Almost complex manifolds

(M, J) $\dim_{\mathbb{R}} M = 2n$ $\quad J : TM \to TM$ with $J^2 = -\mathrm{Id}$

Integrability: J is integrable if the Nijenhuis tensor N vanishes:

$$0 = 4N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

Newlander - Nirenberg: Integrability (i.e. vanishing of N) $\Longleftrightarrow$ J is the almost complex structure defined by the structure of complex manifold on M i.e. for holomorphic coordinates $(z_1, \dots, z_n)$ on $\mathcal{U} \subset M$ if $z_j = x_j + i y_j$ then

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

"<u>holomorphicity</u>"

$f:(M,J)\to(M',J')$ of class C^α, $\alpha\geq 1$ between almost complex manifolds

is (J,J')-<u>holomorphic</u> if

$$\bar\partial_{JJ'}f(v)=0 \qquad \forall v\in TM$$

where
$$\partial_{JJ'}f : TM \longrightarrow TM'$$
$$v \longmapsto f_*(Jv)-J'(f_*v)$$

i.e. iff the differential of f commutes with alm. com. struc.

For $\underset{\text{open}}{M}\subset(\mathbb{C}^n,J_{st})$ write $\bar\partial_{J'}=\bar\partial_{J_{st}J'}$

J-holomorphic disk: $f:\Delta=\{|z|<1\}\to(M,J)$ (J_{st},J)-holomorphic

<u>Fact</u>: $\bar\partial_J f=0 \iff \bar\partial_J f\left(\frac{\partial}{\partial x}\Big|_{z=x+iy}\right)=0$

Canonical lifts of almost complex structures

For a complex manifold M it is easy to define complex structures on TM and T^*M with holomorphic charts valued in $T\mathbb{C}^n \simeq \mathbb{C}^{2n} \simeq T^*\mathbb{C}^n$ "naturally" defined starting with holomorphic charts of M.
For almost complex manifolds, *there are no holomorphic coordinates* thus one needs to do more that a "formal procedure". Fortunately somebody did it!

For example presentation in Yano-Ishihara (1973):

Given a system of real coordinates on (M, J)

$$\xi = (x^1, \dots, x^{2n}) : \mathcal{U} \subset M \longrightarrow \mathbb{R}^{2n},$$

denote

$$\widehat{\xi} = (x^1, \dots, x^{2n}, q^1, \dots, q^{2n}) : \pi^{-1}(\mathcal{U}) \subset TM \longrightarrow \mathbb{R}^{4n} \ ,$$

$$\widetilde{\xi} = (x^1, \dots, x^{2n}, p_1, \dots, p_{2n}) : \widetilde{\pi}^{-1}(\mathcal{U}) \subset T^*M \longrightarrow \mathbb{R}^{4n} \ ,$$

the associated coordinates on $TM|_{\mathcal{U}}$ and $T^*M|_{\mathcal{U}}$, determined by the components q^i of vectors $v = q^i \frac{\partial}{\partial x^i}$ and the components p_j of the covectors $\alpha = p_j dx^j$.
Let $J^i_j = J^i_j(x)$ denote the components of $J = J^i_j \frac{\partial}{\partial x^i} \otimes dx^j$,

The *canonical lifts of J on TM and T^*M* are the almost complex structures $\mathbb{J}$ on TM and $\widetilde{\mathbb{J}}$ on T^*M defined by

$$\mathbb{J} = J_i^a \frac{\partial}{\partial x^a} \otimes dx^i + J_i^a \frac{\partial}{\partial q^a} \otimes dq^i + q^b J_{i,b}^a \frac{\partial}{\partial q^a} \otimes dx^i \ ,$$

$$\widetilde{\mathbb{J}} = J_i^a \frac{\partial}{\partial x^a} \otimes dx^i + J_i^a \frac{\partial}{\partial p_i} \otimes dp_a +$$

$$+ \frac{1}{2} p_a \left(-J_{i,j}^a + J_{j,i}^a + J_\ell^a \left(J_{i,m}^\ell J_j^m - J_{j,m}^\ell J_i^m \right) \right) \frac{\partial}{\partial p_j} \otimes dx^i \ .$$

These tensor fields can be checked to be independent on the chart (x^i) and:

Yano-Ishihara (1973):

i) the standard projections $\pi : T^*M \longrightarrow M$, $\widetilde{\pi} : T^*M \longrightarrow M$ are $(\mathbb{J}, J)$-holomorphic and $(\widetilde{\mathbb{J}}, J)$-holomorphic, respectively;

ii) given a (J, J')-biholomorphism $f : (M, J) \longrightarrow (N, J')$ between almost complex manifolds, the tangent and cotangent maps

$$f_* : TM \longrightarrow TN \qquad \text{and} \qquad f^* : T^*N \longrightarrow T^*M$$

are $(\mathbb{J}, \mathbb{J}')$- and $(\widetilde{\mathbb{J}}', \widetilde{\mathbb{J}})$-holomorphic, respectively;

iii) when J is integrable, $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ coincide with above described integrable complex structures of TM and T^*M, respectively (in the integrable case, all derivatives $J^a_{i,j}$ are 0 in holomorphic coordinates).

