Pluripotential theory and Monge-Ampère foliations

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Report on on joint work with A.Spiro

Main References:

OLD STUFF:

G. Patrizio, *A characterization of complex manifolds biholomorphic to a circular domain*, Math. Z. **189** (1985), 343–363.

G. Patrizio, *Disques extrémaux de Kobayashi et équation de Monge-Ampère complex*, C. R. Acad. Sci. Paris, Série I, **305** (1987), 721–724.

NEW STUFF:

G. Patrizio and A. Spiro, *Monge-Ampère equations and moduli spaces of manifolds of circular type*, Adv. Math. **223** (2010), 174–197.

G. Patrizio and A. Spiro, *Foliations by stationary disks of almost complex domains*, Bull. Sci. Math. **134** (2010), 215–234.

G. Patrizio and A. Spiro, *Stationary disks and Green functions in almost complex domains*, Preprint 2011, arXiv:1103.5383.

1. Example: Circular domains in $\mathbb{C}^n$

$$D \subset \mathbb{C}^n \text{ complete circular domain} \iff \begin{array}{c} \forall Z \in D \text{ then } \lambda Z \in D \\ \forall \lambda \in \mathbb{C} \text{ with } |\lambda| \leq 1. \end{array}$$

Assume D is smoothly bounded and strictly pseudoconvex.

Minkowski functional of D: $\quad m_D \colon \mathbb{C}^n \to \mathbb{R}_+$

$$m_D(Z) = \begin{cases} 0 & \text{if } \ Z = 0 \\ [\sup\{t \in \mathbb{R} \mid tZ \in D\}]^{-1} & \text{if } \ Z \neq 0 \end{cases}$$

– **Monge-Ampère exhaustion for** D: $\rho_D = m_D^2$

Easy to see: $\rho_D(Z) = G(Z)\|Z\|^2$ for some bounded $G \in C^\infty(\mathbb{C}^n \setminus \{0\}$ which is constant on complex lines through the origin (i.e. $G \in C^\infty(\mathbb{CP}^{n-1})$).

– **Moduli space for Circular domains**:

In fact the funtion ρ_D (and hence G) is (almost complete) modular datum. Patrizio-P.M.Wong ('83):

Proposition Two bounded circular domains D_1 and D_2 are biholomorphic $\Longleftrightarrow \rho_{D_1} = \rho_{D_2} \circ A$ for some $A \in GL(n, \mathbb{C})$

and consequently:

Theorem Let

$\mathcal{D} = \{$biholomorphic classes of smootly bd complete circular domains$\}$

$\mathcal{D}^+ = \{[D] \in \mathcal{D} \mid D$ is strictly pseudoconvex$\}$

$[\Omega] = \{$smooth $(1,1)$ forms coomologous to the Fubini-Study form$\}$

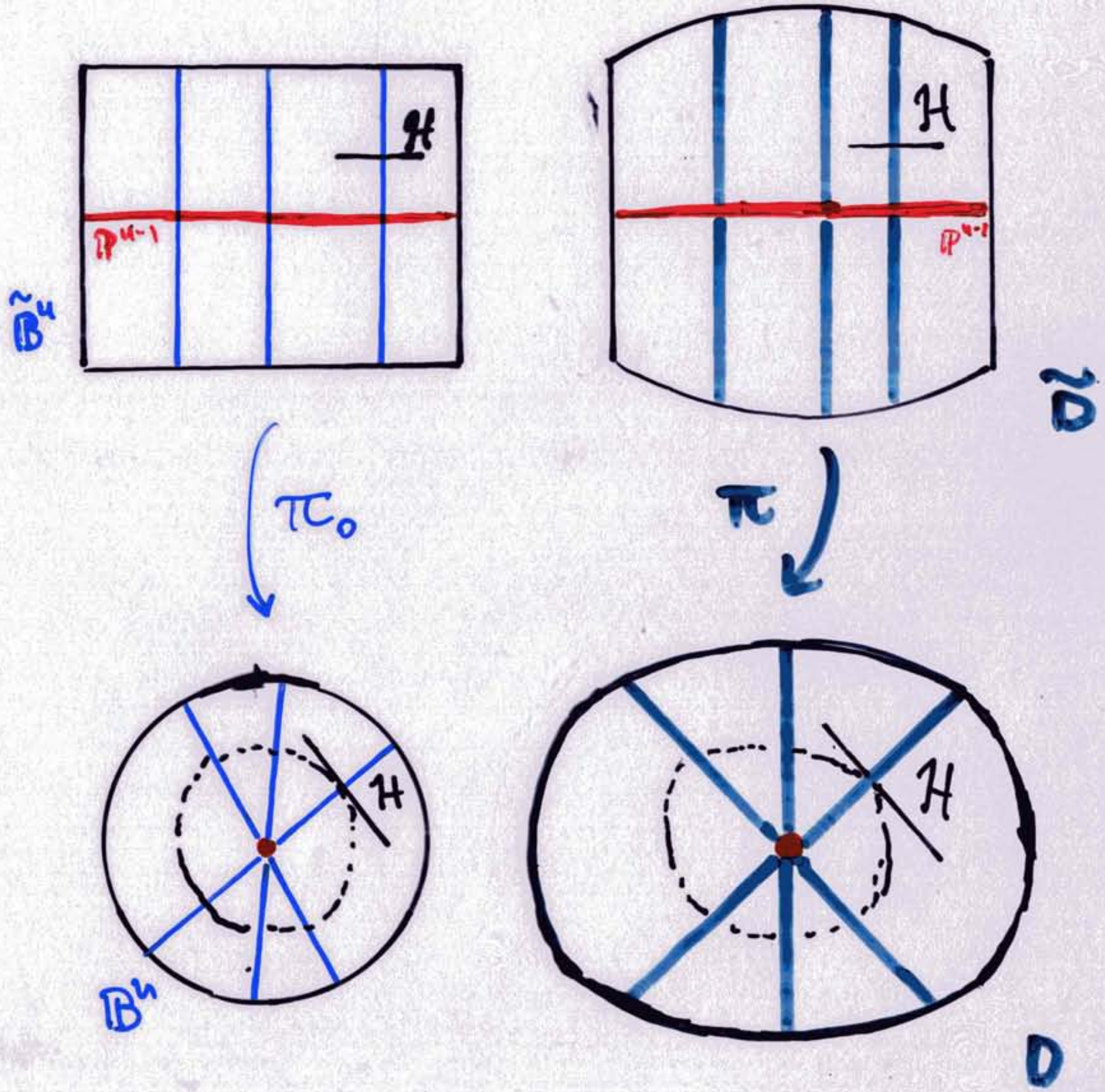
$[\Omega]^+ = \{$positive forms in $[\Omega]\}$

Then

$$\mathcal{D} \cong [\Omega]/Aut(\mathbb{P}^{n-1}) \qquad \mathcal{D}^+ \cong [\Omega]^+/Aut(\mathbb{P}^{n-1})$$

Comparison of complex structures of Ball $\{\|Z\|^2 < 1\}$ and of a Complete Circular Domain $\{G(Z)\|Z\|^2 < 1\}$:

Both are blow down of diffeomorphic disk bundles over $\mathbb{CP}^{n-1}$ with "same" complex structure along fibers, "same" normal bundle $\mathcal{H}$ to the fibers but different structures on $\mathcal{H}$. Difference is completely determined by funtion G



NOTATIONS

On a complex manifold we denote

$$d = \partial + \overline{\partial} \qquad\qquad d^c = i(\overline{\partial} - \partial)$$

$$dd^c = 2i\partial\overline{\partial} \qquad \text{and} \qquad dd^c = -d^c d$$

and for a function u of class C^2 one has:

2. Manifolds of Circular type

Definition (— '85) M complex manifold of dimension n. (M,τ) manifold of circular type (of bd type) with center x_0 if

(i) $\tau\colon M \to [0,1)$ exhaustion with $\{\tau = 0\} = \{x_0\}$ with

$$\begin{cases} \tau \in C^0(M) \cap C^\infty(\{\tau > 0\}) \\ \tau \in C^\infty(\tilde{M}) \end{cases} \qquad (\ \pi\colon \tilde{M} \to M \text{ the blow up at } \{x_0\})$$

$$(ii) \qquad \begin{cases} 2i\partial\overline{\partial}\tau = dd^c\tau > 0 \\ \\ 2i\partial\overline{\partial}\log\tau = dd^c\log\tau \geq 0 \\ \\ (dd^c\log\tau)^n \equiv 0 \text{ (Monge} - \text{Ampère Eq.)} \end{cases} \qquad \text{on } \{\tau > 0\}$$

(iii) near x_0, w.r.t. any local coordinates centered at x_0:

$$\log\tau(Z) = \log\|Z\|^2 + O(1) \text{ (logarithmic singularity)}$$

More often we consider the following situation:

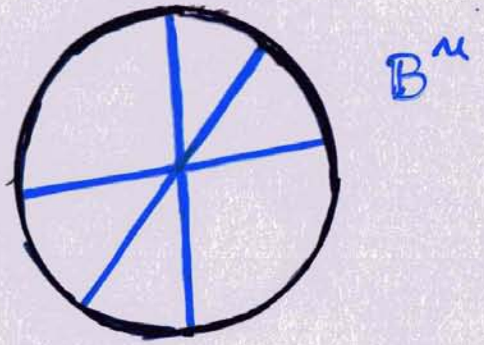
M complex manifold of dimension n, $D \subset M$ smooth,

relatively compact, is a *domain of circular type* with center $x_0 \in D$

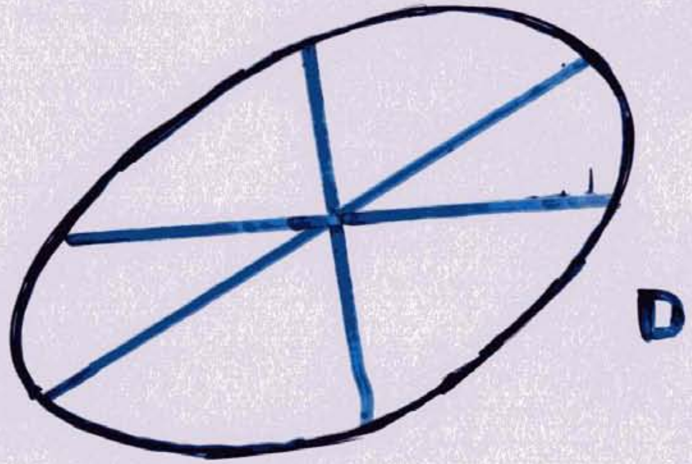
if there exisists a smooth exhaustion $\tau \colon D \to [0,1]$ with $\{\tau = 0\} = \{x_0\}$ such that (D, τ) is a manifold of circular type.

Examples:

– The unit ball $\mathbb{B}^n$ with $\tau = \| \bullet \|^2$



– More generally:
strictly pseudoconvex smoothly bounded complete circular domains $D \subset \mathbb{C}^n$ with $\rho_D = m_D^2 = G(\bullet)\| \bullet \|^2$ squared Minkowski functional.



Lempert Theory for Strictly (linearly) Convex Domains:

Theorem (Lempert '81): $D \subset\subset \mathbb{C}^n$ smooth, bounded strictly (linearly) convex domain, K_D and δ_D its Kobayashi metric and distance

$$\Downarrow$$

– $\delta_D \in C^\infty(D \times D \setminus Diagonal)$

– For $p \in D$ let $\delta_p = \delta_D(p, \bullet)$: then $u = \log(\tanh \delta_p) \in C^\infty(\overline{D} \setminus \{p\})$, it is the unique solution of the problem

$$\begin{cases} det(u_{\mu\overline{\nu}}) = 0 \quad \text{on} \quad D \setminus \{p\} \\ u_{|\partial D} = 0 \quad \text{and} \quad u(z) = \log|z - p| + O(1) \ \text{near} \ D \setminus \{p\} \end{cases}$$

In fact D with the exhaustion $\tau = (\tanh \delta_p)^2$ is manifold of circular type with center p.

Deep proof of geometric nature:

$p \in D$ and $v \in T_p(D)$ with $K_D(p, v) = 1$

$\exists$ unique complex geodesic

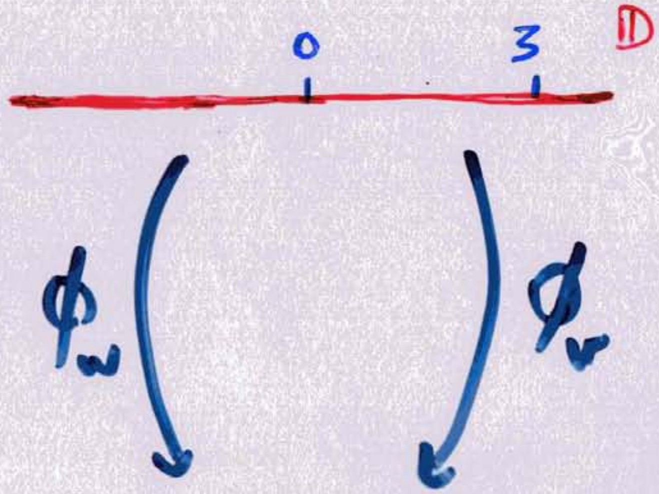
(i.e. holomorphic disk isometric w.r.t. hyperbolic metric on unit disk and Kobayashi metric of D)

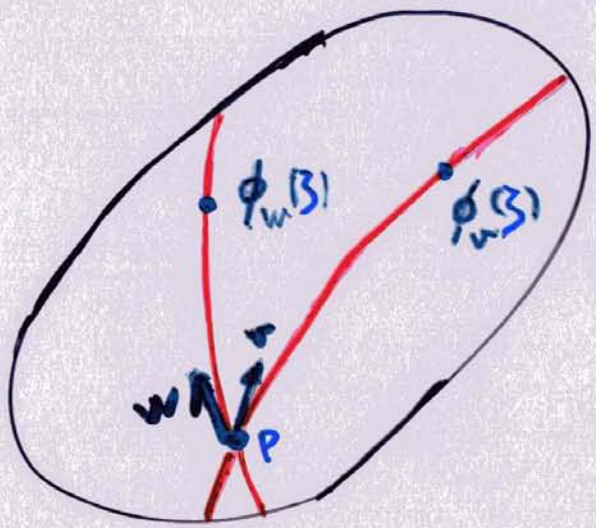
$\phi_v : \mathbb{D} \to D$ with $\phi_v(0)$ and $\phi_v'(0) = v$

smoothly depending on direction v.

