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# Monge–Ampère equations and moduli spaces of manifolds of circular type ☆

Giorgio Patrizio [a], Andrea Spiro [b,*]

[a] *Dipartimento di Matematica "U. Dini", Università di Firenze, Firenze, Italy*
[b] *Dipartimento di Matematica e Informatica, Università di Camerino, Camerino, Italy*

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**Abstract**

A (*bounded*) *manifold of circular type* is a complex manifold $M$ of dimension $n$ admitting a (bounded) exhaustive real function $u$, defined on $M$ minus a point $x_o$, so that: (a) it is a smooth solution on $M \setminus \{x_o\}$ to the Monge–Ampère equation $(dd^c u)^n = 0$; (b) $x_o$ is a singular point for $u$ of logarithmic type and $e^u$ extends smoothly on the blow up of $M$ at $x_o$; (c) $dd^c(e^u) > 0$ at any point of $M \setminus \{x_o\}$. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$.

A set of modular parameters for bounded manifolds of circular type is considered. In particular, for each biholomorphic equivalence class of them it is proved the existence of an essentially unique manifold in normal form. It is also shown that the class of normalizing maps for an $n$-dimensional manifold $M$ is a new holomorphic invariant with the following property: it is parameterized by the points of a finite dimensional real manifold of dimension $n^2$ when $M$ is a (non-convex) circular domain while it is of dimension $n^2 + 2n$ when $M$ is a strictly linearly convex domain. New characterizations of the circular domains and of the unit ball are also obtained.


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* Corresponding author.
*E-mail addresses:* patrizio@math.unifi.it (G. Patrizio), andrea.spiro@unicam.it (A. Spiro).

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## 1. Introduction

In this paper we consider the moduli spaces of a family of complex manifolds, which includes the smoothly bounded strictly linearly convex domains and the smoothly bounded strictly pseudoconvex circular domains in $\mathbb{C}^n$. For these complex manifolds, we prove the existence of normal forms and of modular parameters that generalize those introduced by Bland and Duchamp in [6].

More precisely, we consider a larger class of manifolds, called *(bounded) manifolds of circular type*, which naturally includes the previous two families of domains and are characterized by the property of admitting a (bounded) exhaustive real function $u$, defined on $M$ minus a point $x_o$, so that: (a) it is a smooth solution on $M \setminus \{x_o\}$ to the Monge–Ampère equation $(dd^c u)^n = 0$; (b) $x_o$ is a singular point for $u$ of logarithmic type and $e^u$ extends smoothly on the blow up of $M$ at $x_o$; (c) $dd^c(e^u) > 0$ at any point of $M \setminus \{x_o\}$.

In any biholomorphic equivalence class of such domains, we prove the existence of an essentially unique *manifold in normal form*, consisting of the unit ball $B^n$ together with a non-standard complex structure $J$, which satisfies some suitable conditions. One of them consists of requiring that the non-standard CR structures induced on the spheres $S^{2n-1}(r) = \{|z| = r\}$, $0 < r < 1$, have the same real distribution of the standard ones as underlying distribution of $J$-invariant real subspaces. The other conditions imply that $J$ is uniquely determined by only one of such CR structures. This CR structure is completely determined by a sequence $\{\phi_k\}_{k=0}^{\infty}$ of $(1,1)$-tensor fields on $\mathbb{C}P^{n-1} \simeq S^{2n-1}(r_o)/S^1$, obtained by expanding in Fourier series the tensor field $\phi$ that gives the complex structure of the CR structure as a deformation of the standard one. As applications, we use these results to obtain new characterizations of the circular domains in $\mathbb{C}^n$ and of the unit ball.

The normal forms considered in this paper are essentially the same of the normal forms constructed in [5,7] by Bland and Duchamp only for domains that are small deformations of the unit ball. The major improvement consists in showing that *such normal forms exist for any bounded manifold of circular type*, i.e. for any complex manifold which admits a solution to the described Monge–Ampère differential problem. Such result has been obtained by methods and techniques that are substantially different from those of [5] and [7].

We should also mention that our normal forms can be also considered as alternative to the normal forms constructed by Lempert and Bland and Duchamp in [17,6], where equivalence classes of pointed strictly linearly convex domains were studied; furthermore we complete the description of the moduli giving an explicit treatment of the dependence of the invariants of the domains on the choice of the distinguished point (see Section 5).

In many regards, the properties of normal forms of manifolds of circular type recall those of the well-known Chern–Moser normal forms for Levi non-degenerate real hypersurfaces [10]. For instance, if $D$ is a domain of circular type in a Stein manifold, it turns out that the class $\mathcal{N}(D)$ of the diffeomorphisms $f : D \to B^n$, which map $D$ into a normal form $(B^n, J)$, is naturally parameterized by a subset of the automorphism group $\mathrm{Aut}(B^n, J_o)$ of the standard unit ball $(B^n, J_o)$. As for Chern–Moser normal forms, this fact determines a natural embedding of $\mathrm{Aut}(D)$ as a subgroup of $\mathrm{Aut}(B^n, J_o)$.

On the other hand, in contrast with what occurs for Chern–Moser normal forms, the parameter set for the class $\mathcal{N}(D)$ is *not* independent of $D$ and it represents an important biholomorphic invariant for the domain (we recall that, on the contrary, the Chern–Moser normalizing maps are always parameterized by the isotropy $\mathrm{Aut}_{x_o}(B^n, J_o)$ of a fixed boundary point $x_o \in \partial B^n$).

For example, if $D$ is a smoothly bounded, strictly linearly convex domain in $\mathbb{C}^n$, then $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$, while if $D$ is a generic (non-convex) strongly pseudoconvex circular domain, then

$\mathcal{N}(D) \simeq U_n \subsetneq \mathrm{Aut}(B_n, J_o)$. These facts motivate the following question: *Is it true that a domain of circular type $D$ is biholomorphic to a smoothly bounded, strictly linearly convex domains in $\mathbb{C}^n$ if and only if $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$?*

We conjecture that the answer is "yes", at least when $n \geqslant 3$ and some boundary regularity conditions are assumed. However, besides such conjecture, there is another reason of interest for the domains of circular type for which $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$, namely the abundance of solutions to the quoted Monge–Ampère problem that there exists on any such domain (there is at least one such solution for any point $x_o \in D$—see remarks after Theorem 5.5). In addition, any such domain is endowed with a biholomorphically invariant complex Finsler metric, namely the infinitesimal Kobayashi metric (e.g. [12,21,25,1,27]).

Some applications of the theory of normal forms we developed here can be found in the last section. We obtain f.i. the following result (Theorem 5.9): *a bounded manifold of circular type $M$, with $u : M \setminus \{x_o\} \to [0, r^2)$ satisfying* (a), (b), (c), *is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if there exists at least two subdomains $M_{<c} = \{u < c\}$, $M_{<c'} = \{u < c'\}$, $0 < c < c' < r^2$, which are biholomorphic one to the other by a map fixing $x_o$*. This theorem represents a generalization to any manifold of circular type of results in [18] and [1], originally proved only for smoothly bounded, strictly linearly convex domains in $\mathbb{C}^n$ or complex Finsler manifolds. Such generalization is obtained by an approach which is quite different from the ones in [18] and [1]. By a result in [23], our Theorem 5.9 has also an immediate corollary which gives a new characterization of the unit ball (Corollary 5.11).

The structure of the paper is as follows. Section 2 is devoted to preliminaries and to the so-called *circular representation* of a manifold of circular type, a generalization of the standard circular representation of strictly linearly convex domains (see [23,6]). In Section 3, the normal forms of domains of circular type is defined and the existence of normalizing maps is proved. In Section 4, it is shown that the complex structure of a manifold in normal form is completely determined by the associated "deformation tensor" $\phi$. In the same section, the Bland and Duchamp invariants $\phi^{(k)}$, determined by Fourier series expansion of $\phi$, are defined. In Section 5 we give a geometrical interpretation of the Bland and Duchamp invariant $\phi^{(0)}$, we establish the parameterization by elements of $\mathrm{Aut}(B^n)$ of the family of normalizing maps of a manifold of circular type and we prove the mentioned characterizations of circular domains and of the unit ball.

## 2. Preliminaries

### *2.1. Notation, first definitions and some basic properties*

In all the following, $J_o$ is the standard complex structure of $\mathbb{C}^n$, $B^n \subset \mathbb{C}^n$ the unit ball and $\Delta = B^1$ the unit disc in $\mathbb{C}$. A CR structure $\mathcal{D}^{1,0} \subset T^{\mathbb{C}}M$ on a manifold $M$ will be often indicated by the associated pair $(\mathcal{D}, J)$ of the underlying real distribution $\mathcal{D} \subset TM$ and the family of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$.

We recall that a CR structure $(\mathcal{D}, J)$ of hypersurface type (i.e. with $\operatorname{codim}\mathcal{D} = 1$) is *Levi non-degenerate* if and only if the underlying real distribution $\mathcal{D}$ is a *contact distribution* i.e. so that for any 1-form $\theta$ with $\operatorname{Ker}\theta = \mathcal{D}$, one has that $d\theta_x|_{\mathcal{D}_x \times \mathcal{D}_x}$ is non-degenerate. This definition is completely equivalent to the classical notion of "Levi non-degeneracy". For any Levi non-degenerate real hypersurface $S \subset \mathbb{C}^n$, the induced CR structure $(\mathcal{D}_o, J_o)$ will be called *standard CR structure* and $\mathcal{D}_o$ *standard contact distribution*.

We recall that a domain $D \subset \mathbb{C}^n$ is called *circular* if it is invariant under any transformation $z \mapsto e^{i\theta} \cdot z$, for all $\theta \in \mathbb{R}$, while it is called *circular and complete* if it is invariant under all transformations $z \mapsto \zeta \cdot z$ for all $\zeta \in \bar{\Delta}$. The *Minkowski function* $\mu_D : \mathbb{C}^{n+1} \to \mathbb{R}_{\geqslant 0}$ of a complete circular domain is defined by

$$\mu_D(z) = \frac{1}{t_z}, \quad \text{where } t_z = \sup\{s \in \mathbb{R}:\ s \cdot z \in D\}. \tag{2.1}$$

For instance, $\mu_{B^n}(z) = |z|$. Notice that, $\mu_D(\zeta \cdot z) = |\zeta|\mu_D(z)$ and, if $D$ is smoothly bounded, the function $\tau = \mu_D^2 - 1$ is a defining function for $D$ which is smooth on $\mathbb{C}^{n+1} \setminus \{0\}$. In this case, $D$ is strictly linearly convex if and only if $\tau = \mu_D^2 - 1$ has strictly positive Hessian at any $x \neq 0$.

Given a complete circular domain $D$ with smooth boundary and $v \in T_0\mathbb{C}^n = \mathbb{C}^n$, we call *standard radial disc tangent to* $v$ the map $f_v^{(\mu)} : \bar{\Delta} \to D$ defined by $f_v^{(\mu)}(\zeta) = \zeta \frac{v}{\mu_D(v)}$. Any standard radial disc is indeed a *stationary disc for* $D$ (see [15,22]; for stationary discs of hypersurfaces, see also [30]).

We denote by $\tilde{\pi} : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ and by $\tilde{\pi} : \widetilde{D} \to D$ the blow ups at the origin of $\mathbb{C}^n$ and $D \subset \mathbb{C}^n$, respectively. In fact, $\widetilde{D} = \tilde{\pi}^{-1}(D) \subset \widetilde{\mathbb{C}}^n$. We recall that $\widetilde{\mathbb{C}}^n$ is the total space of the tautological line bundle $\pi : \widetilde{\mathbb{C}}^n = E \to \mathbb{C}P^{n-1}$ and that the exceptional divisor of the blow up coincides with the image of the zero section of $\pi : E \to \mathbb{C}P^{n-1}$. For any point $([v], z)$ of $\pi^{-1}(\{z^i \neq 0\}) \subset E$, we denote by $(v^1, \ldots, v^{n-1}, \zeta)$ the complex coordinates defined by

$$\big([v], z\big) = \Big(\big[v^1, \ldots, \underset{i\text{-th place}}{1}, \ldots, v^{n-1}\big], \zeta \cdot v^1, \ldots, \underset{i\text{-th place}}{\zeta}, \ldots, \zeta \cdot v^{n-1}\Big). \tag{2.2}$$

Any standard radial disc $f_v : \Delta \to B^n$ admits a unique lifted map $\tilde{f}_v : \Delta \to \overline{\widetilde{B}^n}$ with $\pi \circ \tilde{f}_v = f_v$. It can be checked that any map $\tilde{f}_v$ is a stationary disc for the sphere $S^{2n-1}$, considered as real hypersurface in $\widetilde{\mathbb{C}}^n$, and the image is the fiber $\pi^{-1}([v]) \subset \widetilde{B}^n$. In particular, the images of the standard radial discs of $\widetilde{B}^n$ determine a holomorphic foliation. The same claim holds for the lifts in $\widetilde{\mathbb{C}}^n$ of the radial discs $f_v^{(\mu)}$ of a complete circular domain $D$.

### 2.2. *Manifolds of circular type, Monge–Ampère foliations, indicatrices*

We recall now the definition of "manifolds of circular type" introduced in [24] and a few other related concepts, essential for studying the moduli of such manifolds. For a function $\tau : M \to \mathbb{R}$, we denote $M_{\tau=0} = \tau^{-1}(0)$ and $M_{\tau\neq 0} = M \setminus M_{\tau=0}$.