With respect to complex coordinates and their conjugates $(z^A) = (z^a, z^{\overline{a}} \stackrel{\text{def}}{=} \overline{z^a})$, denoting by $(p_A) = (p_a, p_{\overline{a}} \stackrel{\text{def}}{=} \overline{p_a})$ the complex components of real 1-forms $\omega = p_a dz^a + \overline{p}_a dz^{\overline{a}} \in T^*M$, the canonical lift $\widetilde{\mathbb{J}}$ of an almost complex structure J on T^*M is of the form

$$\widetilde{\mathbb{J}} = J_A^B \left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B \right) +$$

$$+ \frac{1}{2} p_C \left(-J^C_{A,B} + J^C_{B,A} + J^C_L \left(J^L_{A,M} J^M_B - J^L_{B,M} J^M_A \right) \right) \frac{\partial}{\partial p_B} \otimes dz^A \,,$$

where J_B^A are the components of J w.r.t. the complex vector fields $\left(\frac{\partial}{\partial z^A}\right)$. So that, when J is integrable and $(z^1, \ldots, z^n)$ are holomorphic coordinates, $\widetilde{\mathbb{J}} = J_A^B \left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B \right)$

Strong pseudoconvexity

(M, J) almost complex manifold, $\Gamma \subset M$ (smooth) hypersurf.

Conormal bundle of Γ:

$$\mathcal{N} = \{\alpha \in T_x^*M,\ x \in \Gamma \mid \underbrace{\alpha|_{T_x\Gamma} \equiv 0}_{\text{i.e. } \operatorname{Ker}\alpha \subset T_x\Gamma}\} \subset T^*M|_\Gamma$$

$\mathcal{N}_* = \mathcal{N} \setminus \{\text{zero section}\}$

CR-structure of Γ: $(\mathcal{D}, J|_\mathcal{D})$ where

$$\mathcal{D} = \bigcup_{x \in \Gamma} \mathcal{D}_x \subset T\Gamma \qquad \mathcal{D}_x = \{v \in T_x\Gamma \mid Jv \in T_x\Gamma\}$$

A 1-form θ is a defining form for $\mathcal{D}$ if $\forall x \in \Gamma$

$$\mathcal{D}_x = \operatorname{Ker} \theta_x$$

Levi form of Γ $\quad x \in \Gamma$, $v \in \mathcal{D}_x$ $\qquad \theta$ defining form for $\mathcal{D}$

$$\mathcal{L}_x(v) := d\theta_x(v, Jv) = \theta_x([v, Jv])$$

Def. An (oriented) smooth hypersurface Γ is strongly pseudoconvex if $\mathcal{L}_x > 0 \quad \forall x \in \Gamma$

$D \subset M$ smooth domain (in our consideration always relatively compact) strongly pseudoconvex $\underset{(\mathrm{def})}{\Longleftrightarrow}$ ∂D is

Fact D strongly pseudoconvex $\Longrightarrow$ admits a $\underbrace{J\text{-pluri SH}}_{\text{to be defined}}$ defining function

For these and other basic properties may see:
survey Coupet - Gaussier - Sukhov (Arxiv 2007)

Stationary disks

Let $D \subset\subset (M, J)$ (strongly pseudoconvex), $\mathcal{N}$ be the conormal bundle of $\Gamma = \partial D$, i.e.

$$\mathcal{N} = \{\ \beta \in T_x^* M\ ,\ x \in \Gamma\ :\ \ker \beta \subset T_x \Gamma\ \}\ \subset T^* M|_\Gamma$$

$f : \overline{\Delta} \longrightarrow M$ is $\mathcal{C}^{\alpha,\varepsilon}$*-stationary disk of* D , $\alpha \geq 1$, $\varepsilon > 0$,

i) $f|_\Delta$ is a J-holomorphic embedding and $f(\partial\Delta) \subset \partial D$;

ii) there exists a $\widetilde{\mathbb{J}}$-holomorphic map $\widetilde{f} : \overline{\Delta} \longrightarrow T^* M$ in $\mathcal{C}^{\alpha,\varepsilon}(\overline{\Delta})$ with $\pi \circ \widetilde{f} = f$, so that

$$\zeta^{-1} \cdot \widetilde{f}(\zeta) \in \mathcal{N} \setminus \{\text{zero section}\}\ \ \text{for any } \zeta \in \partial\Delta \qquad (1)$$