The images of complex geodesics ϕ_v foliate $D \setminus \{p\}$. and $u(\phi_v(\zeta)) = \log|\zeta|$ for all complex geodesics ϕ_v i.e.

$\tau(\phi_v(\zeta)) = |\zeta|^2$





The problem of finding moduli for (pointed) strictly convex domains was addressed – and to a large extent solved – by

Lempert (Annals of Math '88):

He uses special coordinates along Kobayashi extremal disks to get boundary invariants

Bland-Duchamp (Inventiones '91), Bland (Acta '94), Bland-Duchamp ('95):

Their invariants are the Kobayashi indicatrix at the center and certain deformation tensor along the extremal disks (they are able to construct them for strictly convex domains and for "small" deformation of the unit ball.

What is left:

– the moduli seem to belong to a larger class of manifolds.

Problem: determine the “right” class

– For circular domains there is a natural “special” point, for stricly convex domains any point is a natural “special’ point.

Problem: “understand” the set of special points

Extending and simplifying Bland-Duchamp construction, we:

– prove that the right class to have complete (bijective) description is the class of manifolds of circular type

and

– determine a framework to understand the problem of the “special”point.

Technology: Monge-Ampère equation and foliations

M cmpx manifold $dim_{\mathbb{C}} M = n$, $u\colon M \to \mathbb{R}$ of class C^∞

$$dd^c u = 2i\partial\overline{\partial}u =_{loc.} 2i \sum u_{j\overline{k}} dz^j \wedge d\overline{z}^k$$

thus:

$$u \text{ plurisubharmonic} \Longleftrightarrow dd^c u = 2i\partial\overline{\partial}u \geq 0 \Longleftrightarrow_{loc.} (u_{j\overline{k}}) \geq 0$$

Complex (homogeneous) Monge-Ampère equation **(M-A)**:

$$(dd^c u)^n = (2i\partial\overline{\partial}u)^n = (2i)^n \underbrace{\partial\overline{\partial}u \wedge \ldots \wedge \partial\overline{\partial}u}_{n \text{ times}} = 0$$

$$\Updownarrow_{loc.}$$

$$\det(u_{j\overline{k}}) dz^1 \wedge \ldots \wedge dz^n \wedge d\overline{z}^1 \wedge \ldots \wedge d\overline{z}^n = 0 \Longleftrightarrow \det(u_{j\overline{k}}) = 0$$

For u plurisubharmonic and smooth, if for $\tau = e^u$ we have $dd^c\tau > 0$, it follows:

$$\begin{cases} (\partial\overline{\partial}u)^n = 0 & \qquad (\mathbf{M} - \mathbf{A}) \\ \\ (\partial\overline{\partial}u)^{n-1} \neq 0 & \qquad \text{non degeneracy condition} \end{cases}$$

Then at every point $\partial\overline{\partial}u$ (and $(u_{j\overline{k}})$) has rank $n-1$: infact $n-1$ positive eigenvalues and one equal to 0.

To see this, define a vector field $\mathbb{Z}$ of type $(1,0)$ on $\tau > 0$ by

$$\kappa(\mathbb{Z}, \overline{W}) = \overline{\partial}\tau(\overline{W}) \qquad \forall\, W \in T^{1,0}M \qquad (*)$$

where κ is the Kähler metric defined by $dd^c\tau > 0$. From the formula

$$\tau^2 dd^c \log\tau = \tau dd^c\tau - d\tau \wedge d^c\tau \qquad (**)$$

so that

$$0 = \tau^{2n}(dd^c \log\tau)^n = \tau^n(dd^c\tau)^n - \tau^{n-1}(dd^c\tau)^{n-1} \wedge d\tau \wedge d^c\tau$$

i.e. on $M \setminus \{x_0\}$

$$(dd^c \log\tau)^n = 0 \Longleftrightarrow \tau(dd^c\tau)^n = n(dd^c\tau)^{n-1} \wedge d\tau \wedge d^c\tau$$

or, in coordinates: $(dd^c \log\tau)^n = 0 \Longleftrightarrow \tau = \tau_{\overline{\nu}}\tau^{\overline{\nu}\mu}\tau_\mu \qquad$ where $(\tau^{\overline{\nu}\mu}) = (\tau_{\overline{\nu}\mu})^{-1}$

One computes locally:

$$\mathbb{Z} =_{loc} \sum Z^\mu \frac{\partial}{\partial Z^\mu} \qquad \text{where } Z^\mu = \sum_\nu \tau_{\overline{\nu}} \tau^{\overline{\nu}\mu}$$

and

$$dd^c \tau(\mathbb{Z}, \overline{\mathbb{Z}}) = \tau = \partial\tau(\mathbb{Z}) = \overline{\partial}\tau(\overline{\mathbb{Z}})$$

Now, from $(**)$

$$\tau^2 dd^c \log\tau(\mathbb{Z}, \overline{\mathbb{Z}}) = \tau dd^c \tau(\mathbb{Z}, \overline{\mathbb{Z}}) - d\tau \wedge d^c\tau(\mathbb{Z}, \overline{\mathbb{Z}}) = \tau^2 - \tau^2 = 0$$

As $dd^c \log\tau \geq 0$ then $\mathbb{Z} \in \mathrm{Ann} dd^c \log\tau$ so that, with a similar computation it follows that $\mathbb{Z}$ is orthogonal to the holomorphic tangent spaces to the level sets ot τ (and $\log\tau$) which are strongly psedoconvex. On the directions in the holomorphic tangent spaces then $dd^c \log\tau > 0$. Putting all this together we get the claim.

Set, for $u = \log \tau$

$$\mathcal{Z} = \mathrm{Ann} \partial\overline{\partial} u = \mathrm{Ann} dd^c \log \tau \bigcup_{p \in M} \mathcal{Z}_p$$

where $\mathcal{Z}_p =$ eigenspace of 0–eigenvalue of $(u_{j\overline{k}})$ at $p =\mathbb{C}\mathbb{Z}_p$, then

– $\mathcal{Z}$ is an integrable distribution ($\partial\overline{\partial} u$ is closed)

– leaves are holomorphic integral curves of $\mathbb{Z}$ ($\partial\overline{\partial} u$ is $(1,1)$ form!)