**Definition 2.1.** We say that a non-compact complex manifold $M$ of complex dimension $n$ is a *manifold of circular type* if it admits an exhaustion function $\tau : M \to [0, r^2)$, for some $r^2 \in (0, \infty]$, such that $M_{\tau=0} = \{x_o\}$ and

(a) $\tau \in \mathcal{C}^0(M) \cap \mathcal{C}^\infty(M_{\tau\neq 0})$;
(b) on $M_{\tau\neq 0}$, $\tau$ is so that $dd^c\tau > 0$ and $dd^c \log\tau \geqslant 0$;
(c) on $M_{\tau\neq 0}$, $(dd^c \log\tau)^n \equiv 0$;
(d) if $\pi : \widetilde{M} \to M$ is the blow up of $M$ at $x_o$, then $\tau \circ \pi : \widetilde{M} \to \mathbb{R}$ is smooth and, in any coordinate system centered at $x_o$, there exist two positive constant $C_1, C_2$ so that $C_1\|x - x_o\|^2 \leqslant \tau(x) \leqslant C_2\|x - x_o\|^2$ for $x$ near $x_o$.

The point $x_o$ is called *center of $M$ w.r.t.* $\tau$ and $\tau : M \to [0, r^2)$ is called *parabolic exhaustion*. If $r < \infty$ (i.e. $\tau$ is bounded), we call $M$ *bounded*. We denote a manifold of circular type with parabolic exhaustion $\tau$ with $(M, \tau)$ and we use the notation $(M, J, \tau)$ if we need to indicate also the complex structure $J$.

A manifold of circular type is called *domain of circular type* if it is a domain $D$ in a complex manifold, admitting one parabolic exhaustion $\tau$, which is smooth up to the boundary and satisfying (b) also on $\partial D$. For such domains, we consider only these exhaustions and by "parabolic exhaustions" we will always mean such functions.

Any strictly pseudoconvex, complete circular domain $D \subset \mathbb{C}^n$ is a domain of circular type with center $x_o = 0$ and parabolic exhaustion $\tau = \mu_D^2$. Other examples are given by all bounded, strictly linearly convex domain $D \subset \mathbb{C}^n$ with smooth boundary (see [15,16,24]). In fact, if $\kappa_D$ is the Kobayashi pseudo-distance of $D$, for any point $x_o \in D$, the function $\tau_{x_o}(x) = \tanh^2(\kappa_D(x, x_o))$ is a parabolic exhaustion, smooth up to the boundary. In particular, *any point $x_o$ of a smoothly bounded, strictly linearly convex domain $D$ is a center for $D$.*

Let $(M, J, \tau)$ be a bounded manifold of circular type with center $x_o \in M$ and $\widetilde{M}$ the blow up of $M$ at $x_o$. It is well known (see e.g. [2,29,24,31]) that the distribution $\mathcal{Z}$ defined by

$$\mathcal{Z}_x = \ker dd^c \log \tau|_x \tag{2.3}$$

is a $J$-invariant integrable distribution on $M \setminus \{x_o\}$, generated over the reals by $Z$ and $JZ$, where $Z$ is the vector field defined by the condition

$$dd^c\tau(Z, JX) = X(\tau) \quad \text{for any } X \in T\big(M \setminus \{x_o\}\big). \tag{2.4}$$

The associated foliation is usually called *Monge–Ampère foliation*. Using the Monge–Ampère equation, one can check that

$$dd^c\tau(Z, JZ)|_x = d\tau(Z)|_x = \tau(x). \tag{2.5}$$

We also consider the $J$-invariant distribution $\mathcal{H}$ on $M \setminus \{x_o\}$, called *normal distribution to the Monge–Ampère foliation*, defined by

$$\mathcal{H}_x = (\mathcal{Z}_x)^\perp \stackrel{\text{def}}{=} \big\{X \in T_xM\colon\ dd^c\tau(Z, X) = dd^c\tau(JZ, X) = 0\big\}. \tag{2.6}$$

Clearly $T_xM = \mathcal{Z}_x \oplus \mathcal{H}_x$ and by (2.6), (2.4) and $J$-invariance, $\mathcal{H}_x$ coincides with the holomorphic tangent space to $S = \{y\colon\ \tau(y) = \tau(x)\}$. In particular, for any $c > 0$, *the pair $(\mathcal{H}|_{S_c}, J)$ is the CR structure of the level hypersurface $S_c = \{\tau = c\}$.*

The distributions $\mathcal{Z}$ and $\mathcal{H}$ are defined just on $M \setminus \{x_o\}$ but *extend as smooth $J$-invariant distributions on the blow up $\widetilde{M}$* and the extension depends uniquely on the real manifold structure of $\widetilde{M}$. The distribution $\mathcal{Z}$ is integrable everywhere and, after identification of a neighborhood of $\pi^{-1}(x_o) \subset \widetilde{M}$ with an open neighborhood of $\mathbb{C}P^{n-1}$ in $E = \widetilde{\mathbb{C}}^n$, the integral leaves of $\mathcal{Z}$ coincide with the fibers of the projection $\pi : E \to \mathbb{C}P^{n-1}$. On the other hand, $\mathcal{H}$ is not integrable, except only when restricted on $\mathbb{C}P^{n-1} = \pi^{-1}(x_o)$, where $\mathcal{H}|_{\mathbb{C}P^{n-1}} = T\mathbb{C}P^{n-1}$.

The smooth extendibility of $\mathcal{Z}$ follows directly from (2.7) and (2.10) of [24], written in a system of coordinates $(v^1, \dots, v^{n-1}, \zeta)$ as described in (2.2). For what concerns $\mathcal{H}$, standard computations on Monge–Ampère equations (see e.g. [31]) show that $\mathcal{H}$ is preserved by the flows of $Z$ and $JZ$. This allows to extend smoothly $\mathcal{H}$ on $\mathbb{C}P^{n-1} = \pi^{-1}(x_o)$ by setting $\mathcal{H}|_{\mathbb{C}P^{n-1}} = T\mathbb{C}P^{n-1}$.

Standard considerations (see f.i. [20]) imply that any manifold of circular type is a Stein manifold. Furthermore if $\tau$, $\tau'$ are two parabolic exhaustions of a domain of circular type $D$ with the same center $x_o$, then $D$ is a hyperconvex domain and, as a consequence of Theorem 4.3 in [11], there exists a unique plurisubharmonic function $u$ so that $u|_{\partial D} = 0$, $(dd^c u)^n \equiv 0$ and $u(z) \sim \log|z - x_o|$ for $z \to x_o$. This implies that $\log\tau = u + c$ and $\log\tau' = u + c'$ where $c$ and $c'$ are the constant values $c = \log\tau|_{\partial D}$, $c' = \log\tau|_{\partial D}$. From this we get the following

**Proposition 2.2.** (i) *If $D$ is a domain of circular type and $\tau, \tau'$ are two parabolic exhaustions with the same center, then $\tau' = k\tau$ for some constant $k > 0$.*

(ii) *If $M$ is a manifold of circular type and $\tau, \tau'$ are parabolic exhaustions with the same center with $\{\tau = c\} = \{\tau' = c'\}$ for some $c, c'$, then $\tau' = k\tau$, for some $k > 0$.*

The first claim is direct consequence of the uniqueness property mentioned above. The second claim follows from the fact that by (i) we have that $\tau|_D = k\tau'|_D$ on $D = \{\tau < c\} = \{\tau' < c'\}$ and from the uniqueness of $\tau$ along the leaves of the Monge–Ampère foliation.

We now give the following technical lemma, needed later. Let $(M, J, \tau)$ be a bounded manifold of circular type and $\pi : \widetilde{M} \to M$ the blow up at the center $x_o$. Let also $(J', \tau')$ another pair so that also $(M, J', \tau')$ is a domain of circular type with the same center $x_o$ and let $\pi : \widetilde{M}' \to M$ the new blow up at $x_o$. Then, we have the following:

**Lemma 2.3.** *If $(J, \tau)$ and $(J', \tau')$ determine the same vector field $Z$ via* (2.4)*, then $\widetilde{M} = \widetilde{M}'$ (as real differentiable manifolds), $\tau = \tau'$ and the normal distributions of the Monge–Ampère foliations $\mathcal{H} = \mathcal{H}'$ of $(M, J, \tau)$ and $(M, J', \tau')$ are the same. In particular, the CR structures induced by $J$ and $J'$ on any hypersurface $S_c = \{\tau = \tau' = c\}$ have the same underlying real distributions.*

**Proof.** Consider the identity map between $\widetilde{M} \setminus \pi^{-1}(x_o) = M \setminus \{x_o\} = \widetilde{M}' \setminus \pi'^{-1}(x_o)$ and extend it continuously along each integral curve of $Z$. From the uniqueness of the extension of $\mathcal{Z}$, such map is unique and smooth w.r.t. the real manifold structures of $\widetilde{M}$ and $\widetilde{M}'$ and shows that $\widetilde{M} = \widetilde{M}'$ as differentiable manifolds. On the other hand, $\tau|_{\pi^{-1}(x_o)} = \tau'|_{\pi^{-1}(x_o)} = 0$ and hence, by (2.5), for any $x$ of the form $x = \Phi_t^Z(y)$, $y \in \pi^{-1}(x_o)$, where $\Phi_t^Z$ is the flow of $Z$, we have $\tau(x) = \tau'(x)$. Finally, as $\mathcal{H}_y = \mathcal{H}'_y = T_y\mathbb{C}P^{n-1}$ for $y \in \mathbb{C}P^{n-1} = \pi^{-1}(x_o)$, the invariance of $\mathcal{H}, \mathcal{H}'$ under the flow of $Z$ implies as before that $\mathcal{H}_x = \mathcal{H}'_x$ at any $x$.

*2.3. Circular representations*

Here, we illustrate that any manifold of circular type admits a *circular representation* as it is known for the strictly linearly convex domains. For a bounded manifold of circular type $(M, \tau)$, with center $x_o$, consider a system of complex coordinates $z = (z^1, \ldots, z^n)$, around $x_o$, with $z(x_o) = 0$. By Lemma 2.1 in [24], there exists a smooth map $h : S^{2n-1} \to \mathbb{R}_{>0}$ such that $\sqrt{\tau(z)} = |z| h(\frac{z}{|z|}) + o(|z|^2)$. We set

$$\kappa : T_{x_o}M \simeq \mathbb{C}^n \to \mathbb{R}_{\geqslant 0}, \qquad \kappa(v) = \begin{cases} |v| \cdot h(\frac{v}{|v|}) & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases} \tag{2.7}$$

It can be immediately checked that $\kappa(v) = \lim_{t_o \to 0} \frac{d}{dt}\sqrt{\tau(\gamma_t)}|_{t=t_o}$ for any smooth curve $\gamma_t$ with $\gamma_0 = x_o$ and $\dot{\gamma}_0 = v$ and hence it is independent of the coordinates. Furthermore, $\kappa$ is a smooth function on $T_{x_o}M \setminus \{0\}$ and satisfies $\kappa(\lambda v) = |\lambda|\kappa(v)$ for any $\lambda \in \mathbb{C}$.

**Definition 2.4.** We call *indicatrix of $M$ at the center $x_o$ determined by $\tau$* the smoothly bounded, complete circular domain $I_{x_o} \subset T_{x_o}M$ with $\kappa$ as Minkowski function, i.e.

$$I_{x_o} = \{v \in T_{x_o}M \colon \kappa(v) < 1\}. \tag{2.8}$$

**Remark 2.5.** Notice that by [24], Proposition 4.1, $dd^c\kappa^2 > 0$ at all points of $\bar{I}_{x_o} \setminus \{0\}$ and hence $I_{x_o}$ is strongly pseudoconvex. Furthermore, using the arguments of the proof of the main result in [25], it follows that $\kappa$ is the Kobayashi metric of $M$ at $x_o$ and $I_{x_o}$ is the Kobayashi indicatrix of $M$ at $x_o$.

In the next proposition, we collect a few well-known properties, usually stated for strictly linearly convex domains (see e.g. [6,8,15,17,22,23]), but actually true for all domains of circular type.