In (1) " $\cdot$ " denotes the $\mathbb{C}$-action on $T^* M$:

$$\zeta \cdot \alpha \overset{\text{def}}{=} \Re(\zeta)\alpha - \Im(\zeta) J^* \alpha \quad \text{for any } \alpha \in T^* M, \zeta \ \ \in \mathbb{C}\ \ .$$

If f is stationary, maps $\widetilde{f}$ satisfying (ii) are called *stationary lifts of* f.

Remark Roughly speaking Condition (ii) means that the restriction along $f(\partial\Delta)$ of the CR-distribution of ∂D extends to a $\mathbb{J}$-holomorphic bundle on Δ.

Fact: (Using maximum principle for suharmonic function!) For D strongly pseudoconvex and $f : \overline{\Delta} \longrightarrow M$ stationary disk one has $f(\bar{\Delta}) \subset \bar{D}$ and $f(\zeta) \in \partial D \Leftrightarrow \zeta \in \partial\Delta$

Looking for stationary disks

(M, J) almost complex man. $D \subset\subset M$ smooth strongly pseudoconvex

Here we suppose $D \subset \mathcal{U} \subset \mathbb{R}^{2n}$ coordinate nbd.

$$D = \{ x \in \mathcal{U} \mid \rho(x) < 0 \,,\; d\rho_x \neq 0 \text{ on } \partial D \}$$

$\rho : \mathcal{U} \to \mathbb{R}_+$ global defining function

$\mathcal{N}_* = \{\text{Conormal bundle of } \partial D\} \setminus \{\text{zero section}\}$

$\tilde{\rho} : \mathbb{R}_* \times T^*M|_{\mathcal{U}} \to \mathbb{R} \times T^*M \qquad (\pi : T^*M \to M \text{ projection})$

$$\tilde{\rho}(t, \alpha) = \left(\rho(\pi(\alpha)),\; \alpha - t\, d\rho_{\pi(\alpha)}\right)$$

$$\Rightarrow \mathcal{N}_* = \{(t,\alpha) \mid t \neq 0 \;\; \tilde{\rho}(t,\alpha) = (0_{\mathbb{R}}, 0_{T^*_{\pi(\alpha)}M})\}$$

$\tilde{\rho}$ is a "defining function" for $\mathcal{N}_*$

Fact: $\mathcal{N}_*$ is a $2n$-dimensional manifold in a $(4n+1)$-dim manifold

Using suitable coordinates identify $\mathbb{R}_* \times T^*M|_{\mathcal{U}} \simeq \mathcal{V} \subset \mathbb{R}^{4n+1}$ so that

$$\mathcal{N}_* = \{(t,\alpha) \in \mathcal{V} \mid \tilde{\rho}^i(t,\alpha) = 0 \quad i = 1, \dots, 2n+1\}$$

and define
$$r : \mathbb{C} \times \mathcal{V} \subset \mathbb{C} \times \mathbb{R}^{4n+1} \longrightarrow \mathbb{R}^{2n+1}$$
$$(\zeta, t, \alpha) \longmapsto (\tilde{\rho}^1(t, \zeta^{-1}\cdot\alpha), \dots, \tilde{\rho}^{2n+1}(t, \zeta^{-1}\cdot\alpha))$$

Using these identifications

$f : \overline{\Delta} \xrightarrow[C^{\alpha,\varepsilon}]{} \overline{D} \subset \mathcal{U} \subset \mathbb{R}^{2n}$ is a stationary disk

$$\Longleftrightarrow$$

$\exists$ $\hat{f} : \overline{\Delta} \xrightarrow{C^{\alpha,\varepsilon}} \mathbb{C}^{2n} \ (\simeq (T^*M|_{\mathcal{U}}, \mathbb{J}))$ and $\lambda : \partial\Delta \xrightarrow{C^{\varepsilon}} \mathbb{R}_*$ s.t.

$$(*) \begin{cases} \bar{\partial}_{\mathbb{J}} \hat{f}(\zeta) = 0 & \zeta \in \Delta \quad \leftarrow \mathbb{J}\text{-holom on } \Delta \\ r(\zeta, \lambda(\zeta), \hat{f}(\zeta)) = 0 & \zeta \in \partial\Delta \quad \leftarrow \text{All bd conditions + lifting of } f \end{cases}$$

"Generalized Riemann-Hilbert"

There is good theory

Mikhlin - Prosdorf, Wendland ... Vekua ...