The foliation defined by $\mathcal{Z}$ is called **Monge-Ampère foliation** associated to u

To "recognize" leaves of the Monge-Ampère foliation associated to u:

$$\text{a holomorphic curve } L \subset M \text{ is (contained in) a leaf}$$

$$\Longleftrightarrow$$

$$u_{|L} \text{ is harmonic}$$

Main known facts: (—,'85-'87)

(M^n, τ) of circ. type with center x_0

$$\text{i.e.}\qquad \begin{cases} \tau\colon M \to [0,1) \text{ exhaustion with } \{\tau = 0\} = \{x_0\} \\ \tau \circ \pi \in C^\infty(\tilde{M}) (\pi\colon \tilde{M} \to M \text{ blow} - \text{up at } x_0) \\ dd^c\tau > 0, \quad dd^c \log\tau \geq 0, \quad (dd^c \log\tau)^n \equiv 0 \quad \text{on } \{\tau > 0\} \end{cases}$$

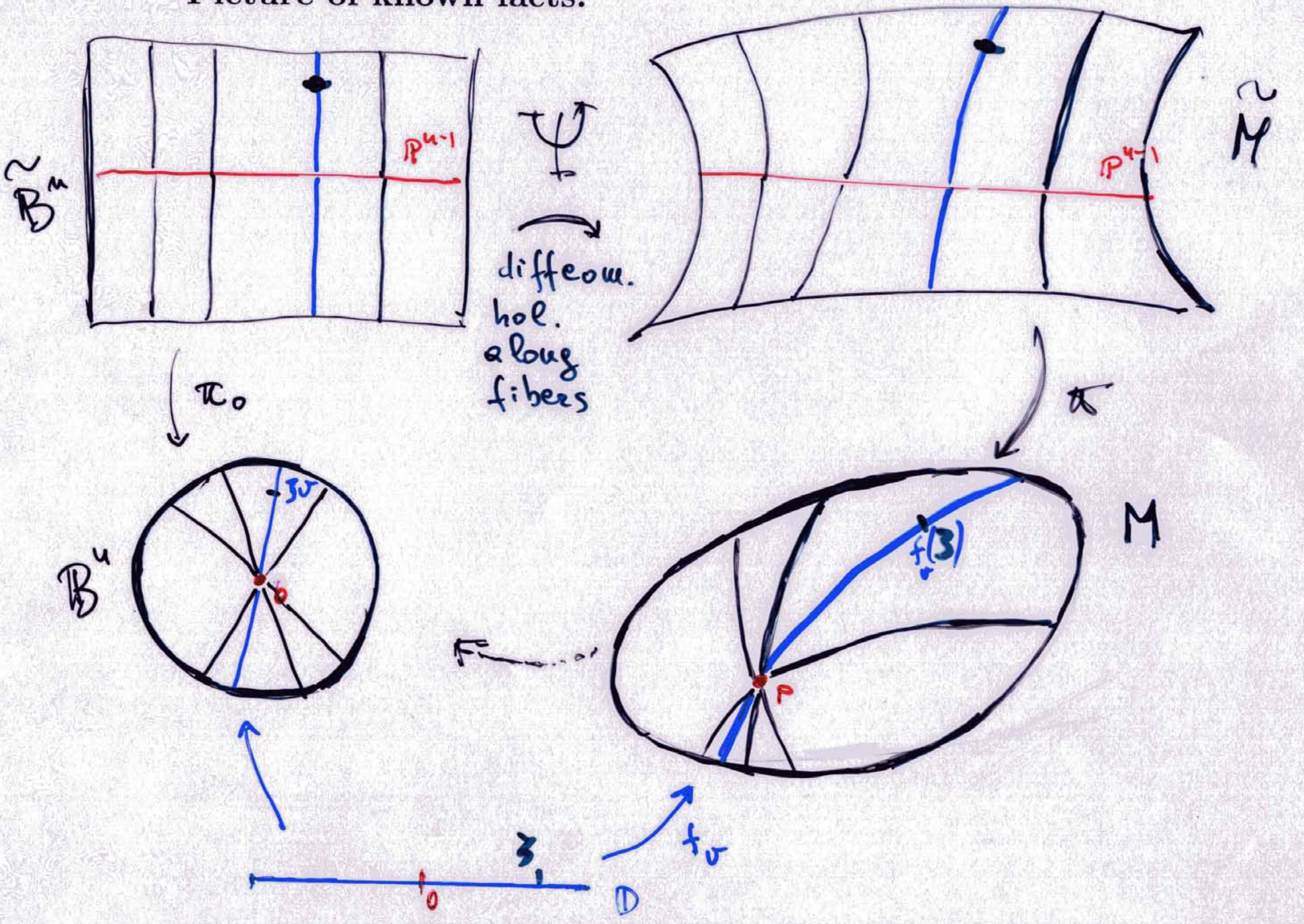
and $\log \tau(Z) = \log \|Z\|^2 + O(1)$ near $0 = x_0$

If $\pi_0\colon \tilde{\mathbb{B}}^n \to \mathbb{B}^n$ blow-up at 0, $\Longrightarrow$

$\exists$ diffeo $\Psi\colon \tilde{\mathbb{B}}^n \to \tilde{M}$ s. t. $\forall\, v \in S^{2n-1}$ the map $\zeta \mapsto f_v(\zeta) = \pi(\Psi(\pi_0^{-1}(\zeta v)))$ is holomorphic (proper and $1-1$) on unit disk $\mathbb{D}$, $f_v(\mathbb{D})$ is (the closure of) leaf of the Monge-Ampère foliation and the complex geodesic for the Kobayashi metric collinear to $v \in S^{2n-1} \subset \mathbb{C}^n \cong T_{x_0}(M)$

Moreover $\tau(\pi(\Psi(\pi_0^{-1}(Z))) = \|Z\|^2$

Picture of known facts:



Under the hypothesis, w.r. to coordinates centered at x_0 $(\simeq 0)$ one gets

$$\tau(Z) = h(Z)\|Z\|^2 + o(\|Z\|^3)$$

for some $h : S^{2n-1} \subset \mathbb{C}^n \simeq T_{x_0}M \longrightarrow \mathbb{R}_+$ of class C^∞ with $h(\lambda Z) = h(Z)$ for $|\lambda| = 1$

If $\mathcal{Z} = \sum_\nu \tau^{\bar\nu\mu}\tau_{\bar\nu}\frac{\partial}{\partial z^\mu}$ is the "complex gradient" of τ with r. to the metric $dd^c\tau$ on $M\setminus\{x_0\}$ then locally

(*) $\mathcal{Z}(z) = z +$ "quadratic"

and in "polar coordinates"

$$\mathcal{Z}(\lambda, Z) = \mathcal{Z}(\lambda Z)$$

(**) $\mathcal{Z}$ is C^∞ for $\lambda \in \Delta(0,\varepsilon) \subset \mathbb{C}$, $Z \in B(0 \simeq x_0, r)$

here need M-A equation + nature of singularity of τ at $x_0 \simeq 0$

Using ⊛ + ⊛⊛ and integrating separately "real part" of Z (in fact $Y = \frac{1}{\sqrt{2}}(Z + \bar{Z})$) and "imaginary part" $W = J(Z + \bar{Z})$ of Z, using the fact that $[Y, W] = 0$ (need M-A equation) one obtains a map

$$F : \Delta(0,1) \times S^{2n-1} \longrightarrow M \quad \text{of class } C^{\infty}$$
$$(z, v) \longmapsto F(z,v) = f_v(z)$$

with:

① $\tau(F(z,v)) = |z|^2$ so that $F(0,v) = x_0$

② $F(\lambda z, v) = F(z, \lambda v)$ $\forall (z,v) \in \Delta \times S$ and $|\lambda| = 1$