**Proposition 2.6.** *Let $(M, \tau)$ be a bounded manifold of circular type with $\tau : M \to [0, 1)$, and $I \subset T_{x_o}M$ the indicatrix determined by $\tau$. Then there exists a unique diffeomorphism $\Psi : \tilde{I} \to \widetilde{M}$ with the following properties:*

(i) $\Psi|_{\pi^{-1}(0)} = Id_{\pi^{-1}(0)}$, *provided that we identify the exceptional divisor $\pi^{-1}(0)$ of $\widetilde{T_{x_o}M}$ with the exceptional divisor $\pi^{-1}(x_o)$ of $\widetilde{M}$;*
(ii) *for any $t \in (0, 1)$, the map $\Phi^{(t)} : \partial\tilde{I} \to \widetilde{M}$, defined by $\Phi^{(t)}([v], z) = \Psi([v], tz)$ is so that $\Phi^{(t)}|_{\partial\tilde{I}}$ is a diffeomorphism between $\partial\tilde{I}$ and the level hypersurface $S^{(t)} = \{\tau = t^2\}$;*
(iii) *for any $t \in (0, 1)$, $\Phi^{(t)}|_{\partial\tilde{I}}$ maps the real distribution of the CR structure of $\partial\tilde{I}$ onto the real distribution of the CR structure of $S^{(t)}$;*
(iv) *for any $([v], z) \in \partial\tilde{I}$ and $t \in ]0, 1[$, the map $\tilde{f}^{(t)}_{([v],z)} : \Delta \to \widetilde{M}$, defined by $\tilde{f}^{(t)}_{([v],z)}(\zeta) = \Psi([v], t\zeta z)$, is proper holomorphic and injective, and so that $\tilde{f}^{(t)}_{([v],z)}(\partial\Delta) \subset S^{(t)}$ and $\tilde{f}^{(t)}_{([v],z)}(\Delta)$ is equal to an integral leaf of the Monge–Ampère foliation of $\widetilde{M}_{<t^2} = \{\tilde{\tau} < t^2\}$.*

*In case $M$ is a domain of circular type, then $\Psi$ extends smoothly up to the boundary with $\Psi(\partial\tilde{I}) \subset \partial\widetilde{M}$ and* (iii), (iv) *are valid also for $t = 1$.*

The proof of (i)–(iv) is indeed contained in [24]. The last claim can be checked observing that, by the properness of the maps $\tilde{f}^{(t)}_{([v],z)}$, one can infer that $\Psi$ extends smoothly up to the boundary with $\Psi(\partial\tilde{I}) = \partial\widetilde{M}$. Properties (iii) and (iv) when $t = 1$ are obtained with the same arguments used for $t < 1$ in [24].

The diffeomorphism $\Psi : \tilde{I} \to \widetilde{M}$ will be called *circular representation of $M$ associated with $\tau$*. It should be viewed as the lift at the level of the blow ups of the circular representation defined in [24]. When $M$ is a smoothly bounded, strictly linearly convex domain in $\mathbb{C}^n$, $\Psi$ is the circular representation considered in [8].

## 3. Normal forms for manifolds of circular type

In the following, we insert the tilde "$\sim$" on top of symbols of manifolds, domains or maps, when we indicate blow ups or lifts of maps on blow ups, and $\widetilde{B}^n$ is the blow up of $B^n$ at 0. Also $\tau_o : \mathbb{C}^n \to [0, \infty)$ is the standard exhaustion $\tau_o(z) = |z|^2$.

### 3.1. Complex structures of Lempert type and manifolds in normal form

**Definition 3.1.** Let $I \subset \mathbb{C}^n$ be a strictly pseudoconvex complete circular domain with Minkowski function $\mu$, and let $\pi : \tilde{I} \to I$ be the blow up of $I$ at 0 and $\tilde{\mu} = \mu \circ \pi$. A complex structure $J$ on $\tilde{I}$ is called *of Lempert type* if

(a) on any hypersurface $S_c = \{z \in \tilde{I}\colon \tilde{\mu}(z) = c\}$, $0 < c < 1$, the distribution $\mathcal{D}$ of the CR structure $(\mathcal{D}, J)$ coincides with the distribution $\mathcal{D}_o$ of the standard CR structure $(\mathcal{D}_o, J_o)$;
(b) the projection $\pi : \tilde{I} \to I$ induces a complex manifold structure on $I \setminus \{0\}$, whose charts are smoothly overlapping with the charts of the standard manifold structure of $\mathbb{C}^n \setminus \{0\}$, i.e. the projected complex structure is given by a smooth tensor field $J$ of type $(1, 1)$ on $I \setminus \{0\}$;
(c) the restriction of $J$ on any tangent space of a standard radial disc of $I$ preserves that tangent space and it coincides with the standard complex structure $J_o$.

Notice that, from (a), the level hypersurfaces $S_c$ of the function $\tilde{\tau} \overset{\text{def}}{=} \mu^2 \circ \pi$ are Levi non-degenerate. When they are strictly pseudoconvex (for instance, when $J$ is L-isotopic to $J_o$; see next definition), $\tilde{\tau}$ is indeed plurisubharmonic on $\tilde{I}$ and strictly plurisubharmonic on the complement of the exceptional set. By Narasimhan's result in [20], if $I$ is endowed with a suitable complex manifold structure, this implies that $\tilde{I}$ is a proper modification of $I$. Such complex structure coincides with the one described in (b) on $I \setminus \{0\}$, but *need not* to smoothly overlap with the standard complex structure at 0.

We call such complex structure the *projected complex structure of Lempert type on $I$* and it will be indicated by the associated tensor field $J$, even if such tensor is a smooth tensor w.r.t. the standard coordinates of $\mathbb{C}^n$ only at the points of $I \setminus \{0\}$.

Two complex structures $J$ and $J'$ of Lempert type are called *Lempert isotopic* (or, shortly, *L-isotopic*) if there exists a smooth family $J_t$, $t \in [0, 1]$, of complex structures of Lempert type on $\tilde{I}$, such that $J_0 = J$ and $J_1 = J'$.

**Theorem 3.2.** *Any complex structure $J$ on $\widetilde{B}^n$, which is of Lempert type and L-isotopic to $J_o$, projects onto a non-standard complex manifold structure $J$ on $B^n$ which makes $(B^n, J, \tau_o)$ a bounded manifold of circular type.*

**Proof.** Since $(B^n, J_o, \tau_o)$ is a domain of circular type, we only need to prove that conditions (b) and (c) of Definition 2.1 are still true after replacing $J_o$ with $J$. Let $\mathcal{Z}_o$ and $\mathcal{H}_o$ be the tangent and normal distributions of the Monge–Ampère foliation of $\widetilde{B}^n$ determined by $(J_o, \tau_o)$ and $Z$ the vector field defined in (2.4). By (a) of Definition 3.1, the distribution $\mathcal{H}_o$ is $J$-invariant. Let $d^{c'} = -\frac{1}{4\pi} J^{-1} \circ d \circ J$ be the $J$-twisted differential (see e.g. [3, p. 68]). Now, $dd^{c'}\tau_o|_{\mathcal{Z}_o \times \mathcal{Z}_o} = dd^c\tau_o|_{\mathcal{Z}_o \times \mathcal{Z}_o} > 0$, because $(B^n, J_o, \tau_o)$ satisfies (b) of Definition 2.1 and $J|_{\mathcal{Z}_o} = J_o|_{\mathcal{Z}_o}$. On the other hand, recall that $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o}$ coincides with the Levi form (w.r.t. $J$) of the hypersurfaces $\{\tau_o = const.\}$. Since $\mathcal{H}_o$ is also the real distribution underlying the CR structure induced by $J_o$ and the hypersurfaces $\{\tau_o = const.\}$ are strongly pseudo-convex (they are spheres), $\mathcal{H}_o$ is a contact distribution over each such hypersurface (see Section 2.1). This implies that, at any point, $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o}$ is a *non-degenerate* $J$-Hermitian form. The same claim is true for all complex structures $J_t$ of an L-isotopy between $J$ and $J_o$. A trivial continuity argument implies that $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o} > 0$. Moreover, for any vector field $X$ in $\mathcal{H}_o$,

$$dd^{c'}\tau_o(Z, X) = \frac{1}{4\pi}\big\{Z\big(JX(\tau_o)\big) - X\big(JZ(\tau_o)\big) - J\big([X, Z]\big)(\tau_o)\big\} = 0, \tag{3.1}$$

since $\mathcal{H}_o$ is preserved by the flow of $Z$ and, by (b) of Definition 3.1, $JZ(= J_oZ)$ and $JX$ are both tangent to the level hypersurfaces of $\tau_o$. From this we conclude that $dd^{c'}\tau_o > 0$. Since $\tau_o^2 dd^{c'}\log\tau_o = \tau_o dd^{c'}\tau_o - d\tau_o \wedge d^{c'}\tau_o$ and the structures $J_o$ and $J$ coincide along $\mathcal{Z}$, one has $dd^{c'}\log\tau_o|_{\mathcal{Z}\times\mathcal{Z}} = 0$, so that $dd^{c'}\log\tau_o \geqslant 0$ and $(dd^{c'}\log\tau_o)^n = 0$, i.e. (b) and (c) of Definition 2.1 are true also when $J_o$ is replaced by $J$. □

**Definition 3.3.** We call *manifold of circular type in normal form* any bounded manifold of circular type of the form $(B^n, J, \tau_o)$, where $\tau_o$ is the standard exhaustion $\tau_o = |\cdot|^2$ and $J$ is a complex structure of Lempert type that is L-isotopic to $J_o$.

For a bounded manifold of circular type $(M, J, \tau)$ with center $x_o$, a biholomorphism $\Phi : \widetilde{M} \to \widetilde{B}^n$ between the blow up $(\widetilde{M}, J)$ at $x_o$ and the blow up $(\widetilde{B}^n, J')$ of a manifold in normal form $(B^n, J', \tau_o)$ is called *normalizing map w.r.t. $\tau$ and $x_o$* if:

(a) $\Phi$ induces a diffeomorphism between the exceptional divisors;
(b) $\tilde{\tau} = \tilde{\tau}_o \circ \Phi$, where $\tilde{\tau}$ and $\tilde{\tau}_o$ are the lifts of $\tau$ and $\tau_o$ at the blow ups.

### *3.2. Existence and uniqueness of normalizing maps*

**Theorem 3.4.** *Let $(M, J, \tau)$ be a bounded manifold of circular type, with $x_o$ center of $M$ associated with $\tau$ and blow up $\pi : \widetilde{M} \to M$ at $x_o$. Then, there exists at least one normalizing map $\Phi : (\widetilde{M}, J) \to (\widetilde{B}, J')$ relative to $\tau$ and $x_o$. Moreover, any two normalizing maps $\Phi$ and $\Phi'$, both relative to $\tau$ and $x_o$, are equal if and only if $(\Phi' \circ \Phi^{-1})|_{\pi^{-1}(x_o)} = Id$ and $(\Phi' \circ \Phi^{-1})_*$ induces the identity map on the tangent spaces of the leaves of the Monge–Ampère foliation at the points of $\pi^{-1}(x_o)$.*

*If $(M, J, \tau)$ is a domain of circular type and $\tau$ extends up to the boundary, then there exists a normalizing map which extends smoothly up to the boundary.*

**Proof.** Fix a complex basis $(e_0, \dots, e_{n-1})$ for $T_{x_o}M$ and consider the unique isomorphism of complex vector spaces $\imath : T_{x_o}M \to \mathbb{C}^n$ which maps each vector $e_i$ into the corresponding vector of the standard basis $e_i^o = \imath(e_i)$ of $\mathbb{C}^n$. In what follows, we constantly identify $T_{x_o}M$ with $\mathbb{C}^n$ by means of such isomorphism. In particular, use such isomorphism in order to identify the indicatrix $I \subset T_{x_o}M$ associated with $\tau$ with the corresponding circular domain $I \subset \mathbb{C}^n$ with Minkowski function $\kappa$.

Let $\Psi : \tilde{I} \subset \widetilde{\mathbb{C}}^n \to \widetilde{M}$ be the circular representation associated with $\tau$ and consider the complex structure $J_M = \Psi_*^{-1}(J)$ on $\tilde{I}$. Keep in mind that, by (i) of Proposition 2.6 and definition of $J_M$, the blow up at the origin of $(I, J_M)$ is precisely the blow up at the origin of $(I, J_o)$ and that $J_M|_{\mathbb{C}P^n} = J_o|_{\mathbb{C}P^n}$. Moreover, by construction of $\Psi$ (it is the map $\Psi(v) = F(\frac{v}{|v|}, \kappa(v))$, where $F$ is the map defined in Theorem 3.4 in [24]) and by property (d) in the proof of Proposition 2.6, it follows immediately that $\tau \circ \Psi = \kappa^2$ and hence that $(I, J_M, \kappa^2)$ is a bounded manifold of circular type. Moreover, using the proof of (iii) in Proposition 2.6 and by (iv) of the same proposition, it is quite direct to check that $J_M$ satisfies all three conditions for being a complex structure of Lempert type on $I$. We claim that $J_M$ is also L-isotopic to $J_o$. For this, it suffices to consider the diffeomorphisms

$$\Psi^{(t)} : \tilde{I} \to \widetilde{M}_{<t} = \big\{\tau < t^2\big\}, \qquad \Psi^{(t)}\big([v], z\big) = \Psi\big([v], tz\big), \quad 0 < t < 1,$$

and the complex structures $J^{(t)} = \Psi_*^{(t)-1}(J)$ on $\tilde{I}$. The same arguments of before show that each complex structure $J^{(t)}$ is of Lempert type. If we set $J^{(0)} = J_o$ and $J^{(1)} = J_M$, using explicit coordinate expressions for the maps $\Psi$ and $\Psi^{(t)}$, it can be checked that $J^{(t)}$, $t \in [0, 1]$, is a smooth family of complex structures even at $t = 0$ and $t = 1$, proving that $J_M$ is L-isotopic to $J_o$.

From these remarks, the proof can be done assuming that $M$ is a smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$, the parabolic exhaustion $\tau$ is equal to the $\tau = \mu^2$ where $\mu$ is the Minkowski function of $I$, and $J$ is a complex structure of Lempert type on $M = I$, which is L-isotopic to the standard one and so that the blow up of $(I, J)$ at the origin coincides with the blow up of $(I, J_o)$ and with $J|_{\mathbb{C}P^n} = J_o|_{\mathbb{C}P^n}$.