Want to define suitable operator to study (*)

Fix a. comp. str. J_0 on $\mathbb{R}^{2n}$, $x_0 \in D$ $(\subset \mathbb{R}^{2n})$ $v_0 \in T_{x_0} D \simeq \mathbb{R}^{2n}$

For $\hat{f} \in C^{\alpha,\varepsilon}(\bar{\Delta}, \mathbb{C}^{2n})$, $\lambda \in C^{\varepsilon}(\partial\Delta)$, $\mu \in \mathbb{R}_*$

the operator → Measures

$R_1(\hat{f}, \lambda, \mu) = \bar{\partial}_{J_0} \hat{f} \in C^{\alpha-1,\varepsilon}(\bar{\Delta}, \mathbb{C}^{2n})$ → J-hol. of $\hat{f}$

$R_2(\hat{f}, \lambda, \mu) = r(\zeta, \lambda(\zeta), \hat{f}(\zeta)) \in C^{\varepsilon}(\partial\Delta, \mathbb{R}^{2n+1})$ → "boundary data"

$R_3(\hat{f}, \lambda, \mu) = \pi(\hat{f})\big|_{\zeta=0} - x_0 \in \mathbb{C}^n$ → "initial point" of f

$R_4(\hat{f}, \lambda, \mu) = \pi(\hat{f})_* \left(\frac{\partial}{\partial x}\Big|_{\zeta=0}\right) - \mu v_0 \in \mathbb{C}^{2n}$ → "initial tg vector" of f

$R_5(\hat{f}, \lambda, \mu) = \hat{f}\left(\pi(\hat{f})_* \left(\frac{\partial}{\partial x}\Big|_{\zeta=0}\right)\right) - 1 \in \mathbb{R}$ → normalization of lift

If $R = (R_1, R_2, R_3, R_4, R_5) = R_{J_0, x_0, v_0}$

$\exists\, f : \bar{\Delta} \to \bar{D}$ stationary disk with $f(0) = x_0$, $f_*\left(\frac{\partial}{\partial x}\Big|_0\right) \in \mathbb{R} v_0$ $\iff$ $R_{J_0, x_0, v_0}(\hat{f}, \lambda, \mu) = 0$ has solution (**)

Well Known (Lempert's theory) For $D \subset \mathbb{C}^n$ strictly (linearly) convex, smoothly bounded with J_0 standard $\forall x_0 \in D$, $\forall v_0 \in T_{x_0} D$ the problem (**) has unique solution smoothly dependent on x_0, v_0.

"Definition" $f_0 : \bar{\Delta} \to \bar{D} \subset (M, J_0)$ stationary disk for (D, J_0), $f(0) = x_0$ $v_0 = f_*\left(\frac{\partial}{\partial x}\big|_0\right)$. ∂D is a good boundary for (J_0, f_0) if there exists a lift $\hat{f}_0$ of f_0 and function λ_0 s.t. $(\hat{f}_0, \hat{\lambda}_0, 1)$ is a solution of (**) such that R has invertible linearization at $(\hat{f}_0, \hat{\lambda}_0, 1)$

Proposition If ∂D is good for $(J_0, f_0) \Longrightarrow \exists$ $x_0 \in \mathcal{U} \subset D$ open nbd $v_0 \in \mathcal{V} \subset T_{x_0} D$ open and $\varepsilon > 0$ such that for all J c.s. with $\|J - J_0\|_{C^1(\bar{D})}$, $x \in \mathcal{U}$, $v \in T_x D \cap \mathcal{V}$ there $\exists$ unique stationary disk f with $f(0) = x$ and $f_*\left(\frac{\partial}{\partial x}\big|_0\right) = \mu v$ $\mu \neq 0$ and f depends smoothly on x, v, J

~~Implicit~~ Function Theorem type of arguments.

Remark To get a "true" theorem one must give conditions to ensure that ∂D is "good". It is not easy but something can be done. We'll see later

For the moment let us see what can be done if ∂D is good.