③ $f_v = F(\cdot, v)$ maps biholomorphically Δ onto a (closure) of a leaf and $\forall$ leaf there is a v s.t. f_v covers it

④ $Z(F(z,v)) = z(f_v)_*\left(\frac{\partial}{\partial x}\Big|_z\right) = z f_v'(z)$

⑤ $f_v'(0) = \sqrt{h(v)}\, v$ here h is as in ⊛ : $\tau(Z) = h(v)\|Z\|^2 + \cdots$

The required map $\Psi : \widetilde{\mathbb{B}} \longrightarrow \widetilde{M}$

$\pi_0 : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{B}$

$E = \mathbb{P}^{n-1} \longmapsto 0$

is defined by

$\pi : \widetilde{M} \longrightarrow M$

$E = \mathbb{P}^{n-1} \longrightarrow x_0$

$$\pi\left(\Psi\left(\pi_0(\zeta v)\right)\right) = f_v(\zeta) = F(\zeta, v)$$

- Notice that identifying exceptional set $\Psi|_E = Id_E$

Define $\sigma(z) = h(z)\|z\|^2$

Recall: near $x_0 \cong 0$

$\tau(z) = \sigma(z) + o(\|z\|^3)$

$$I_{x_0} = \{ v \in \mathbb{C}^n \simeq T_{x_0} M \mid \sigma(z) < 1 \}$$

I_{x_0} is the Kobayashi indicatrix of

M at x_0 :

Fact:

f_v is the unique Kobayashi extremal disk in the direction v

Pf: for $g : \Delta \to M$ with $g(0) = x_0$, $g'(0) = t v$ $t \in \mathbb{R}_+$

need to show $t \leq \| f_v'(0) \| = \sqrt{h(v)}$

Let $\ell(z) = \log \tau(g(z))$ subharmonic on Δ with

$|\ell(z)| \leq 0$ and $|\ell(z) - 2\log|z|| = o(1)$

$\Rightarrow$ $\log |z|^2$ is harmonic majorant for $\ell(z)$

$$\Rightarrow \quad \tau(g(z)) \leq |z|^2 = \tau(f_v(z))$$

Since τ lifts to a smooth lift to blow up $\tilde{M}$ at x_0

one may take limits along any holom. disk $\varphi : \Delta \to M$

with $\varphi(0) = x_0$, $\varphi'(0) \in \mathbb{R} v$ of $\tau_{\mu\bar{\nu}}(\varphi(\zeta))$ for $\zeta \to 0$

and

$$\tau^{v}_{\mu\bar{\nu}}(0) = \lim_{z \to 0} \tau_{\mu\bar{\nu}}(\varphi(z))$$ DEPENDS ONLY ON v (not on φ)

$0 \le \tau(g(\zeta)) \le |\zeta|^2 \Rightarrow \tau(g(\zeta)) = r(\zeta)$ with $0 \le r \le 1$

Differentiating: $\tau_{\mu\bar\nu}(g(\zeta))\, g^{\mu\prime}(\zeta)\, \overline{g^{\nu\prime}(\zeta)} = r_{\zeta\bar\zeta}(\zeta)|\zeta|^2 + r_\zeta(\zeta)\zeta + r_{\bar\zeta}(\zeta)\bar\zeta + r(\zeta)$

$\zeta \to 0$

$\Rightarrow t^2 \tau^{v}_{\mu\bar\nu}(0) v^\mu \bar v^\nu = r(0) \le 1 = \lim_{\zeta \to 0} \frac{\partial}{\partial\zeta\partial\bar\zeta} \overbrace{\tau(f_v(\zeta))}^{|\zeta|^2}$

$= \underbrace{h(v)}_{\|f_v'(0)\|^2} \tau^{v}_{\mu\bar\nu}(0) v^\mu \bar v^\nu$

$\Rightarrow t \le \|f_v'(0)\|$

Uniqueness: If $t = \|f'_v(0)\| \Rightarrow r(0) = 1$

Fact 1 $\log r$ is subharmonic: $\Delta \log r = \Delta \log r + \Delta \log |z|^2 = \Delta \log \tau(g) \ge 0$

Fact 2 $\log r \le 0 \Longrightarrow \log r \equiv 0 \Rightarrow r \equiv 1$

max princ.

$\Rightarrow \tau(g(\zeta)) = |\zeta|^2 \Longrightarrow g(\Delta) \subset$ leaf defined by f_v

little computation or $\log \tau|_{g(\Delta)}$ harmonic

$\Longrightarrow g \equiv f_v$

Suppose now that M is a smooth domain in a larger manifold $M \subset\subset N$ and that τ extends smoothly up to the bd ∂M + $dd^c \tau > 0$ on $\overline{M}$

Then also $\tilde{\Psi} : \widetilde{\mathbb{B}}^n \to \hat{M}$ extends to $\Psi : \widetilde{\overline{\mathbb{B}}^n} \to \widetilde{\overline{M}}$ and all $f_v : \overline{\Delta} \to \overline{M}$ extend.

<u>Fact</u>: $\forall v$ f_v is stationary (Lempert, Poletski ... ... Tumanov ...)