Now, notice that if we replace $J$ by $J_o$, then $(I, J_o, \mu^2)$ remains a manifold of circular type. We claim that if $\Phi : (\tilde{I}, J_o) \to (\widetilde{B}^n, J')$ is a normalizing map for $(I, J_o, \mu^2)$ relative to $\mu^2$ and center $x_o = 0$, then it is also a normalizing map also for $(I, J, \mu^2)$. In fact, it is enough to consider the complex structure $J'' = \Phi_*(J_o)$ on $\widetilde{B}^n$ and observe that:

(1) $\Phi$ is $(J, J'')$-biholomorphic map by construction;
(2) $\tilde{\tau}_o \circ \Phi = \mu^2 \circ \pi$ because $\Phi$ is a normalizing map for $(I, J_o, \mu^2)$;
(3) $J''$ is of Lempert type because $J$ is of Lempert type on $I$;
(4) $J''$ is L-isotopic to $J'$ (because $J$ is L-isotopic to $J_o$) and $J'$ is L-isotopic to $J_o$; from this it follows that $J''$ is L-isotopic to $J_o$.

So, if we show the existence of a normalizing map for any smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$, relative to $\tau = \mu^2$ and $x_o = 0$, we automatically prove the existence of normalizing maps for any other manifold of circular type.

This is done using the lemma that follows. In order to state it, we have to fix some notation. As usual, the blow-up of $\mathbb{C}^n$ at the origin is identified with the tautological line bundle $\pi : \widetilde{\mathbb{C}}^n = E \to \mathbb{C}P^{n-1}$ and we set $E_* = E \setminus \{\text{zero section}\} = \mathbb{C}^n \setminus \{0\}$. We remark that $E_*$ is a holomorphic principal $\mathbb{C}_*$-bundle over $\mathbb{C}P^{n-1}$.

Let now $\mu : \mathbb{C}^n \to \mathbb{R}_{\geqslant 0}$ be the Minkowski function of a smoothly bounded, complete circular domain $D \subset \mathbb{C}^n$ and let $\tilde{\mu} = \mu \circ \pi : E_* \to \mathbb{R}_{\geqslant 0}$. It is quite direct to realize that $\tilde{\mu}^2$ is the quadratic form of an Hermitian metric $h^{(\mu)}$ on $\pi : E \to \mathbb{C}P^{n-1}$ and that the distribution $\mathcal{H}$, defined by

$$\mathcal{H}_u = \{ v \in T_u E^* \colon\ d\tilde{\mu}_u(v) = d\tilde{\mu}_u(J_o v) = 0 \}, \tag{3.2}$$

is a connection on the principal $\mathbb{C}_*$-bundle $E_*$. The associated curvature 2-form $\tilde{\omega}^{(\mu)}$ is a $\mathbb{C}_*$-invariant, horizontal 2-form on $E_*$ and projects down onto a closed, 2-form $\omega^{(\mu)}$ on $\mathbb{C}P^{n-1}$. A direct computation shows that, for any two Minkowski functions $\mu$, $\mu'$, the associated 2-forms $\omega^{(\mu)}$ and $\omega^{(\mu')}$ are cohomologous (see e.g. [26], Theorem 3.3).

From the definition, it is clear that, for any hypersurface $S(c) = \{\mu = const.\}$, the restriction $\mathcal{H}|_{S(c)}$ coincides with the real distribution underlying the CR structure of $S(c)$. When $D$ is strongly pseudo-convex, the function $\tau = \mu^2$ is a parabolic exhaustion for $D$ and hence $\mathcal{H}|_D$ coincides with the normal distribution of the Monge–Ampère foliation of $D$ determined by $\tau$. Moreover, being each hypersurface $S(c)$ strongly pseudo-convex, it is simple to check that the associated 2-form $\omega^{(\mu)}$ is a Kähler form.

In the following, $\mu_o$ is the Minkowski function of $B^n$, i.e. $\mu_o = |\cdot|$, $\tilde{\mu}_o = \mu_o \circ \pi$ and $\mathcal{H}_o$ is the corresponding connection on $E_*$ as defined in (3.2).

**Lemma 3.5.** *Let $D \subset \mathbb{C}^n$ be a smoothly bounded, strongly pseudoconvex circular domain with Minkowski function $\mu$. Set $\tilde{\mu} = \mu \circ \pi$ and let $\mathcal{H}$ be the corresponding connection on $E_*$ defined in* (3.2). *Then, there exists a diffeomorphism $\phi : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n$ with the following properties:*

(i) *it is a fiber preserving map for the bundle $\pi : \widetilde{\mathbb{C}}^n = E \to \mathbb{C}P^{n-1}$ which is holomorphic on any fiber;*
(ii) $\tilde{\mu}_o = \tilde{\mu} \circ \phi$;
(iii) $\phi_*(\mathcal{H}_o) = \mathcal{H}$.

*Moreover, if $\{\mu^{(t)},\ 0 \leqslant t \leqslant 1\}$ is a smooth* 1*-parameter family of Minkowski functions of smoothly bounded, strongly pseudoconvex circular domains, then it is possible to choose a family of diffeomorphisms $\phi^{(t)} : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n$, satisfying* (i)–(iii) *for $\mu = \mu^{(t)}$, which depends smoothly on $t$.*

**Proof.** Let $\omega_o$ and $\omega$ be the Kähler forms on $\mathbb{C}P^n$ determined by the curvatures on $E_*$ of the connections $\mathcal{H}_o$ and $\mathcal{H}$, respectively. Since $\omega_o$ and $\omega$ are cohomologous, by Moser's theorem [19] there exists a diffeomorphism $\psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ such that

$$\psi^* \omega = \omega_o. \tag{3.3}$$

Indeed, by the proof of Moser's theorem, there exists a smooth 1-parameter family of diffeomorphisms $\psi_t$, $t \in [0, 1]$, such that $\psi_0 = Id_{\mathbb{C}P^{n-1}}$, $\psi_1 = \psi$. Such family of diffeomorphisms is obtained by integrating a family of vector fields $X_t = \dot{\psi}_t$ satisfying a particular system of differential equations. Consider the vector fields $\hat{X}_t$ on $E_* \simeq \mathbb{C}^n \setminus \{0\}$, which are horizontal w.r.t. $\mathcal{H}_o$ and project onto the vector fields $X_t$. Since any space of the distribution $\mathcal{H}_o$ is tangent to some spheres in $E_* = \mathbb{C}^n \setminus \{0\}$, it is possible to integrate such vector fields and obtain another 1-parameter family of diffeomorphisms $\hat{\psi}_t : E_* \to E_*$ so that $\hat{X}_t = \dot{\hat{\psi}}_t$. We then define

$$\tilde{\psi}_t : E \to E, \qquad \tilde{\psi}_t\big([v], z\big) = \begin{cases} (\psi_t([v]), \hat{\psi}_t(z)) & \text{if } z \neq 0, \\ (\psi_t([v]), 0) & \text{if } z = 0, \end{cases}$$

which can be easily checked to be a family of diffeomorphisms and we set $\tilde{\psi} \stackrel{\text{def}}{=} \hat{\psi}_1$. By construction, $\tilde{\psi} : E \to E$ commutes with the action of $\mathbb{C}$ on $E$ and restricts to $\psi$ on $\mathbb{C}P^{n-1}$. Consider now a diffeomorphism $\phi : E \to E$ of the form

$$\phi\big([v], z\big) \stackrel{\text{def}}{=} \left( \psi\big([v]\big), e^{i\lambda([v])} \frac{|z| \cdot \hat{\psi}(z)}{\mu(\hat{\psi}(z))} \right) \tag{3.4}$$

where we denote by $\lambda : \mathbb{C}P^{n-1} \to \mathbb{R}$ a smooth function to be fixed later. It is clear that a map of the form (3.4) satisfies (i). But it satisfies also (ii), since

$$\tilde{\mu}\big(\phi\big([v], z\big)\big) = \mu \left( e^{i\lambda([v])} \frac{|z| \cdot \hat{\psi}(z)}{\mu(\hat{\psi}(z))} \right) = |z| = \tilde{\mu}_o\big([v], z\big),$$

and we claim that there exists a $\lambda$ so that it satisfies also (iii). To check this, observe that, since $\phi|_{E_*}$ commutes with the action of $\mathbb{C}_*$, it maps the connection $\mathcal{H}_o$ into a connection $\mathcal{H}'$ on $E_*$ whose curvature 2-forms projects down on $\mathbb{C}P^{n-1}$ onto the $\mathbb{C}$-valued 2-form

$$\omega' = \psi_* \omega_o = \omega. \tag{3.5}$$

Moreover, by (ii), all spaces of $\mathcal{H}'$ are tangent to the hypersurfaces $\{\mu = const.\}$.

By standard arguments of theory of connections, we obtain that, for any $u \in E_*$ there exists a linear map

$$\nu_u : T_{\pi(u)}\mathbb{C}P^{n-1} \to \mathbb{R} \tag{3.6}$$

so that any space $\mathcal{H}'_u$ is of the form

$$\mathcal{H}'_u = \left\{ w = v + \nu_u\big(\pi_*(v)\big) \cdot \left( i \frac{\partial}{\partial \zeta}\bigg|_u - i \frac{\partial}{\partial \bar{\zeta}}\bigg|_u \right) \text{ for some } v \in \mathcal{H}_u \right\} \tag{3.7}$$

(here we denoted by $\zeta$ the standard coordinate of $\mathbb{C}$ and by $\frac{\partial}{\partial \zeta}$ the vertical holomorphic vector field on $E$ induced by the holomorphic action of $\mathbb{C}$ on $E$). Equivalently, we may say that if $\varpi : TE_* \to \mathbb{C}_*$ is the connection form for $\mathcal{H}$, then the connection form $\varpi'$ of $\mathcal{H}'$ is

$$\varpi' = \varpi - i\nu. \tag{3.8}$$

By the invariance of $\mathcal{H}'$ and $\mathcal{H}$ under the $\mathbb{C}_*$-action, the linear map $\nu_u$ depends only on the point $x = \pi(u) \in \mathbb{C}P^{n-1}$ and we may consider $\nu$ as a 1-form on $\mathbb{C}P^n$. Computing the curvature, we get from (3.8) and (3.5) that

$$d\nu = i(\omega' - \omega) = 0. \tag{3.9}$$

Since $H^1(\mathbb{C}P^{n-1}) = 0$, there exists a smooth function $\tilde{\lambda} : \mathbb{C}^n \to \mathbb{R}$ such that $\nu = d\tilde{\lambda}$. Now, let us replace the function $\lambda$ in (3.4) with the function $\lambda - \tilde{\lambda} \circ \psi^{-1}$. By construction, in (3.7) the function $\nu_u$ has to be replaced by the function $\tilde{\nu}_u = \nu_u - d\tilde{\lambda}|_{\pi(u)} = 0$ and the new map $\phi$ satisfies (iii).

It remains to prove the final part of the statement. First of all, notice that by the first part of the proof, the function $\lambda : \mathbb{C}P^{n-1} \to \mathbb{R}$ is uniquely determined (up to a constant) by the diffeomorphism $\psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ which satisfies (3.3). By choosing some suitable normalization condition for $\lambda$, we may assume that the map $\phi$ is uniquely determined by $\psi$. If $\mu^{(t)}$ is a smooth family of Minkowski functions of strongly pseudoconvex, complete circular domains, then also the corresponding Kähler forms $\omega^{(t)}$ are smoothly depending on $t$ (it suffices to see the explicit expression of $\omega^{(t)}$ in term of $\mu^{(t)}$—see e.g. [26, p. 27]). Now, the proof of Moser's theorem in [19] shows that there exists a smooth family of diffeomorphisms $\psi^{(t)}$, $t \in [0, 1]$, each of them satisfying $\psi^{(t)*}\omega^{(t)} = \omega_o$. This automatically implies the existence of a smooth family $\phi^{(t)}$ satisfying (i)–(iii) for any $t$. □

We may now conclude the proof of the theorem. Given a smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$ with Minkowski function $\mu$, let $\Phi = \phi^{-1}|_{\tilde{I}} : \tilde{I} \to \widetilde{B}^n$, where $\phi$ is diffeomorphism of the previous lemma, and let $J' = \Phi_*(J_o)$. From (i)–(iii) of the lemma, it follows immediately that $J$ is of Lempert type and that $\tilde{\tau} = \tilde{\tau}_o \circ \Phi$ with $\tau = \mu^2$. Moreover, we may consider the smooth 1-parameter family of Minkowski functions $\mu^{(t)} = (1-t)\mu + t\mu_o$ with $t \in [0, 1]$, a corresponding family of diffeomorphisms $\Phi_t = \phi_t^{-1}$ with $\phi_t$ associated by the lemma with $\mu^{(t)}$ and smoothly depending on $t$, and the 1-parameter family of complex functions

$J_t \stackrel{\text{def}}{=} \Phi_{t*}(J_o)$. By construction, $J_t$ is an L-isotopy between $J'$ and $J_o$ and we may conclude that $\Phi$ is a normalizing map for $(I, J_o, \tau = \mu^2)$, as needed.