$D \subset (M, J)$ smooth, relativ. compact strongly pseudoconvex

for $x_0 \in D$ let $\tilde{D} \xrightarrow{\pi} D$ be the blow up at x_0

and stationary disk $f : \bar{\Delta} \longrightarrow \bar{D}$

$$0 \longmapsto x_0 \qquad f_*\left(\frac{\partial}{\partial x}\Big|_0\right) = w$$

CAUTION: Needs to be checked that $\tilde{D}$ it makes sense in almost compl. case!

may define J-holomorp. lift to the blow-up

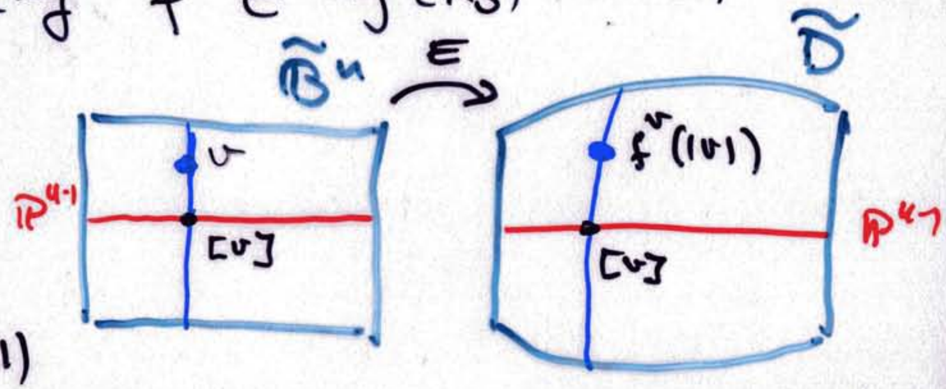
$$\begin{tikzcd} & \tilde{\bar{D}} \subset M \times \mathbb{C}P^{n-1} \arrow[d, "\pi"] \\ \bar{\Delta} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & \bar{D} \end{tikzcd}$$

$$\tilde{f}(\zeta) = \begin{cases} (f(\zeta), [f(\zeta)] & \zeta \neq 0 \\ (x_0, [w] & \zeta = 0 \end{cases}$$

Thus this gives "meaning" to the following:

<u>Definition</u> $\mathcal{F}(x_0) = \{ \forall$ stationary disk $f : \bar{\Delta} \to \bar{D}$, $f(0) = x_0 \}$
is called foliation of circular type of (D, x_0) if

① $\forall\, v \in T_{x_0} D$ there exists uniquely $f^{(v)} \in \mathcal{F}(x_0)$ with
$f^{(v)}_* \left(\frac{\partial}{\partial x}\Big|_0 \right) \in \mathbb{R}_+ v$



② The map $E : \tilde{B}^n \longrightarrow \tilde{D}$
$$(v, [v]) \longmapsto \tilde{f}^{(v)}(|v|)$$
is a diffeom. (that extends to boundaries)

<u>Definition</u> If $\mathcal{F}(x_0)$ is a foliation of circular type
D domain of circular type with center x_0.

If ∂D is "good" one has a stability result for foliation of circular type with respect to small deformations.

Proposition D domain of circular type w.r.to an almost complex structure J_0 with center x_0 s.t. ∂D is good for all $f \in \mathcal{F}(x_0)$. Then there exists $\varepsilon > 0$ and an open set $\mathcal{U} \subset D$, $x_0 \in \mathcal{U}$, such that for all complex structures J (defined on a neigh. of $\bar{D}$) with $\| J - J_0 \|_{C^1(\bar{D})} < \varepsilon$ and for any $x \in \mathcal{U}$ D is a domain of circular type w.r. to J with center in $x \in \mathcal{U}$ (i.e. $\mathcal{F}(x)$ is foliation of circular type for all $x \in \mathcal{U}$ and any such J's)

The expected "compactness" argument on the directions at x_0 works!!

Main results on "EXISTENCE" of foliations in stationary disks:

Theorem D smooth, relatively compact, strongly pseudoconvex in an almost complex manifold (M, J_0). If there exists a diffeomorphism $\varphi : \mathcal{U} \to \varphi(\mathcal{U}) \subset \mathbb{C}^n$ of open $\mathcal{U} \supset \overline{D}$ s.t. $\varphi(D)$ is a strictly (linearly) convex domain and $\varphi_*(J_0)$ is sufficiently close to $J_{st.}$ in C^1 norm on $\overline{D}$, then D is a domain of circular type w.r.to J_0 with center $x \in D$ (Any $x \in D$!)