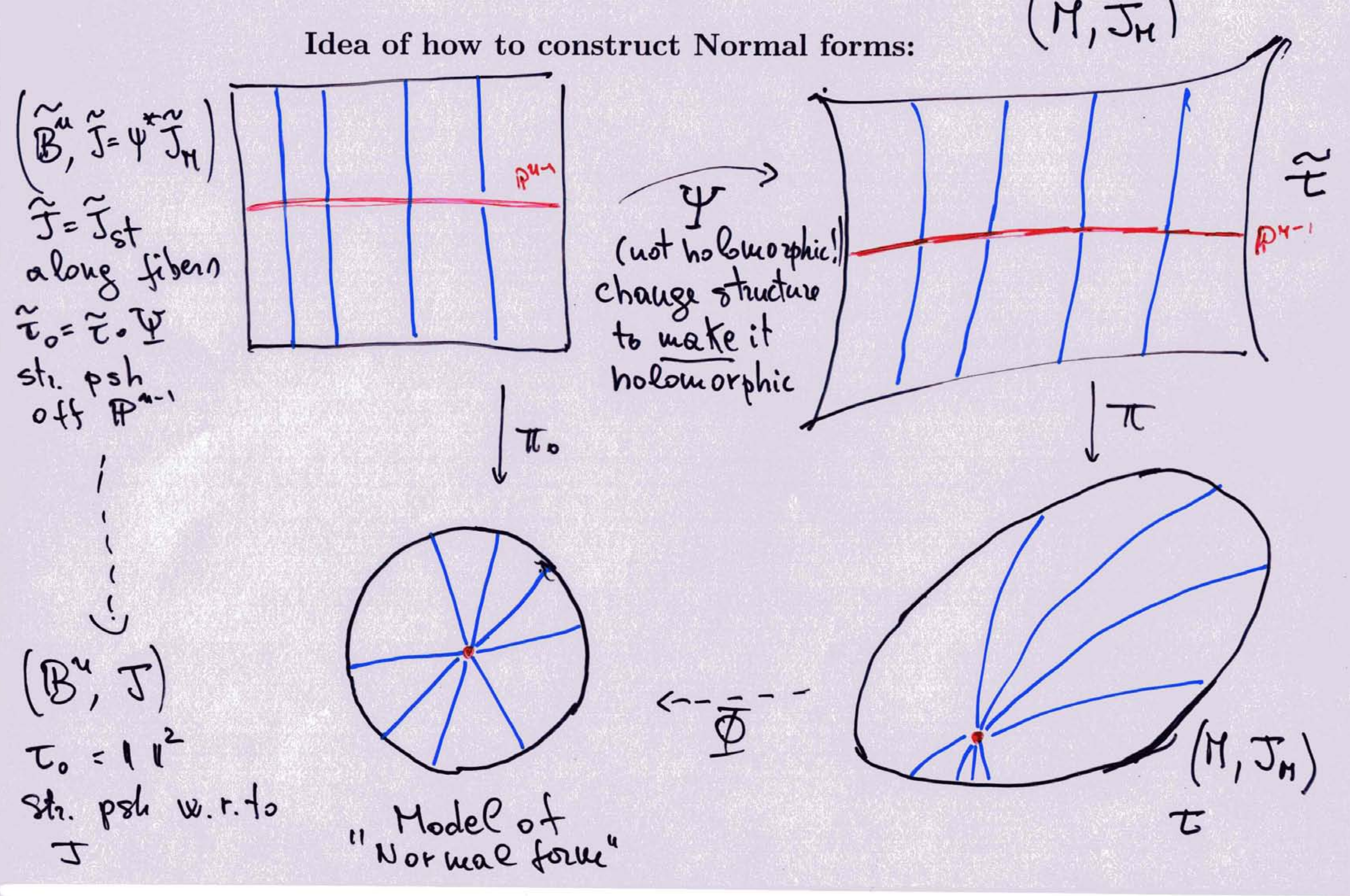
i.e. there is a holomorphic map (on Δ)

$\hat{f}_v : \overline{\Delta} \longrightarrow T^*N$ which is a smooth lift of f

such that $\zeta^{-1} \cdot \hat{f}_v(\zeta) \in \underbrace{\text{Conormal}(\partial M)}_{\text{1-form which annih. } T(\partial M)} \setminus \{\text{zero section}\}$ $\forall \zeta \in \partial\Delta$

$\hat{f}_v(\zeta) = \zeta \, \partial \log \tau (f_v(\zeta))$ hol. along the disk as $\log \tau$ is harmonic along $f_v(\Delta)$.

Idea of how to construct Normal forms:



3. Normal forms and deformations of CR Structures

Notations: $\mathbb{B}^n$ unit ball in $\mathbb{C}^n$ with standard complex strct J_0, $\tilde{\mathbb{B}}^n$ its blow-up at 0, $\tau_0 = \| \bullet \|^2$

let $\mathcal{Z}$ be the distribution tangent to the Monge-Ampère foliation associated to $\log \tau_0$ ("radial disks") and $\mathcal{H}$ be normal distribution (actually defined on $\tilde{\mathbb{B}}^n$!)

Definition: J is an *L–Complex Structure* on $\tilde{\mathbb{B}}^n$ if

(i) $\mathcal{Z}$ and $\mathcal{H}$ are J–invariant

(ii) $J_{|\mathcal{Z}} = J_{0|\mathcal{Z}}$ (i.e. J and J_0 differ only for their action on $\mathcal{H}$!)

(iii) there exists a smooth homotopy $J(t)$ of L–complex structures on $\tilde{\mathbb{B}}^n$ with $J(1) = J$, $J(0) = J_0$

Definition: A complex manifold M is said *of circular type in normal form* if $M = (\mathbb{B}^n, J)$ where $(\mathbb{B}^n, J)$ is the blow-down of $(\tilde{\mathbb{B}}^n, \tilde{J})$ for some L–Complex Structure $\tilde{J}$ on $\tilde{\mathbb{B}}^n$.

Proposition A manifold $M = (\mathbb{B}^n, J)$ in normal form with the exahustion τ_0 is a manifold of circular type (if J is smooth up to the boundary, "everything" is up to the boundary!).

Idea: The only thing that really needs to be checked is the fact that τ_0 is strictly plurisubharmonic w.r. to the structure J. By construction the "radial" distribution $\mathcal{Z}$ and the "normal" $\mathcal{H}$ are "orthogonal" with respect to the $dd^c\tau_0$ (here dd^c is w.r. to J!). Since $J_{|\mathcal{Z}} = J_{0|\mathcal{Z}} \Longrightarrow dd^c\tau_{0|\mathcal{Z}} > 0$. On the other hand $dd^{c'}\tau_o|_{\mathcal{H}\times\mathcal{H}}$ coincides with the Levi form (w.r.t. to J) of the hypersurfaces $\{\ \tau_o = const.\ \}$. which are strongly J_0-pseudo-convex and hence $\mathcal{H}$ is a contact distribution over each such hypersurface. This implies that, at any point, $dd^c\tau_o|_{\mathcal{H}\times\mathcal{H}}$ is a *non-degenerate* J-Hermitian form. The same claim is true for all complex structures J_t of an L-isotopy between J and J_o. By continuity it follows that that $dd^c\tau_o|_{\mathcal{H}\times\mathcal{H}} > 0$.

Conversely:

Theorem (existence of normalizing maps) For each manifold of circular type M, with exhaustion τ and center x_0, there is a biholomorphism $\Phi\colon M \to (\mathbb{B}^n, J)$ to a manifold in normal form $(\mathbb{B}^n, J)$ with:

– $\Phi\,(x_0) = 0$ and $\tau = \tau_0 \circ \Phi$.

– Φ maps leaves of the Monge-Ampère foliation to disks trough the origin

Remark Two normalizing maps at the same center differ only for the action on the leaf space. In fact the set of normalizing maps $\mathcal{N}(M)$ is naturally parametrized by a subset $Aut(\mathbb{B}^n)$ containing $Aut(\mathbb{B}^n)_0 = U_n$ (we'll come back to this).