Assume now that $\Phi, \Phi' : \widetilde{M} \to \widetilde{B}^n$ are two normalizing maps so that $(\Phi'^{-1} \circ \Phi)|_{\pi^{-1}} = Id_{\pi^{-1}}$ or, equivalently, that $(\Phi' \circ \Phi^{-1})|_{\mathbb{C}P^{n-1}} = Id|_{\mathbb{C}P^{n-1}}$. Now, $\Phi$ and $\Phi'$ map the leaves of the Monge–Ampère foliation of $(M, J, \tau)$ into the leaves of the Monge–Ampère foliation of $(B^n, \Phi_*(J), \tau_o)$ and of $(B^n, \Phi'_*(J), \tau_o)$, which are in both cases the images of the standard radial discs. This means that $\Phi' \circ \Phi^{-1}$ maps biholomorphically any standard radial disc into itself, since $(\Phi' \circ \Phi^{-1})|_{\mathbb{C}P^n} = Id|_{\mathbb{C}P^{n-1}}$. If in addition $(\Phi'^{-1} \circ \Phi)_*$ induces the identity map on any tangent space of a leaf of the Monge–Ampère foliation at the points of $\pi^{-1}(x_o)$, we get that $\Phi' \circ \Phi^{-1}$ maps any radial disc into itself by a biholomorphism which fixes the origin and with derivative equal to 1 at 0. By Schwarz lemma, $\Phi' \circ \Phi^{-1} = Id$ on any radial disc and hence on the whole $B^n$.

It remains to check the smooth extendibility up the boundary of $\Phi$ if $\tau$ is smoothly extendible. But this is a consequence of the fact that $\Phi$ is obtained by composing the inverse of the circular representation (which is smoothly extendible to the boundary because of Proposition 2.6) and the diffeomorphism between $\widetilde{I} \subset \widetilde{T}_{x_o}M = \widetilde{\mathbb{C}}^n$ and $\widetilde{B}^n$, which is given in Lemma 3.5 and which is trivially smooth up to the boundary. □

We remark that one can prove a stronger statement about the uniqueness of normalizing maps. We will come back on this topic in Section 6.

## 4. Bland and Duchamp's invariants

In this section, we consider only manifolds of circular type in normal form, i.e. of the form $(B^n, J, \tau_o)$ with $\tau_o = |\cdot|^2$ and $J$ complex structure on $\widetilde{B}^n$ of Lempert type and L-isotopic to the standard one. The distribution in $\widetilde{B}^n$, which is normal to the standard Monge–Ampère foliation, will be denoted $\mathcal{H}$. We recall that, for any sphere $S(c) = \{\tau_o = c^2\}$, the restriction $\mathcal{H}|_{S(c)}$ is the distribution underlying the standard CR structure of $S(c)$ and that, for any $x \in \widetilde{B}^n$,

$$T_x\widetilde{B}^n = \mathcal{Z}_x \oplus \mathcal{H}_x$$

where $\mathcal{Z}$ is the distribution tangent to the standard radial disc. We also denote by $Z$ the vector field defined in (2.4): in the standard coordinates of $\mathbb{C}^n$, the corresponding holomorphic and anti-holomorphic parts of $Z$ are

$$Z^{1,0} = \frac{1}{2}(Z - iJ_oZ) = z^i \frac{\partial}{\partial z^i}, \qquad Z^{0,1} = \bar{z}^i \frac{\partial}{\partial \bar{z}^i}. \tag{4.1}$$

Recall also that any complex structure $J$ of Lempert type is uniquely determined by its action on the vector fields on $\mathcal{H}$, since the action on the vector fields in $\mathcal{Z}$ is the same of the standard complex structure $J_o$.

Let $H^{1,0}$ and $H^{0,1} = \overline{H^{1,0}}$ be the $J_o$-holomorphic and $J_o$-anti-holomorphic subbundles of $\mathcal{H}^{\mathbb{C}}$. For any other complex structure $J$ of Lempert type, we denote the corresponding $J$-holomorphic and $J$-anti-holomorphic subbundles of $\mathcal{H}^{\mathbb{C}}$ by $H^{1,0}_J$ and $H^{0,1}_J = \overline{H^{1,0}_J}$.

**Definition 4.1.** Let $J$ be a complex structure on $\widetilde{B}^n$ of Lempert type. We call *deformation tensor associated with $J$* any smooth section

$$\phi : \widetilde{B}^n \to \bigcup_{x \in \widetilde{B}^n} \mathrm{Hom}\big(H^{0,1}, H^{1,0}\big) = H^{0,1\,*} \otimes H^{1,0}$$

so that $H^{0,1}$ can be expressed as

$$H^{0,1}_J|_x = \big\{ w + \phi_x(w),\ w \in H^{0,1}|_x \big\} \quad \text{for any } x \in \widetilde{B}^n. \tag{4.2}$$

Notice that not necessarily every complex structure of Lempert type has an associated deformation tensor. However if $J$ has an associated deformation tensor, then any sufficiently small deformation $J'$ of $J$, which is also of Lempert type, has an associated deformation tensor. Indeed, we will shortly see that *any complex structure $J$ of Lempert type and L-isotopic to $J_o$ admits an associated deformation tensor*.

We now want to exhibit some differential equations which characterize the deformation tensors. In order to do this, we first need to recall the definition of two important operators on the tensor fields in $H^{0,1*} \otimes H^{1,0}$.

We recall that $\widetilde{B}^n \subset \widetilde{\mathbb{C}}^n$ is a holomorphic bundle $\hat{\pi} : \widetilde{B}^n \to \mathbb{C}P^{n-1}$, with fibers given by the radial discs (if endowed with the standard complex structure). Since $\mathcal{H}$ is a connection in such a bundle, the holomorphic and anti-holomorphic distributions are generated by vector fields $X^{1,0} \in H^{1,0}$ and $Y^{0,1} \in \mathcal{H}^{0,1}$ that can be locally chosen so that $\hat{\pi}_*([X^{1,0}, Y^{0,1}]) = [\hat{\pi}_*(X^{1,0}), \hat{\pi}_*(Y^{0,1})] = 0$. Let us call such vector fields *holomorphic* and *anti-holomorphic* vector fields of the distribution $\mathcal{H}^{\mathbb{C}}$, respectively. It can be easily checked that if $\phi$ is a deformation tensor associated with a complex structure of Lempert type, then for any two anti-holomorphic vector fields $X, Y \in H^{0,1}$

$$\big[X, \phi(Y)\big] \in H^{1,0} + \mathcal{Z}^{\mathbb{C}}. \tag{4.3}$$

Hence, if we denote by $(\cdot)_{\mathcal{H}^{\mathbb{C}}}$ the projection onto the distribution $\mathcal{H}^{\mathbb{C}}$, we surely have that $[X, \phi(Y)]_{\mathcal{H}^{\mathbb{C}}} \in H^{1,0}$ for any pair of anti-holomorphic vector fields. Now, consider the following two operators (see [14]):

$$\begin{gathered} \bar{\partial}_b : H^{0,1\,*} \otimes H^{1,0} \to \Lambda^2 H^{0,1\,*} \otimes H^{1,0}, \\ \bar{\partial}_b \alpha(X, Y) \stackrel{\text{def}}{=} \big[X, \alpha(Y)\big]_{\mathcal{H}^{\mathbb{C}}} - \big[Y, \alpha(X)\big]_{\mathcal{H}^{\mathbb{C}}} - \alpha\big([X, Y]\big), \end{gathered} \tag{4.4}$$

and

$$\begin{gathered} [\,\cdot\,,\cdot\,] : \big(H^{0,1\,*} \otimes H^{1,0}\big) \times \big(H^{0,1\,*} \otimes H^{1,0}\big) \to \Lambda^2 H^{0,1\,*} \otimes H^{1,0}, \\ [\alpha, \beta](X, Y) \stackrel{\text{def}}{=} \frac{1}{2}\big(\big[\alpha(X), \beta(Y)\big] - \big[\alpha(Y), \beta(X)\big]\big) \end{gathered} \tag{4.5}$$

for any pair of anti-holomorphic vector fields $X, Y$ in $H^{0,1}$.

**Proposition 4.2.** *Let $J$ be a complex structure on $\widetilde{B}^n$ of Lempert type that admits an associated deformation tensor $\phi$. Then:*

(i) $dd^c\tau_o(\phi(X), Y) + dd^c\tau(X, \phi(Y)) = 0$ *for anti-holomorphic $X, Y \in H^{0,1}$;*
(ii) $\bar{\partial}_b\phi + \frac{1}{2}[\phi, \phi] = 0$*;*
(iii) $\mathcal{L}_{Z^{0,1}}(\phi) = 0$.

*Conversely, any tensor field* $\phi \in H^{0,1*} \otimes H^{1,0}$ *that satisfies* (i)–(iii) *is the deformation tensor of a complex structure of Lempert type.*

*In addition, a complex structure* $J$ *of Lempert type, associated with a deformation tensor* $\phi$*, is so that* $(B^n, J, \tau_o)$ *is a manifold of circular type if and only if*

(iv) $dd^c\tau_o(\phi(X), \overline{\phi(X)}) < dd^c\tau_o(\bar{X}, X)$ *for any* $0 \neq X \in H^{0,1}$.

**Proof.** First of all, recall that by the $J_o$-invariance of the 2-form $dd^c\tau_o$, for any two vector fields in $H^{0,1} \oplus \mathcal{Z}^{0,1}$ or in $H^{1,0} \oplus \mathcal{Z}^{1,0}$,

$$dd^c\tau_o(W, W') = dd^c\tau_o(J_oW, J_oW') = -dd^c\tau_o(W, W') = 0.$$

So, from the proof of Theorem 3.2, the reader can check that a complex structure $J$ of Lempert type is so that $(B^n, J, \tau_o)$ is a manifold of circular type, if and only if for any $0 \neq X \in H^{0,1}$

$$dd^c\tau_o\big(\bar{X} + \overline{\phi(X)}, X + \phi(X)\big) = dd^c\tau_o(\bar{X}, X) + dd^c\tau_o\big(\overline{\phi(X)}, \phi(X)\big) > 0.$$

This proves (iv). For checking the necessity and sufficiency of (i)–(iii), we only need to show that those properties give necessary and sufficient conditions for the integrability of the unique almost complex structure $J$, which coincides with $J_o$ on the radial discs, leaves the distribution $\mathcal{H}$ invariant and have an associated anti-holomorphic distribution $H_J^{0,1}$ which is as in (5.2). Such almost complex structure $J$ is integrable if and only if for any anti-holomorphic vector fields $X, Y \in H^{0,1}$ one has

$$\big[X + \phi(X), Y + \phi(Y)\big] \in \mathcal{Z}^{0,1} + H_J^{0,1}, \qquad \big[Z^{0,1}, X + \phi(X)\big] \in \mathcal{Z}^{0,1} + H_J^{0,1}. \tag{4.6}$$

But conditions (4.6) are satisfied if and only if

$$\big[X + \phi(X), Y + \phi(Y)\big]_{\mathcal{H}^{\mathbb{C}}} = [X, Y] + \phi\big([X, Y]\big) \quad \Leftrightarrow \quad \bar{\partial}_b\phi(X, Y) + \frac{1}{2}[\phi, \phi](X, Y) = 0, \tag{4.7}$$

$$\big[X + \phi(X), Y + \phi(Y)\big]_{\mathcal{Z}^{\mathbb{C}}} = 0 \quad \Leftrightarrow \quad dd^c\tau_o\big(X + \phi(X), Y + \phi(Y)\big) = 0, \tag{4.8}$$

$$\big[Z^{0,1}, X + \phi(X)\big] = \big[Z^{0,1}, X\big] + \phi\big(\big[Z^{0,1}, X\big]\big) \quad \Leftrightarrow \quad \mathcal{L}_{Z^{0,1}}\phi(X) = 0 \tag{4.9}$$

for any anti-holomorphic $X, Y \in H^{0,1}$, i.e. if and only if (i)–(iii) are true. □

Let $J$ be a complex structure of Lempert type and $J_t$, $t \in [0, 1]$, an L-isotopy between $J$ and $J_o$. By the previous remark, the set of $t$'s, for which $J_t$ has an associated deformation tensor, is open, while (iv) of the previous lemma implies that it is also closed. From this, we conclude that also $J = J_1$ has a deformation tensor and hence that *there is a natural injective map between the class of manifolds in normal form* $(B^n, J, \tau_o)$ *and the class of tensor fields* $\phi \in H^{0,1*} \otimes H^{1,0}$ *on* $\widetilde{B}^n$ *which satisfy* (i)–(iv) *of Proposition* 4.2.