- For $D = \mathbb{B}^n$: essentially due to Coupet-Gaussier-Sukhov 2003

- General D: P.-Spiro (Bull. Sc. Math. 2010) and independently Gaussier-Joo (Ann. Sc. Nor. Pisa 2010)

The proof is very technical. May outline the key ingredients:

- Of course one needs to show that the boundary ∂D of a strictly (linearly) convex domain $D \subset (\mathbb{C}^4, J_{st})$ is good for any stationary disk through any $x_0 \in D$

(Existence and uniqueness in this case is guaranteed by Lempert's ~~results~~ results)

- If $R = (R_1, R_2, R_3, R_4, R_5)$ is the Riemann-Hilbert operator and $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5)$ the linearization at $J_{st} = J_0$, x_0, $v_0 \in T_{x_0} D$ one shows that

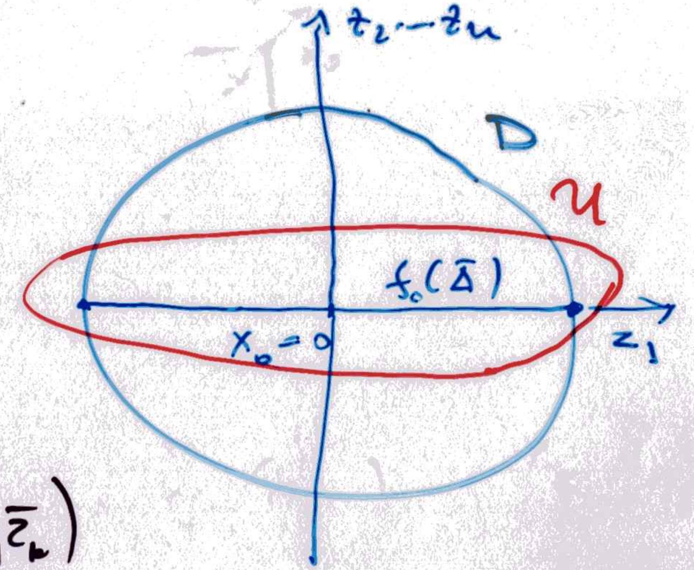
- $\overline{\mathcal{R}} = (\mathcal{R}_1, \mathcal{R}_2)$ is surjective of finite dimensional kernel (Key Fact!)
- the other components $\mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ "nail down" the isomorphism! (initial data + norm.)

$\overline{R} = (R_1, R_2)$ linearizes

$$\overline{R}(\hat{f}, \lambda, \mu) = \Big(\underbrace{\bar{\partial}_J \hat{f}}_{\substack{J\text{-hol} \\ \text{of } \hat{f}}}\,,\ \underbrace{r(\zeta, \lambda(\zeta), \hat{f}(\zeta))}_{\text{boundary data}} \Big)$$

To study this operator (i.e. prove surjectivity of the Riemann-Hilbert operator) due to results of Globevnik (1994) and "classical" Riemann-Hilbert theory one needs to compute certain "topological" data the Maslov index of the conormal bundle along the boundary of stationary disks and check that satisfy suitable estimates

Key for the computation : "Flatten stationary disks" i.e. (Lempert and later Pang) There exists global holomorphic coordinates $(z_1 \dots z_n)$ in a neighborhood $\mathcal{U}$ of $f_0(\overline{\Delta})$ (closed stationary disk) and a defining function ρ for D such that for all $\zeta \in \overline{\Delta}$



- $f(\zeta) = (\zeta, 0, \dots, 0)$
- $\rho(z) = -1 + \|z'\|^2 + \sum_{j,k=2}^{n} A_{jk} z_j \bar{z}_k + \mathrm{Re}\left(\sum_{j,k=2}^{n} B_{jk} z_j \bar{z}_k\right)$
$$+ O(\|z'\|^3)$$