Interpretation as deformation of CR-Structures:

Trivialize locally $\pi: \tilde{\mathbb{B}}^n \to \mathbb{P}^{n-1}$ setting

$$Z \in \tilde{\mathbb{B}}^n \Longleftrightarrow Z = ([v], \zeta) \quad [v] \in \mathbb{P}^{n-1}, \zeta \in \mathbb{D}$$

Furthermore set $\tau_0(Z) = |\zeta|^2 = \|Z\|^2$

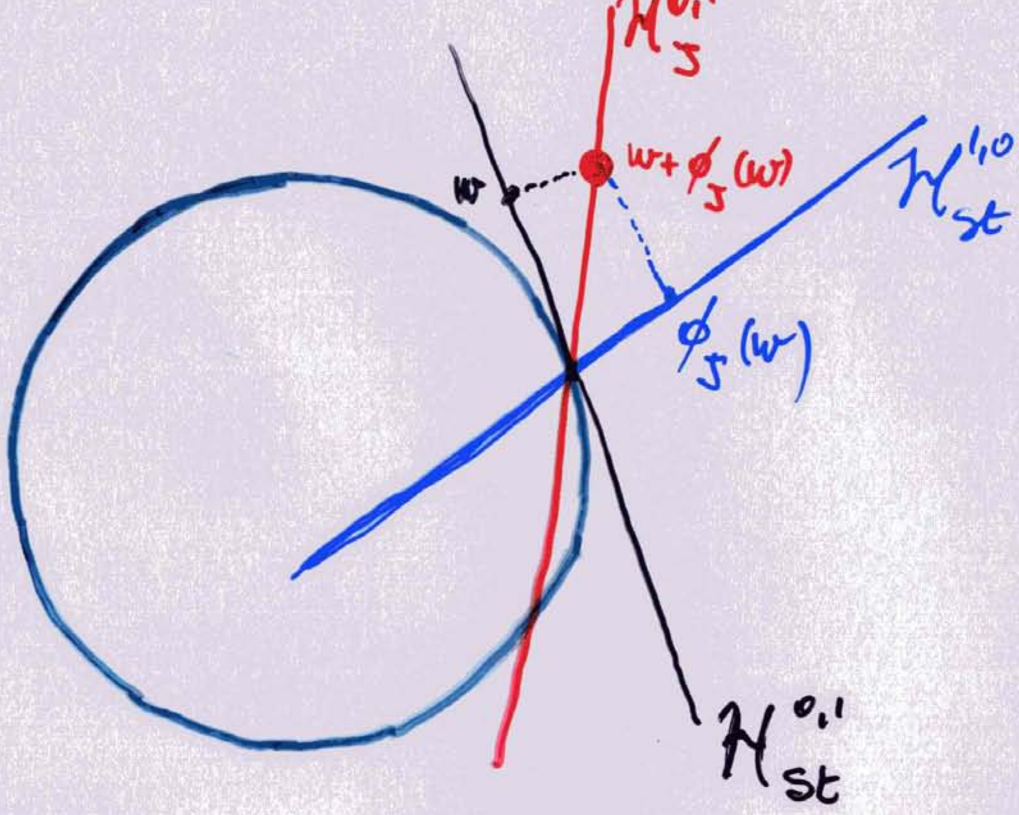
Let $M = (\mathbb{B}^n, J, \tau_0)$ be a manifold of circular type in normal form and let $(\tilde{\mathbb{B}}^n, J)$ be the blow up.

J is completely determined by

$$\phi_J \in (\mathcal{H}^{0,1})^* \otimes \mathcal{H}^{1,0} = \cup Hom(\mathcal{H}^{0,1}, \mathcal{H}^{1,0})$$

which is defined by

$$\mathcal{H}_J^{0,1} = \{w + \phi_J(w) \mid w \in \mathcal{H}^{0,1}\}$$



Remark: The existence of a deformation tensor for an L-complex structure J is an open condition i.e. "nearby" L-complex structure J' are defined by such a deformation tensor

Try to characterize the deformation tensors using differential equations (classical idea: Kodaira...... Here we adapt ideas of Bland-Duchamp)

If ϕ is a deformation tensor of an L-complex structure, then for $X, Y \in \mathcal{H}^{0,1}$

$$[X,\phi(Y)] \in \mathcal{H}^{1,0} + \mathcal{Z}^{\mathbb{C}}$$

Thus, if $(\cdot)_{\mathcal{H}^{\mathbb{C}}}$ is the projection onto the distribution $\mathcal{H}^{\mathbb{C}}$, $[X,\phi(Y)]_{\mathcal{H}^{\mathbb{C}}} \in H^{1,0}$ for or $X, Y \in \mathcal{H}^{0,1}$. Consider (see Kodaira-Morrow e.g.) (here X, Y in $\mathcal{H}^{0,1}$) :

$$\overline{\partial}_b : \mathcal{H}^{1,0*} \otimes \mathcal{H}^{1,0} \to \Lambda^2 \mathcal{H}^{1,0*} \otimes \mathcal{H}^{1,0} \ ,$$

$$\overline{\partial}_b \alpha(X, Y) = [X,\alpha(Y)]_{\mathcal{H}^{\mathbb{C}}} - [Y,\alpha(X)]_{\mathcal{H}^{\mathbb{C}}} - \alpha([X, Y])$$

and

$$[\cdot,\cdot] : \left(\mathcal{H}^{0,1*} \otimes \mathcal{H}^{1,0}\right) \times \left(\mathcal{H}^{0,1*} \otimes \mathcal{H}^{1,0}\right) \longrightarrow \Lambda^2 \mathcal{H}^{0,1*} \otimes \mathcal{H}^{1,0} \ ,$$

$$[\alpha,\beta](X, Y) = \frac{1}{2}\left([\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\right)$$

Theorem Let J be a complex structure on $\widetilde{B}^n$ of Lempert type that admits an associated deformation tensor ϕ. Then:

(i) $dd^c\tau_o(\phi(X), Y) + dd^c\tau_o(X, \phi(Y)) = 0 \; \forall \; X, Y \in \mathcal{H}^{0,1}$; **($J$ cmplx str.)**

$(ii) \overline{\partial}_b\phi + \frac{1}{2}[\phi, \phi] = 0$; **(integrability)**

(iii) $\mathcal{L}_{Z^{0,1}}(\phi) = 0$. **(holomorphicity along radial disks)**

Conversely, any tensor field $\phi \in \mathcal{H}^{0,1*} \otimes \mathcal{H}^{1,0}$ that satisfies $(i) - (iii)$ is the deformation tensor of a complex structure of Lempert type. In addition, an L-complex structure J, associated with a deformation tensor ϕ, is so that (B^n, J, τ_o) is a manifold of circular type (i.e. the standard exhaustion τ_o is strictly plurisubharmonic if and only if

(iv) $dd^c\tau_o(\phi(X), \overline{\phi(X)}) < dd^c\tau_o(\overline{X}, X) \; \forall \; 0 \neq X \in H^{0,1}$. **(positivity)**

Remark Condition (iv) of the theorem may be interpreted as an a-priori estimate for the deformation tensor ϕ in particular it is "bounded" and hence the condition for an L-complex structure J of (B^n, J, τ_o) a manifold of circular type to have a deformation tensor is also closed. Being all smoothly homotopic to the standard complex structure, by continuity they all have a deformation tensor.