The correspondence between normal forms and deformation tensors satisfying (i)–(iv) is a priori only injective, not surjective. However, for any deformation tensor satisfying (i)–(iv), the associated complex structure $J$ defines a manifold of circular type and hence there exists some normalizing map $\Phi : \widetilde{B}^n \to \widetilde{B}^n$ for which $\hat{J} = \Phi_*(J)$ is in normal form and whose associated deformation tensor is $\hat{\phi} = \Phi_*(\phi)$. In other words, we may say that *any deformation tensor satisfying* (i)–(iv) *is, up to a diffeomorphism, the deformation tensor of some normal form.*

Proposition 4.2(iii) has also the following consequence. Consider $n-1$ holomorphic vector fields $(e_1, \ldots, e_{n-1})$, defined on some open subset of $\mathbb{C}P^{n-1} \subset \widetilde{B}^n$ and linearly independent at all points where they are defined. Extend them to $\widetilde{B}^n$ as $Z$ and $JZ$ invariant vector fields on taking values in $H^{1,0}$. Let also $(e^1, \ldots, e^{n-1}, Z^{1,0*})$ be the holomorphic field of $(1,0)$-forms, which is dual to the frame field $(e_1, \ldots, e_{n-1}, Z^{1,0})$. Then any tensor field $\phi \in H^{0,1*} \otimes H^{1,0}$ is of the form $\phi = \sum_{a,b=1}^{n-1} \phi^a_b \overline{e^b} \otimes e_a$ and satisfies Proposition 4.2(iii) if and only if the restrictions of the functions $\phi^a_b$ on the radial discs are holomorphic. In particular, using a system of coordinates $(v^1, \ldots, v^{n-1}, \zeta)$ for $\widetilde{B}^n$ as in (2.2), we have that $\phi$ satisfies (iii) if and only if it is of the form

$$\phi = \sum_{j=0}^{\infty} \phi_j \zeta^j, \qquad \phi_j = \sum_{a,b=1}^{n-1} \phi^a_{bj} \overline{e^b} \otimes e_a$$

where $\phi^a_{bj} = \phi^a_{bj}(v^1, \ldots, v^{n-1})$ are the coefficients of the series expansion in powers of $\zeta$ of the functions $\phi^a_b(v^1, \ldots, v^{n-1}, \zeta)$. It can be checked that the deformation tensors $\phi^{(k)} \stackrel{\text{def}}{=} \phi_k \zeta^k \in H^{0,1*} \otimes H^{1,0}$ are independent on the choice of the coordinates and of the frame field $(e_1, \ldots, e_{n-1})$. Moreover, $\phi$ satisfies (i) and (iii) of Proposition 4.2 if and only each tensor field $\phi^{(i)}$ satisfies (i) and that the following equations:

$$\bar{\partial}_b \phi^{(k)} + \frac{1}{2} \sum_{i+j=k} \left[\phi^{(i)}, \phi^{(j)}\right] = 0 \quad \text{for any } 0 \leqslant k < \infty. \tag{4.10}$$

Summarizing, we have the following theorem, which can be considered as an extension to arbitrary manifolds of circular type some of the main results in [5,7] (see next remark).

**Theorem 4.3.** *Let $D$ be a manifold of circular type in normal form, i.e. $D = (B^n, J, \tau_o)$, with $J$ complex structure of Lempert type and L-isotopic to $J_o$. Then $J$ is uniquely determined by an associated sequence of deformation tensors $\phi^{(k)} \in H^{0,1*} \otimes H^{1,0}$, $0 \leqslant k < \infty$, which satisfy* (4.10) *and* (i) *of Proposition* 4.2 *for any $k$, and with the series $\phi = \sum_k \phi^{(k)}$ uniformly converging on compacta.*

In [5,7], Bland and Duchamp considered small deformations of the standard CR structure of the $S^{2n-1}$ and proved that, for $n > 1$, any such CR structure is embeddable in $\mathbb{C}^{n-1}$ as boundary of a domain which is biholomorphic to a domain in normal form $(B^n, J, \tau_o)$. In particular, they associated with any small deformation of the CR structure of $S^{2n-1}$ a sequence of tensors, which correspond to the restrictions to $S^{2n-1}$ of the tensors $\phi^{(k)}$ appearing in Theorem 4.3.[1] It is therefore natural to name such sequence of deformation tensors $\phi^{(k)}$ the *Bland and Duchamp invariants of* $(B^n, J, \tau_o)$.

We conclude with the following concept, which will turn out to be quite useful in the applications of the next section.

[1] Be aware that our deformation tensor is minus the deformation tensor considered in [5,7].

**Definition 4.4.** Let $(D, J, \tau)$ be a domain of circular type. We say that $D$ is *stable* if it admits a parabolic exhaustion $\tau$, whose associated circular representation $\Psi : \tilde{I} \to D$ extends smoothly up to the boundary inducing a diffeomorphism between $\partial \tilde{I}$ and $\partial D$.

Unless differently stated, for any given stable domain $D$, we will call *parabolic exhaustive functions of $D$* only those, whose associated circular representation satisfies the above condition.

We also recall that, by Lempert's and the first author's result (see e.g. [15,23]), the class of stable domains of circular type naturally includes the smoothly bounded, strictly linearly convex domains of $\mathbb{C}^n$ and the smoothly bounded, strongly pseudoconvex circular domains. Indeed, by Theorem 4.4 in [24], the complete circular domains may be characterized as the unique (stable) domains of circular type whose Monge–Ampère foliation is holomorphic.

Moreover, the "stability" property is invariant under biholomorphisms between domains of circular type. In fact:

**Lemma 4.5.** *Let $(D, J, \tau)$ and $(\hat{D}, \hat{J}, \hat{\tau})$ be two biholomorphic domains of circular type. Then $D$ is stable if and only if $\hat{D}$ is stable. Furthermore they have a normal form $(B^n, J, \tau_o)$, with a complex structure $J$ which is smoothly extendible up to the boundary and makes $\overline{B^n}$ a stable domain of circular type.*

**Proof.** The first claim follows from the fact that any biholomorphism $f : D \to \hat{D}$ between stable domains extends smoothly up to the boundary. This property can be checked using the local regularity results of Berteloot [4, Proposition 3], which imply that any such $f$ admits an Hölder continuous extension up the boundary. In fact, from Hölder boundary regularity, the standard arguments of Lempert's proof of Fefferman theorem (see [15,30]) imply that $f$ extends smoothly up to the boundary.

The last claim is a consequence of the proof of Theorem 3.4. In fact, if $D$ is stable, we may construct a normalizing map which is smooth up to the boundary and induces a diffeomorphism between $\partial D$ and $S^{2n-1} = \partial B^n$. In particular, the complex structure of $\widetilde{D}$, which extends up to $\partial D$, is mapped onto a complex structure on $\widetilde{B}^n$, which extends smoothly up to $\partial B^n$ (and hence also to a small neighborhood of $\overline{B^n}$). This implies that the associated normal form $(B^n, J, \tau_o)$ is a stable domain of circular type since the circular representation of $(B^n, J, \tau_o)$ coincides with the one of $(B^n, J_o, \tau_o)$. $\square$

By the previous lemma, the normal forms of stable domains of circular type correspond to Bland and Duchamp's invariants $\{\phi^{(k)}\}$ converging uniformly on the closure $\overline{\widetilde{B}^n}$.

## 5. Miscellaneous results

### 5.1. The geometrical meaning of the Bland and Duchamp invariant $\phi^{(0)}$

Let $(B^n, J, \tau_o)$ be a manifold in normal form, $\phi^{(k)}$ the Bland and Duchamp invariants, $I \subset T_0 B^n \simeq \mathbb{C}^n$ the indicatrix at 0 and $\Psi : \tilde{I} \to \widetilde{B}^n$ the circular representation. We identify $T_0 B^n \simeq \mathbb{C}^n$ so that we may assume $J|_{T_0 B^n} = J_o$. The following proposition indicates the information carried by $\phi^{(0)}$.

**Proposition 5.1.**

(a) *The pull-backed complex structure* $J' = \Psi^*(J)$ *on* $\tilde{I}$ *is of Lempert type.*
(b) *The tensor field* $\phi - \phi^{(0)} = \sum_{k\geqslant 1}^{\infty} \phi^{(k)}$ *is identically vanishing if and only if the circular representation is a biholomorphism between* $I$ *and* $(B^n, J, \tau_o)$*, i.e.* $(B^n, J)$ *is biholomorphic to a circular domain.*
(c) *The invariant* $\phi^{(0)}$ *is always the deformation tensor of a manifold in normal form* $(B^n, J^{(0)}, \tau_o)$*, more precisely, of a normal form of the indicatrix* $I$*.*

**Proof.** (a) is a direct consequence of definitions and Proposition 2.6. For (b), we remark that $\phi^{(k)} = 0$ for all $k \geqslant 1$ if and only if the projection $\pi : \widetilde{B}^n \to \mathbb{C}P^{n-1}$ is $J$-holomorphic, i.e. if and only if the Monge–Ampère foliation of $(B^n, J, \tau_o)$ is holomorphic. Then (b) follows from [24], Proposition 3.4.

For (c), notice that $\phi^{(0)}$ satisfies (i)–(iv) of Proposition 4.2 and defines a complex structure which is L-isotopic to $J_o$, because $\phi$ does it. So, by the remarks after Proposition 4.2, the first claim follows immediately. Now, consider the circular representation $\Psi : \tilde{I} \to \widetilde{B}^n$. It is straightforward to realize that $J^{(0)}|_{\mathbb{C}P^n} = \Psi_*(J_o|_{\mathbb{C}P^n})$ and hence, by invariance along the leaves of the Monge–Ampère foliations, $J^{(0)} = \Psi_*(J_o)$ on $\tilde{I}$. This implies that the corresponding projected structures on $I$ and on $B^n$ are biholomorphic. □

*5.2. The parameterization of normalizing maps and the automorphisms group of a manifold of circular type*

**Definition 5.2.** Let $\tau$ be a parabolic exhaustion function for $M$ and $x_o$ and $I_{x_o} \subset T_{x_o}M \simeq \mathbb{C}^n$ the corresponding center and indicatrix. We call *special frame at* $x_o$ *associated with* $\tau$ a complex basis $(e_0, e_1, \ldots, e_{n-1})$ for $T_{x_o}M$ defined as follows:

(i) $e_0 \in \partial I_{x_o}$ (i.e. $\kappa(e_0) = 1$, where $\kappa$ is the Minkowski function of $I_{x_o}$);
(ii) $(e_1, \ldots, e_{n-1})$ is a unitary basis w.r.t. $dd^c\kappa^2$ for the holomorphic tangent space $\mathcal{D}^{1,0}_{e_0} \subset T_{e_0}\partial I_{x_o}$ of the CR structure of $\partial I_{x_o} \subset T_{x_o}M \simeq \mathbb{C}^n$.

Recall that if $D$ is a *domain* of circular type, for any center $x_o$ there is a unique parabolic exhaustion function $\tau : D \to [0, 1)$, smoothly extendible at the boundary, for which the center is exactly $x_o$ (see Lemma 2.2). For this reason, for any such domain the following set is well defined

$$P = \bigcup_{x_o \text{ is a center}} P_{x_o}, \quad \text{where } P_{x_o} = \{\text{special frames at } x_o\}.$$

We call it the *pseudo-bundle of special frames of* $D$. We also denote by $\pi : P \to D$ the map which associates to any special frame the base point and $\mathfrak{C}(D) = \pi(P) \subset D$ denote the set of centers of $D$.

It should be observed that, relatively to the biholomorphisms that are smooth up to the boundary (in case of stable domains, any biholomorphism is in such a class), the pseudo-bundle $P$ is a biholomorphic invariant of $D$. In case $\mathfrak{C}(D)$ is discrete, the pseudo-bundle $P$ is a bundle over such a set. Any fiber $P_{x_o} = \pi^{-1}(x_o)$ has a structure of $\mathrm{U}_{n-1}$-principal bundle with basis equal to the boundary of the indicatrix $\partial I_{x_o}$. In case $D$ is a smoothly bounded, strictly linearly convex domain in $\mathbb{C}^n$ (and hence $D = \mathfrak{C}(D)$), $P$ is a bundle over $D$, coinciding with the *unitary frame bundle of the Kobayashi metric of D* [27].

Notice also that, via Gram–Schmidt orthonormalization, any special frame of a manifold in normal form is uniquely associated with a basis which is unitary w.r.t. the standard Poincarè–Bergman metric of the unit ball $B^n \subset \mathbb{C}^n$. Using such correspondence, it is possible to embed the pseudo-bundle $P$ into the bundle $\mathrm{U}_n(B^n)$ of the unitary frames of $B^n$. Since $\mathrm{Aut}(B_n, J_o)$ acts transitively and freely on $\mathrm{U}_n(B^n)$, $\mathrm{U}_n(B^n)$ can be identified with $\mathrm{Aut}(B_n, J_o)$ and the previous immersion $P \hookrightarrow \mathrm{U}_n(B^n)$ can be considered as an immersion $P \hookrightarrow \mathrm{Aut}(B_n, J_o)$. In case $\mathfrak{C}(D) = D$, such immersion is actually a diffeomorphism (see e.g. [27,28]).

Consider a manifold of circular type $(M, J, \tau)$ of dimension $n$ and denote by $\widetilde{\mathcal{N}(M)}$ the class of all normalizing maps $\Phi : \widetilde{M} \to \widetilde{B}^n$ between a blow up of $M$ at a center $x_o$ and the blow up of $B^n$ at the origin. A priori, $\widetilde{\mathcal{N}(M)}$ is a very large class of maps. In fact, by known properties on symplectomorphisms [13] and the arguments in the proof of Lemma 3.5, one can lift symplectomorphisms of $\mathbb{C}P^{n-1}$ and construct smooth families $\tilde{\psi}_t : \widetilde{B}^n \to \widetilde{B}^n$ of bundle automorphisms of $\pi : \widetilde{B}^n \to \mathbb{C}P^{n-1}$, with $\psi_0 = Id$ and preserving the sets $\{\tau_o = c\}$, the distribution $\mathcal{H}_o$ and the complex structure of the fibers. If $\Phi$ is in $\widetilde{\mathcal{N}(M)}$, also the maps $\psi_t \circ \Phi$ are in $\widetilde{\mathcal{N}(M)}$ and this shows that $\widetilde{\mathcal{N}(M)}$ is a priori a class of quite large cardinality, comparable with the cardinality of symplectomorphisms of $\mathbb{C}P^{n-1}$.