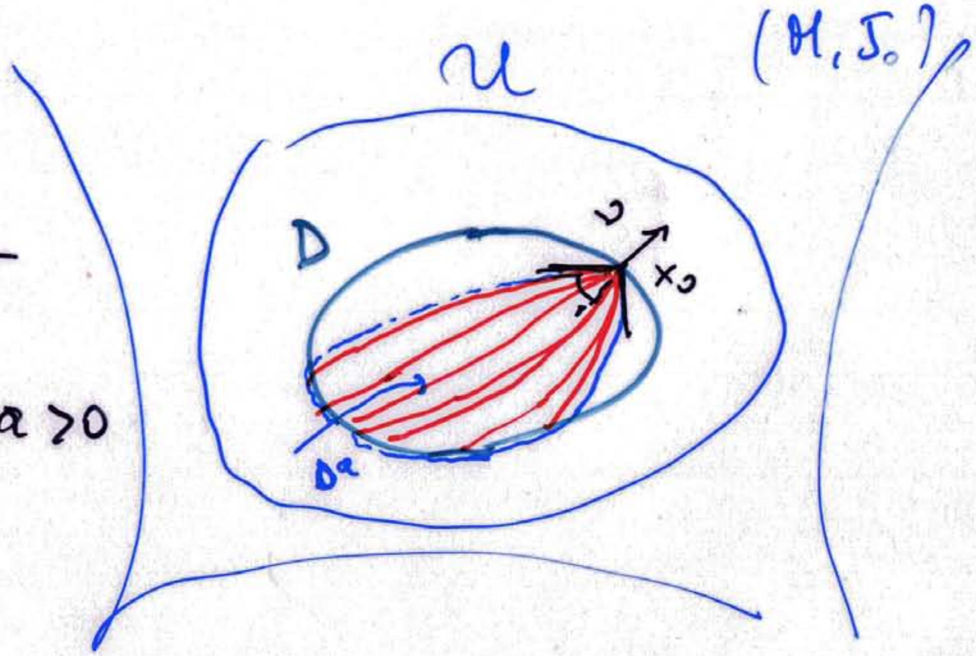
where $\|z'\|^2 = \sum_2^n |z_j|^2$

One computes efficiently in these coordinates under which $f_0(\overline{\Delta})$ "looks" as a straight disk through the origin in $\mathbb{B}^n$.

Boundary version:

ν normal to ∂D at x_0 w.r.to a Riem. metric compatible with J

$\mathcal{C}^{(a)} = \{ v \in T_{x_0} M \mid \langle v, \nu \rangle > a \}$ $\quad a > 0$



Theorem $D \subset (M, J_0)$ s.t. $\exists\ \varphi : \mathcal{U} \to \varphi(\mathcal{U})$ $(\bar{D} \subset \mathcal{U})$ diffeomorphism s.t. $\varphi(D)$ is strictly (linearly) convex and $\varphi_*(J_0)$ is sufficiently close to J_{st} on $\bar{D}$ w.r.to C^1 norm. Then $\forall\, a > 0$ and for any $x_0 \in \partial D$ there exists a foliation in stationary disks of a subdomain $D^a(x_0) \subset D$ s.t. all this disks map $1 \in \partial\Delta$ onto x_0 and have "boundary tangent vector" in x_0 contained in $\mathcal{C}^{(a)}$

Normal forms of almost complex domains of circular type

Idea: repeat the story told in the integrable case and show that there are many of them and are interesting for pluripotential theory.

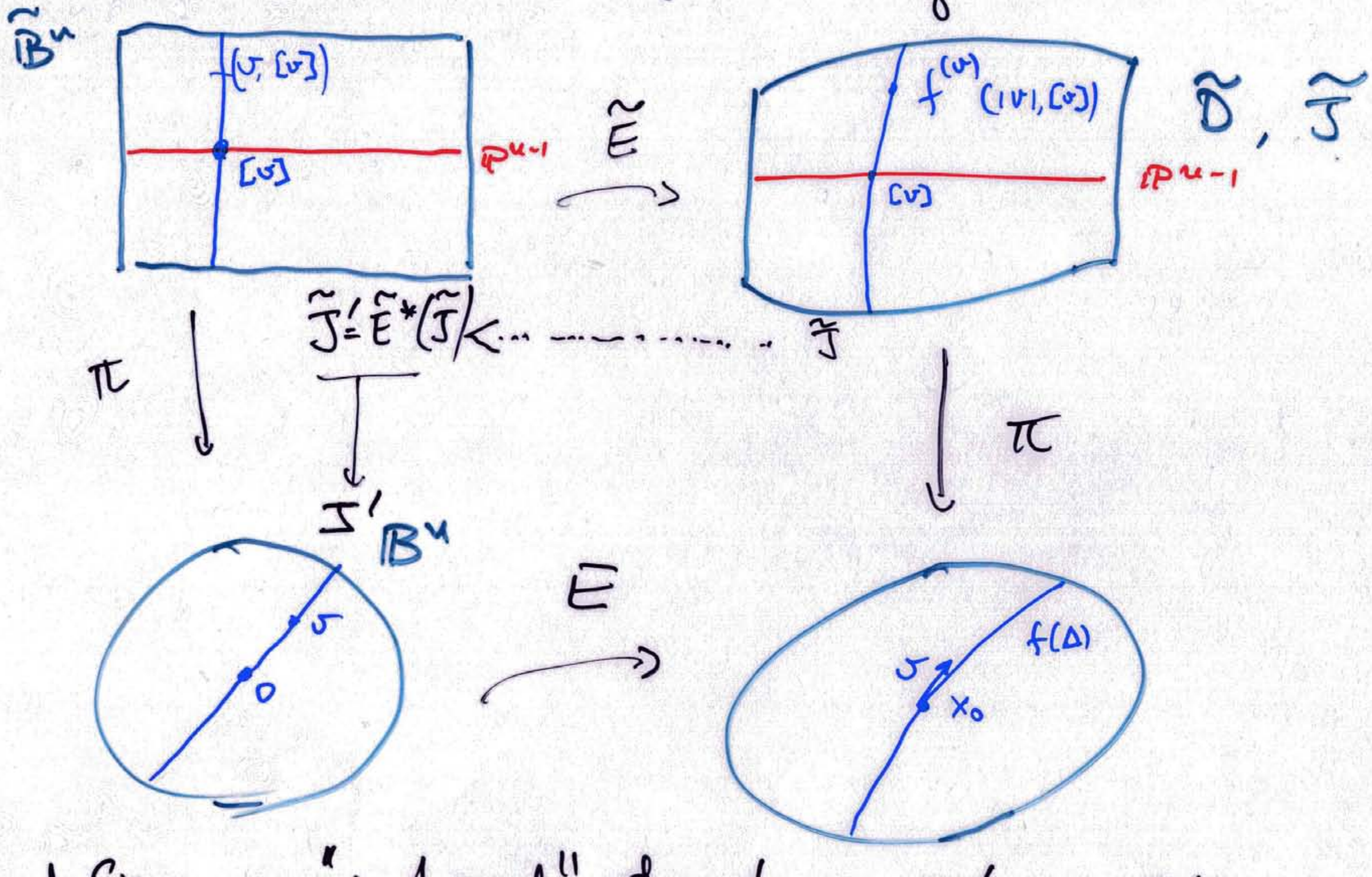
$D \subset (M, J)$ smooth, (relatively compact)
Alm. Comp. domain of circular type with centre x_0

