{j=k+r} Q_{kr}(A_1(Z),\ldots,A_{r+1}(Z)).$$

Since $k \geqq 2$, we have $r+1 \leqq j-1$ and hence there exists a vector polynomial R_j for all $j \geqq 2$ such that

$$(j-1)\,A_j(Z) = R_j(A_1(Z), \ldots, A_{j-1}(Z)). \tag{4.10}$$

Since $A_1 = dX(0) = \mathbf{Id}$ is holomorphic, using (4.10) it follows by induction on j that A_j is holomorphic for all $j \geqq 1$. On the other hand there exists a number $\varepsilon > 0$ such that $\mathbb{B}(2\varepsilon) \subset G(\Delta)$, and $\sum_{j=1} A_j(Z)\,z^j$ converges uniformly if $\|Z\| < 2\varepsilon$ and $|z| < 1/2$. If $\|Z\| < \varepsilon$ then we have

$$h(Z) = A(1/2, 2Z) = \sum_{j=1}^{\infty} A_j(2Z)\,(1/2)^j$$

where the series converges uniformly. Therefore h is holomorphic on a neighbourhood of the origin and hence of class C^∞ there. By Proposition 4.2 it follows that in fact h is biholomorphic. q.e.d.

We can now state our main result.

Theorem 4.4. *Let (M, τ) be a manifold of circular type of radius Δ. Then*

(i) *If $\Delta = \infty$ there exists a biholomorphic map $h: \mathbb{C}^m \to M$ such that $(\tau \circ h)^{1/2}$ is the Minkowski functional of some circular domain $G \subset \mathbb{C}^m$.*

(ii) *If $\Delta < \infty$ and the Monge-Ampère foliation associated to τ is holomorphic, then there exists a circular domain $G(\Delta) \subset \mathbb{C}^m$ and a biholomorphic map $h: G(\Delta) \to M$ such that $\Delta^{-1}(\tau \circ h)^{1/2}$* is the Minkowski functional of $G(\Delta)$.

Proof. Proposition 4.2 and 4.3 yield immediately (ii). To prove (i) we have only to recall that according to a theorem of Burns [4], when $\Delta = \infty$ the Monge-Ampère foliation associated to τ is always holomorphic. q.e.d.

References

1. Bedford, E., Kalka, M.: Foliations and complex Monge-Ampère equations. Commun. Pure Appl. Math. **30**, 510–538 (1977)
2. Bedford, E., Taylor, B.A.: The Dirichlet problem for a complex Monge-Ampère equation. Invent. Math. **37**, 1–44 (1976)
3. Bedford, E., Taylor, B.A.: Variational properties of the complex Monge-Ampère equation, II Intrinsic norms., Amer. J. Math. **101**, 1131–1166 (1979)
4. Burns, D.: Curvature of the Monge-Ampère foliations and parabplic manifolds. Ann. of Math. **115**, 349–373 (1982)
5. Griffiths, P., King, J.: Nevanlinna theory and holomorphic mappings between algebraic varieties. Acta Math. **130**, 145–220 (1973)
6. Lempert, L.: La métrique de Kobayshi et la represéntation des domaines sur la boule. Bull. Soc. Math. Fr. **109**, 427–474 (1981)
7. Lempert, L.: Solving the Degenerate Complex Monge-Ampère Equation with one Concentrated Singularity. Math. Ann. **263**, 515–532 (1983)
8. Milnor, J.: Sommes de variétés différentiables et structure différentiables des spheres. Bull. Soc. Math. Fr. **87**, 439–444 (1959)
9. Partrizio, G.: Parabolic exhaustions for strictly convex domains. Manusc. Math. **47**, 271–309 (1984)

10. Patrizio, G., Wong, P.M.: Stability of the Monge-Ampère foliation. Math. Ann. **263**, 13–29 (1983)
11. Stoll, W.: Value distribution on parabolic spaces, Lecture Notes in Math. **600**. Berlin-Heidelberg-New York: Springer 1977
12. Stoll, W.: The characterization of strictly parabolic manifolds. Ann. Scuola Norm. Super. Pisa, **7**, 81–154 (1980)
13. Stone, D.A.: The exponential map at an isolated singular point. Mem. Am. Math. Soc. **256** (1982)
14. Wong, P.M.: Geometry of the Homogeneous Complex Monge-Ampère Equation. Invent. Math., **67**, 261–274 (1982)
15. Wong, P.M.: On Umbilical Hypersurfaces and Uniformization of Circular Domains, in Complex Analysis of Several Variables, Y.T. Siu ed., Symp. Pure Math. **41**, Am. Math. Soc. Providence, 1984

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