Any normalizing map $\Phi : \widetilde{M} \to \widetilde{B}^n$ induces a complex structure on $\widetilde{B}^n$, which in turn projects down onto a complex structure on $B^n$ (see remarks in Definition 3.1). The charts of two distinct complex manifold structures of this kind in general *belong to two distinct real manifolds structures*, even if they smoothly overlap when restricted to $B^n \setminus \{0\}$. To clarify this point, let $(\widetilde{M}, J) = (\widetilde{B}^n, J_o)$ and $\tilde{\psi}_t : \widetilde{B}^n \to \widetilde{B}^n$ a family of fiber preserving diffeomorphism as described above, which differs from the identity only on $\pi^{-1}(\mathcal{U})$ for some open set $\mathcal{U} \subset \mathbb{C}P^{n-1}$. The induced diffeomorphisms $\widehat{\psi}_t : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$, $t \neq 0$, are not $J_o$-biholomorphisms, while the blow-down map $\tilde{\pi} : \widetilde{B}^n \to B^n$ determines smooth maps $\psi_t = \tilde{\pi} \circ \tilde{\psi}$: $B^n \setminus \{0\} \to B^n \setminus \{0\}$, which extend *non-smoothly* to homeomorphisms of $B^n$ (otherwise $\widehat{\psi}_t([v]) = [J\psi_t|_0(v)]$ and $\widehat{\psi}_t$ would be forced to be $\widehat{\psi}_t = Id_{\mathbb{C}P^{n-1}}$ for all $t$, if $\mathbb{C}P^{n-1} \setminus \mathcal{U}$ is not dense in $\mathbb{C}P^{n-1}$). The maps $\Phi = Id_{\widetilde{B}^n}$ and $\Phi' = \tilde{\psi}_1 \circ \Phi$ are both in $\widetilde{\mathcal{N}}(M)$, but the complex structure determined by $\Phi'$ on $B^n$ is non-standard, since otherwise $\Phi' \circ \Phi^{-1} = \tilde{\psi}_1$ would determine a biholomorphism of $(\mathbb{C}P^{n-1}, J_o)$. Moreover, the new complex structure is given by charts of the form $\xi \circ \psi_1^{-1}$, with $\xi$ standard complex chart of $(B^n, J_o)$, and hence the underlying atlas of real charts is diffeomorphic but *not equal* to the standard atlas, since $\psi_1$ is not smooth at 0.

In the following, given a manifold $M$ of circular type, we fix *once and for all* one of such real manifold structures on $B^n$ and we denote by $\mathcal{N}(M) \subset \widetilde{\mathcal{N}}(M)$ the class of normalizing maps which induce on $B^n$ that real manifold structure. So, a normalizing map $\Phi : \widetilde{M} \to \widetilde{B}^n$ belongs to $\mathcal{N}(M)$ if and only if the corresponding projected map $\phi : M \to B^n$ is a diffeomorphism w.r.t. the real manifold structure of $M$ and the fixed manifold structure of $B^n$. For a given parabolic exhaustion $\tau$ on $M$ with center $x_o$, we also denote by $\mathcal{N}(M, \tau, x_o) \subset \mathcal{N}(M)$ the subclass of normalizing maps associated with $\tau$ and $x_o$.

**Lemma 5.3.** *Let $\Phi, \Phi' \in \mathcal{N}(M, \tau, x_o)$ and denote by $\phi, \phi' : M \to B^n$ the corresponding projected maps. Then $\Phi = \Phi'$ if and only if $(\phi' \circ \phi^{-1})_*|_{T_0 B^n} = Id$.*

**Proof.** The necessity is immediate. For the sufficiency, notice that the hypothesis implies that the lifted maps $\Phi, \Phi' : \widetilde{M} \to \widetilde{B}^n$ are so that $\Phi' \circ \Phi^{-1}$ is the identity when restricted to the exceptional divisor $\pi^{-1}(0) = \mathbb{C}P^{n-1}$ and $(\Phi' \circ \Phi^{-1})_*$ induces the identity map on each tangent space of a standard radial disc at the intersection with $\mathbb{C}P^{n-1} = \pi^{-1}(0)$. So, $Id_{\widetilde{B}^n}$ and $\Phi' \circ \Phi^{-1}$ are normalizing maps for $B^n$ that satisfy the hypothesis of Theorem 3.4. Hence $\Phi' \circ \Phi^{-1} = Id_{\widetilde{B}^n}$. $\square$

With the help of the previous lemma, we can prove the following.

**Proposition 5.4.** *Fix a normalizing map $\Phi_o \in \mathcal{N}(M, \tau, x_o)$ and a special frame $(e_0^o, e_1^o, \dots, e_{n-1}^o)$ at $x_o \in M$, associated with $\tau$. Also, for any $\Phi \in \mathcal{N}(M, \tau, x_o)$ denote by $\phi : M \to B^n$ the corresponding projected map and let $(e_i^\phi)$ be the frame at $T_{x_o}M$ defined by $e_i^\phi = (\phi^{-1} \circ \phi_o)_*(e_i^o)$.*

*Then the frames of the form $(e_i^\phi)$ are special frames, relatively to $x_o$ and $\tau$, and the correspondence $\Phi \to (e_i^\phi)$ is a one to one map between $\mathcal{N}(M, \tau, x_o)$ and the class of special frames at $x_o$.*

**Proof.** By definition of normalizing maps, the map $f = \phi \circ \phi_o^{-1}$ is a biholomorphism of $(M, J)$ into itself, fixing $x_o$ and so that $\tau \circ f = \tau$. This implies that $f_*|_{T_{x_o}M}$ maps the class of special frames at $x_o$ into itself. Moreover, if $\Phi$ and $\Phi'$ are so that $(e_i^\phi) = (e_i^{\phi'})$, it follows from definitions that $(\phi' \circ \phi^{-1})_*|_{T_0 B^n} = Id$ and hence that $\Phi = \Phi'$ by the previous lemma. It remains to show that the correspondence $\Phi \to (e_i^\phi)$ is surjective.

Let $J^{(o)}$ be the complex structure on $B^n$ obtained by pushing forward the complex structure of $M$ and denote by $(f_i^o)$ and $(f_i)$ the special frames of $(B^n, \tau_o, J^{(o)})$ obtained as images of $(e_i^o)$ and of another special frame $(e_i)$, respectively. Let also $([f_i^o])$ and $([f_i])$ be the corresponding points in $\mathbb{C}P^{n-1} \subset \widetilde{B}^n$. Observe that the restriction $J^{(o)}|_{\mathbb{C}P^{n-1}}$ coincides with the complex structure $J^{(o,0)}$ on $\widetilde{B}^n$ defined by the Bland and Duchamp invariant $\phi^{(0)}$ (see Section 5.1) and it is diffeomorphic to the standard complex structure of $\mathbb{C}P^{n-1}$ (to see this, simply use the biholomorphism between $(\widetilde{B}^n, J^{(o,0)})$ and the blow up $\tilde{I}$ of its indicatrix — see Proposition 5.1). Hence $\mathrm{Aut}(\mathbb{C}P^{n-1}, J^{(o)})$ is isomorphic to $\mathrm{Aut}(\mathbb{C}P^{n-1}) = \mathrm{PGL}_n(\mathbb{C})$ and there exists a unique $J^{(o)}$-biholomorphism $\Psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ mapping $[f_i]$ into $[f_i^o]$ for any $0 \leqslant i \leqslant n-1$. Let us extend such a map to a diffeomorphisms $\Psi : \widetilde{B}^n \to \widetilde{B}^n$ in such a way that $\zeta \cdot \Psi(w) = \Psi(\zeta w)$ for any $\zeta \in \Delta$. We stress the fact that even if $\Psi|_{\mathbb{C}P^{n-1}}$ is by construction a biholomorphism of $(\mathbb{C}P^{n-1}, J^{(o)}|_{\mathbb{C}P^{n-1}})$, $\Psi$ is not in general a biholomorphism of $(\widetilde{B}^n, J^{(o)})$.

Now, observe that $\Psi : \widetilde{B}^n \to B^n$ is a normalizing map which maps the $[f_i]$'s into the $[f_i^o]$'s. Assume for the moment that $\Psi$ projects onto a map $\psi : B^n \to B^n$ which preserves the real manifold structure of $B^n$. Then, $\psi_*(f_j) = e^{i\theta_j} f_j^o$ for some suitable complex numbers $e^{i\theta_j}$ and, by a suitable adjustment of the definition of $\Psi$, we may always assume that $e^{i\theta_j} = 1$ for any $0 \leqslant j \leqslant n-1$. From this and its construction, it follows that $\Phi = \Psi \circ \Phi_o$ is a normalizing map in $\mathcal{N}(M, \tau, x_o)$ so that $(e_i^\phi) = (e_i)$. This implies the surjectivity of the map $\Phi \to (e_i^\phi)$.

To conclude, we only need to show that $\Psi$ projects onto a map $\psi : B^n \to B^n$ which preserves the real manifold structure of $B^n$. This is equivalent to check that there is a chart on $B^n$ on a neighborhood of the origin, which belongs to the real manifold structure of $(B^n, J^{(o)})$ and in which $\psi$ is smooth. This can be done using the circular representation. In fact, we may use it to identify the differentiable manifolds $(B^n, J^{(o)})$ and $(\widetilde{B}^n, J^{(o)})$ with the indicatrix $I$ and its blow up $\tilde{I} \subset \widetilde{\mathbb{C}}^n$, respectively, both endowed with a suitable complex structure $\hat{J}^{(o)}$. The circular representation is a diffeomorphism between a domain in $\mathbb{C}^n$ and the manifold of circular type. Thus the real manifold structure on $I$, determined by projection from the manifold structure of $\tilde{I}$, is the standard manifold structure of $I$ as open subset of $\mathbb{C}^n$. The explicit expression of $\Psi$

as a map $\Psi : \tilde{I} \to \tilde{I}$ shows directly that the projected map $\psi : I \to I$ is smooth in any standard coordinates system of $\mathbb{C}^n$. $\square$

The main result of the section can be now immediately inferred.

**Theorem 5.5.** *Let $(D, J, \tau)$ be a stable domain of circular type and $\mathcal{N}(D)$ the class of all normalizing maps, which induce on $B^n$ the same real manifold structure. Fix also a frame $(e^o_i) \subset T_x D$ belonging to the pseudo-bundle of special frames $\pi : P \to D$ and a normalizing map $\Phi_o \in \mathcal{N}(D)$. Then, the correspondence $\Phi \to (e^\phi_i)$ of the previous proposition gives a continuous one to one map between $\mathcal{N}(M)$ and the pseudo-bundle of special frames $P$.*

We remark that if $P$ is identified with a subset of $\mathrm{U}_n(B^n) \simeq \mathrm{Aut}(B^n)$, the previous result can be stated saying that: *$\mathcal{N}(D)$ is parameterized by a special subset of* $\mathrm{Aut}(B^n)$. In particular, we have that if the set of centers $\mathfrak{C}(D)$ is a singleton, then $\mathcal{N}(D)$ is parameterized by $\mathrm{U}_n = \mathrm{Aut}(B^n)_0$, while if $\mathfrak{C}(D) = D$, $\mathcal{N}(D)$ is parameterized by $\mathrm{Aut}(B^n)$. It is interesting to observe the analogy (and the difference) between this class of normalizing maps and the class of Chern–Moser normalizing maps for Levi non-degenerate hypersurfaces of $\mathbb{C}^n$.

Let $M$ be a circular domain of circular type. By means of a normalizing map, there is no loss of generality if we assume that $M$ is $(B^n, J, \tau_o)$, where $J$ is a complex structure of Lempert type, L-isotopic to the standard one. In this case, any automorphism $\Phi \in \mathrm{Aut}(M) = \mathrm{Aut}(B^n, J)$ is also a normalizing map and, if we set $\Phi_o = Id_{B^n}$, Theorem 5.5 implies that the action of $G = \mathrm{Aut}(M)$ on special frames determines a one to one map between $G$ and the points of any orbits $G \cdot (e^o_i)$. If $M$ is a stable domain of circular type, by the results in [9], it possesses a Bergman metric and $G$ is therefore a Lie group. In particular, it is diffeomorphic to any of its orbits $G \cdot u$ in the pseudo-bundle $P$. In this case, if $\mathfrak{C}(M) \subset M$ is contained in a real submanifold of dimension $a$ ($\leqslant 2n = \dim_{\mathbb{R}} M$), we get that

$$\dim G \leqslant a + n^2 \leqslant 2n + n^2,$$

which is a refinement of the classical upper bound $2n + n^2$.

### *5.3. Characterizations of circular domains and of the unit ball*

**Definition 5.6.** Let $(D, J, \tau)$ be a stable domain of circular type. An element $g \in G = \mathrm{Aut}(D, J)$ is called *rotational at the center $x_o$* if $g_*|_{x_o} = \lambda\, Id_{T_{x_o} D}$ for some $\lambda \in \mathbb{C}^n$ so that $\lambda^n \neq 1$ for any $n \in \mathbb{Z}$. We say that *$M$ is rotational at $x_o$* if $G$ contains a rotational element at $x_o$.

The next two theorems extend to stable domains a pair of results in [23].