Remark Condition (iii) of the theorem imply that, with respect to the trivializing coordinates on the blow up, then if ϕ_J is the deformation tensor of an L-complex structure J

$$\phi_J = \sum_{k=0}^{\infty} \phi_J^k([v], \zeta) = \sum_{k=0}^{\infty} \phi_J^k([v])\zeta^k, \qquad \phi_J^k \in (\mathcal{H}^{0,1})^* \otimes \mathcal{H}^{1,0}$$

with the series uniformely covergent on compaact sets

Theorem A manifold in circular type in normal form $M = (\mathbb{B}^n, J, \tau_0)$ determines uniquely the sequence of tensors $\phi_J^k \in (\mathcal{H}^{0,1})^* \otimes \mathcal{H}^{1,0}$, such that if

$$\phi_J = \sum_{k=0}^{\infty} \phi_J^k([v], \zeta) = \sum_{k=0}^{\infty} \phi_J^k([v])\zeta^k, \qquad (*)$$

satisfy the following conditions:

(i) $dd^c\tau_0(\phi_J(X), Y) + dd^c\tau_0(X, \phi_J(Y)) = 0 \; \forall X, Y \in \mathcal{H}^{0,1}$

(ii) $\overline{\partial}_b \phi_J^k + \frac{1}{2} \sum_{i+j=k} [\phi_J^i, \phi_J^j] = 0 \qquad \forall k \geq 0$ (integrability)

(iii) $dd^c\tau_0(\overline{X} + \overline{\phi_J(X)}, X + \phi_J(X)) > 0 \qquad \forall 0 \neq X \in \mathcal{H}^{0,1}$ (positivity)

Conversely a sequence $\phi_J^k \in (\mathcal{H}^{0,1})^* \otimes \mathcal{H}^{1,0}$, with (*) converging uniformely on compact sets and satisfying (i),(ii),(iii), determines uniquely a manifold of circular type.

Remark: ϕ_J restricted to any sphere $S^{2n-1}(r) = \{\|Z\| = r\}, 0 < r < 1$ is a deformation of the CR–structure of $S^{2n-1}(r)$. Viceversa a a deformation of the CR–structure of $S^{2n-1}(r)$ for any $0 < r < 1$ determines uniquely all the terms of the Fourier-type of series:

$$\phi_J = \sum_{k=0}^{\infty} \phi_J^k([v], \zeta) = \sum_{k=0}^{\infty} \phi_J^k([v])\zeta^k$$

Conclusion: there exists a bijection

$$\{\mathrm{Manifolds\,of\,Circular\,type\,in\,normal\,form\,with\,fixed\,center}\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \mathrm{Deformations\,of\,the\,CR-Structure\,of\,S^{2n-1}(r)} \\ \mathrm{satisfying\,suitable\,(explicit)\,conditions} \end{array} \right\}$$

The deformation tensor ϕ_J and its Fourier developement were considered first by Bland ('94) and Bland-Duchamp ('95) for small deformation of the standard CR-Structure of S^{2n-1}.

In fact they where primarily concerned with the embeddability as bounded domain in $\mathbb{C}^n$

Parametrization of normalizing maps:

For a given manifold of circular type M there are many different normalizing maps. The class $\mathcal{N}(M)$ of normalizing maps is the other biholomorphic datum for the class of manifold of circular type. We want parametrize the class $\mathcal{N}(M)$ of normalizing maps .

Idea to compare two different normalizing maps at given center:

M mnfold of circ. type with exhaustion τ and center $x_0 \in M$.

Near $x_0 \Longrightarrow \tau = \mu + higher\, order$

μ is squared Minkowski functional of Kobayashi indicatrix I_{x_0} of M at x_0.

Set "special frames at x_0"

$$P_{x_0} = \{(e_0, e_1, \ldots, e_{n-1})\}$$

where $e_0 \in \partial I_{x_0}$ and $e_1, \ldots, e_{n-1}$ is a unitary frame w.r.t. $dd^c\mu$ of holomorphic tangent space $\mathcal{D}^{1,0}_{e_0}(\partial I_{x_0})$

Fix $(e_0^o, e_1^o, \dots, e_{n-1}^o) \in P_{x_0}$ and a normalizing map$\Phi\ ^o\colon M \to (\mathbb{B}^n, J)$ at x_0. If $\Phi\ M \to (\mathbb{B}^n, J)$ is any other normalizing map. Then, by constuction

$$\left((\Phi^{-1} \circ \Phi^o)_*(e_0^o) = e_0^\Phi, \dots, (\Phi^{-1} \circ \Phi^o)_*(e_{n-1}^o) = e_{n-1}^\Phi\right) \in P_{x_0}$$

Fact: $\Phi \mapsto (e_0^\Phi, \dots, e_{n-1}^\Phi)$ is bijective.

Pseudo-bundle of special frames of M: $P(M) = \displaystyle\bigcup_{x_0 \text{ is a center}} P_{x_0}$

Warning: base of $P(M)$ need not be a manifold! But: If M is a strictly convex domain, $P(M)$ is the unitary frame bundle of the Kobayashi metric of M.

Fact: $\mathcal{N}(M) \cong P(M)$

Identified $P(M)$ with a subset of $U_n(\mathbb{B}^n) = Aut(\mathbb{B}^n)$, the class $\mathcal{N}(M)$ of normalizing maps is naturally parametrized by a subset $Aut(\mathbb{B}^n)$ containing $Aut(\mathbb{B}^n)_0 = U_n$.

4. Some geometrical interpretation and applications

Let $M = (\mathbb{B}^n, J, \tau_0)$ be a manifold of circular type in normal form

Let $\phi_J = \sum_{k=0}^{\infty} \phi_J^k$ be the associated deformation tensor

Let $I = I_0(M)$ be the indicatrix of M at the center 0 and μ its Minkowski functional squared.

Theorem:

(i) ϕ_J^0 is the deformation tensor of a normal form of the manifold of circular type (I, μ)

(i) the tensor field $\phi_J - \phi_J^0$ vanishes identically $\Longleftrightarrow$ M is biholomorphic to the circular domain I

Applications: Generalization of results of Patrizio-P.M.Wong-K.Leung ('87) for strictly convex domains and of Abate-Patrizio ('94) for Kähler-Finsler manifolds

(M,τ) manifold of circular type with center x_0. Set $M(x_0, r) = \{\tau < r\}$ For $0 < r < 1$

Theorem: A manifold of (M,τ) of circular type is biholomorphic to a circular domain $\Longleftrightarrow$

$(*)$ there exist distinct $r_1, r_2 \in (0,1)$ such that $M(x_0, r_1) \cong M(x_0, r_2)$

Theorem: A complex manifold manifold M is biholomorphic to the unit ball $\mathbb{B}^n$ (with standard compx structure!) $\Longleftrightarrow$

M has at least two structures of manifold of circular type (M,τ), (M,τ') relative to different centers x_0, x_0' for which $(*)$ holds.

5. Final Remarks and Questions

Some interesting open question:

(i) Find the geometric meaning of (possibly all!) terms of the Fourier series $\phi_J = \sum_{k=0}^{\infty} \phi_J^k$ of the deformation tensor of a manifold of circular type $(\mathbb{B}^n, J, \tau_0)$ in normal form.

(ii) Starting with modular data construct explicitly manifods of circular type with prescribed properties. E.G.:

– with only one center or a discrete set of centers (if there are!)

– with an open set of centers

– not embeddable in $\mathbb{C}^n$ (if it exists!)

(iii) Find conditions so that every point is a center.

P.M. Wong ('87) proved that any manifold of circular type admits non constant bounded holomorphic functions. In fact such manifolds are hyperbolic and he proves that the Caratheodory metric is bounded below by a multiple of the Kobayashi metric.

In this regard:

(iv) Find conditions on modular that characterize manifold of circular type bibolomorphic to a strictly linearly convex domain or to just a bounded domain in $\mathbb{C}^n$.