**Theorem 5.7.** *A stable domain of circular type $(D, J, \tau)$ is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if it is rotational at some point $x_o \in D$.*

**Proof.** With no loss of generality, we assume that $D$ is in normal form, i.e. that $(D, J, \tau) = (B^n, J, \tau_o)$, and $x_o = 0$. If $g$ is rotational, the lifted map $\tilde{g} : \widetilde{B}^n \to \widetilde{B}^n$ is so that $\tilde{g}|_{\mathbb{C}P^{n-1}} = Id_{\mathbb{C}P^{n-1}}$. Moreover, by Lemma 2.2, $\tau_o \circ \tilde{g} = \tau_o$ and hence $\tilde{g}$ induces on any standard radial disc a biholomorphism that fixes the origin and with derivative at 0 equal to $\lambda$. This implies that $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and, in a system of coordinates as described before (2.2), $\tilde{g}$ is of the form $\tilde{g}(v,\zeta) = (v, e^{i\theta}\zeta)$. Moreover, since $\tilde{g}$ is a $J$-biholomorphism and induces on $\mathbb{C}P^{n-1}$ the identity map, all Bland and Duchamp invariants do not change under the action of $\tilde{g}$. In particular any invariant $\phi^{(k)}$, $k \geqslant 1$, is so that $\phi^{(k)} - e^{ik\theta}\phi^{(k)} = 0$. Since $\lambda^k = e^{ik\theta} \neq 1$ for all $k$, we get that $\phi^{(k)} = 0$ for any $k \geqslant 1$ and we conclude by Proposition 5.1(b). □

By previous theorem and Theorem 9.4 in [23], the next theorem follows immediately.

**Theorem 5.8.** *A stable domain $(D, J, \tau)$ is biholomorphic to the unit ball $B^n$ if and only if it is rotational at two distinct centers $x_o, x'_o \in D$.*

In the following statement, given a manifold of circular type $(M, J, \tau)$ with $\tau : M \to [0, r^2)$ and center $x_o$, for any $c \in (0, r^2)$ we denote by $D_{<c}$ the domain $D_{<c} = \{x \in M,\ \tau(x) < c\} \subset M$. We remark that the indicatrix at $x_o$ of $(M, J, \tau)$ coincides (up to rescaling) with the indicatrix of $(D_{<c}, J, \tau)$. From this and Proposition 2.6, it follows that $(D_{<c}, J, \tau)$ is a stable domain if $D_{<c} \subsetneq M$.

**Theorem 5.9.** *Let $(M, J, \tau)$ be a manifold of circular type of dimension $n$. Then $M$ is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if for some given parabolic exhaustion $\tau$ there exist two domains $D_{<c}$, $D_{<c'}$, $0 < c < c' < r^2$, that are biholomorphic one to the other by means of a map fixing the center of $\tau$.*

**Proof.** The necessity is direct: If $M \subset \mathbb{C}^n$ is a circular domain with Minkowski function $\mu$ and we set $\tau = \mu^2$, then the map $f(z) = \frac{c'}{c}z$ determines a biholomorphism between $D_{<c}$ and $D_{c'}$ for any two $c, c'$.

To prove the sufficiency, we may assume that $M$ is in normal form, i.e. $M = (B^n, J, \tau_o)$ and that the parabolic exhaustion which defines the two biholomorphic stable domains $D_{<c}$ and $D_{<c'}$ is the function $\tau = \tau_o$. Notice also that the restriction on $D_{<c'}$ of the map

$$\psi : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n, \qquad \psi\bigl([z], z\bigr) = \left([z], \frac{1}{c'}z\right)$$

is a normalizing map for $D_{<c'}$ (i.e. maps $(D_{<c'}, J, \tau_o)$ into $(B^n, \tilde{J}, \tau_o)$) and maps $D_{<c}$ into $D_{<k}$, $k = c/c'$.

It follows directly from definitions that the indicatrix $I$ at $x_o = 0$ of $(B^n, \tilde{J}, \tau_o)$ is the same of the $(D_{<k}, \tilde{J}, \tau_o)$ (up to rescaling) and hence all special frames of the first domain at 0 coincide with the special frames of the second domain up to multiplication by $k$. On the other hand, by hypothesis and Proposition 2.2, we have a biholomorphism of domains of circular type $f : (D_{<k}, \tilde{J}, \tau_o) \to (B^n, \tilde{J}, \tau_o)$ so that $f(0) = 0$. The differential $f_*|_0 : T_0 D_{<k} = T_0 B^n \to T_0 B^n$ is a $\mathbb{C}$-linear map mapping $k \cdot I$ into $I$ and hence mapping a fixed special frame $(e_i)$ into another special frame $(e'_i)$ rescaled by the factor $1/k$. Let us denote by $\hat{f} : B^n \to B^n$ the unique normalizing map of $(B^n, \tilde{J}, \tau_o)$ which transforms $(e_i)$ into $(e'_i)$ (see Theorem 5.5). The lift at the blow up level of $f^{-1} \circ \hat{f}$ is a diffeomorphism between $\widetilde{B}^n$ and $\widetilde{D}_{<k} \subset \widetilde{B}^n$ which induces the identity map on $\mathbb{C}P^{n-1}$ and so that, when restricted to any radial disc, it is a holomorphic map (w.r.t. the standard complex structure) fixing the center, mapping the unit disc into the disc of radius $k$ and with derivative equal to $k$ at the origin. Therefore $f^{-1} \circ \hat{f}$ is of the form

$$\bigl(f^{-1} \circ \hat{f}\bigr)\bigl([z], z\bigr) = \psi_k\bigl([z], z\bigr), \quad \text{where } \psi_k\bigl([z], z\bigr) \stackrel{\text{def}}{=} \bigl([z], kz\bigr).$$

The same argument can be repeated for any iterated map $f^n = f \circ \cdots \circ f$ and we obtain that, for any $n \in \mathbb{N}$, the map $\hat{f}^{(n)} = f^n \circ \psi_k^n$ is a normalizing map of $(B^n, \tilde{J}, \tau_o)$, fixing the origin. Since the normalizing maps fixing the origin are continuously parameterized by the compact set of special frames at the origin ($\simeq \mathrm{U}_n$), we may consider a subsequence $n_j$ so that the sequence of normalizing maps $\hat{f}^{(n_j)}$ converges uniformly on compacta to a normalizing map $\hat{f}^{(\infty)}$. In particular, the sequence of complex structures $\tilde{J}^{(n_j)} \stackrel{\text{def}}{=} \hat{f}^{(n_j)*}(\tilde{J})$ converges to a complex structure $\tilde{J}^{(\infty)} = \hat{f}^{(\infty)*}(\tilde{J})$. On the other hand, since $f$ is a $(\tilde{J}, \tilde{J})$-biholomorphism, we have that

$$\tilde{J}^{(n_j)} = \hat{f}^{(n_j)*}(\tilde{J}) = \psi_k^{n_j *}(\tilde{J}).$$

A direct computation shows that, for any point $([z], z) \in \widetilde{B}^n$, the deformation tensor $\phi^{(n_j)}|_{([z],z)}$ of $\psi_k^{n_j *}(\tilde{J})$ coincides with the deformation tensor $\phi$ of $\tilde{J}$, but evaluated at the point $([z], k^{n_j} z)$. It follows that

$$\phi^{(\infty)}|_{([z],z)} = \lim_{n_j \to \infty} \phi^{(n_j)}|_{([z],z)} = \lim_{n_j \to \infty} \phi|_{([z],k^{n_j} z)} = \phi|_{([z],0)},$$

i.e. the deformation tensor $\phi^{(\infty)}$ of $\tilde{J}^{(\infty)}$ is independent on the coordinate of the radial discs. By Proposition 5.1, this means that the domains $(B^n, \tilde{J}^{(\infty)}, \tau_o) \simeq (B^n, \tilde{J}, \tau_o) \simeq (D_{<c'}, J, \tau_o)$ are all circular. In particular, the restriction to $D_{<c'}$ of the deformation tensor of the manifold from which started, i.e. of $(B^n, J, \tau_o)$, is independent on the coordinates of the radial discs. By Proposition 4.2(iii) (i.e. analyticity along the radial discs), we get independence on the coordinates of the radial discs over the entire $(B^n, J, \tau_o)$. Using once again Proposition 5.1, we get the claim, i.e. $(B^n, J, \tau_o)$ is circular. □

**Remark 5.10.** Notice that the proof of previous theorem implies that, if we assume that $(M, J, \tau)$ is a stable domain (in particular if it is a strictly linearly convex domain in $\mathbb{C}^n$), the claim is true also when $c' = r^2$ and $D_{<c'} = M$.

By previous theorem and remark and by Theorem 5.8, we have the following.

**Corollary 5.11.** *A stable domain $D$ is biholomorphic to the unit ball if and only it has at least two centers $x_o \neq x'_o$ and it is biholomorphic, by means of two maps fixing $x_o$ and $x'_o$, respectively, to two proper subdomains $D_{<c} = \{\tau < c\}$, $D'_{<c'} = \{\tau' < c'\}$, with $\tau$, $\tau'$ parabolic exhaustions associated with $x_o$ and $x'_o$, respectively.*

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## References

[1] M. Abate, G. Patrizio, Finsler Metrics — A Global Approach, Lecture Notes in Math., vol. 1591, Springer-Verlag, 1994.
[2] E. Bedford, M. Kalka, Foliation and complex Monge–Ampère equations, Comm. Pure Appl. Math. 37 (1977) 510–538.
[3] A.L. Besse, Einstein Manifolds, Springer-Verlag, 1986.

[4] F. Berteloot, Attraction des disques analytiques et continuité Höldérienne d'applications holomorphes propres, in: Topics in Complex Analysis, in: Banach Center Publ., vol. 31, 1995, pp. 91–98.
[5] J. Bland, Contact geometry and CR structures on $S^3$, Acta Math. 173 (1994) 1–49.
[6] J. Bland, T. Duchamp, Moduli for pointed convex domains, Invent. Math. 104 (1991) 61–112.
[7] J. Bland, T. Duchamp, Contact geometry and CR structures on spheres, in: Topics in Complex Analysis, in: Banach Center Publ., vol. 31, 1995, pp. 99–113.
[8] J. Bland, T. Duchamp, M. Kalka, On the automorphism group of strictly convex domains in $\mathbb{C}^n$, in: Contemp. Math., vol. 49, 1986, pp. 19–29.
[9] B.-Y. Chen, J.-H. Zhang, The Bergman metric on a Stein manifold with a bounded plurisubharmonic function, Trans. Amer. Math. Soc. 354 (8) (2002) 2997–3009.
[10] S.S. Chern, J. Moser, Real hypersurfaces in complex manifolds, Acta Math. 133 (1974) 219–272.
[11] J.-P. Demailly, Mesures de Monge–Ampère et mesures pluriharmoniques, Math. Z. 194 (1987) 519–564.
[12] J.J. Faran, Hermitian Finsler metrics and the Kobayashi metric, J. Differential Geom. 31 (3) (1990) 601–625.
[13] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, 1972.
[14] K. Kodaira, J. Morrow, Complex Manifolds, Holt, Rinehart and Winston, 1958.
[15] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France 109 (1981) 427–474.
[16] L. Lempert, Intrinsic distances and holomorphic retracts, in: Complex Analysis and Applications, Varna, 1981, Publ. House Bulgar. Acad. Sci., Sofia, 1984, pp. 341–364.
[17] L. Lempert, Holomorphic invariants, normal forms and the moduli space of convex domain, Ann. of Math. 128 (1988) 43–78.
[18] K.-W. Leung, G. Patrizio, P.-M. Wong, Isometries of intrinsic metrics on strictly convex domains, Math. Z. 196 (1987) 343–353.
[19] J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965) 286–294.
[20] R. Narasimhan, The Levi problem for complex spaces II, Math. Ann. 146 (1962) 195–216.
[21] M.-Y. Pang, Finsler metrics with properties of the Kobayashi metric on convex domains, Publ. Math. 36 (1992) 131–155.
[22] M.-Y. Pang, Smoothness of the Kobayashi metric of non-convex domains, Internat. J. Math. 4 (6) (1993) 953–987.
[23] G. Patrizio, Parabolic exhaustions for strictly convex domains, Manuscripta Math. 47 (1984) 271–309.
[24] G. Patrizio, A characterization of complex manifolds biholomorphic to a circular domain, Math. Z. 189 (1985) 343–363.
[25] G. Patrizio, Disques extrémaux de Kobayashi et équation de Monge–Ampère complex, C. R. Acad. Sci. Paris Sér. I 305 (1987) 721–724, Ann. Sci. Ecole Norm. Sup. (4) XIII (2) (1986) 267–279.
[26] G. Patrizio, P.-M. Wong, Stability of the Monge–Ampère foliation, Math. Ann. 263 (1983) 13–19.
[27] A. Spiro, The structure equations of a complex Finsler manifold, Asian J. Math. 5 (2) (2001) 291–326.
[28] A. Spiro, S. Trapani, Eversive maps of bounded convex domains in $\mathbb{C}^{n+1}$, J. Geom. Anal. 12 (4) (2002) 695–715.
[29] W. Stoll, The characterization of strictly parabolic manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) VII (1) (1980) 87–154.
[30] A. Tumanov, Extremal discs and the regularity of CR mappings in higher codimension, Amer. J. Math. 123 (2001) 445–473.
[31] P.-M. Wong, Geometry of the complex homogeneous Monge–Ampère equation, Invent. Math. 67 (1982) 261–274.