i.e. the collection $\mathcal{F}(x_0) = \{$stationary disks through $x_0\}$ is such that

1 $\forall v \in T_{x_0}D$ $\exists$ unique $f^{(v)} \in \mathcal{F}(x_0)$ with $f_*\left(\frac{\partial}{\partial x}\big|_0\right) \in \mathbb{R}_+ v$

2 The "generalized Riemann map" $\tilde{E} : \tilde{\mathbb{B}}^n \longrightarrow \tilde{D}$
$(v, [v]) \longmapsto \tilde{f}^{(v)}(|v|)$
is a diffeomorphism, (J_{st}, J) hol along fibers, which extends up to boundaries.

Repeat the construction of the integrable case:



One defines an "induced" structure J' on $\mathbb{B}^n$

<u>Important</u> E is only continuous at 0. Need to define suitable differentiable structure at 0 which makes it diffeomorphism.

Thus, after all the necessary verifications, one concludes

"Every of almost complex domain $D \subset (M, J)$ of circular type has a normal form $(\mathbb{B}^n, J')$ (i.e. D is (J', J)-biholomorphic to $(\mathbb{B}^n, J')$) where $\mathbb{B}^n$ with J' has the property that radial disks through the origin are J'-stationary disks.

We like to provide characterization of such J' on $\mathbb{B}^n$, possibly via "equations"

The key is to impose that radial disks are stationary.

L-Almost complex structures

On the unit ball $\mathbb{B}^n$ let $\mathcal{Z}$ be radial distribution

i.e. $\forall z \in \mathbb{B}^n \setminus \{0\}$ $\quad \mathcal{Z}_z = \mathrm{Span}_{\mathbb{C}}\left(\sum_j z^j \frac{\partial}{\partial z^j}\right)$

Theorem A pair $(\mathbb{B}^n, J)$ is a domain of circular type in normal form $\Longleftrightarrow$

1. $\mathcal{Z}$ is J-invariant $(\mathcal{Z} = J\mathcal{Z})$ and $J|_{\mathcal{Z}} = J_{st}|_{\mathcal{Z}}$
2. $\forall$ straight disks through 0 are J-stationary
3. $\tilde{\mathbb{B}} \longrightarrow \mathbb{B}$ determined by J is diffeom. to the usual one

In this case we call J L-Almost complex structure on $\mathbb{B}^n$

CRUCIAL (2) holds $\Longleftrightarrow$ the "coefficients" of J are in the range of a explicit Fredholm operator which is finite codimensional among the possible ones

In other words: It is possible to give a finite number of conditions ("Equations") which such J must satisfy.

Consequences

1. There are "many" such structures

2. Expressing J in terms of deformation tensor one may give special conditions under which the "Equations" are automatically satisfied.

2. is very useful to provide examples.
We will illustrate such kind of constructions.

Deformation and L-Almost Complex Structures

On $\mathbb{B}^n \setminus \{0\}$

$\mathcal{Z}$ radial distribution $\qquad \mathcal{Z} = \mathbb{C}\left(\sum_j z^j \frac{\partial}{\partial z^j}\right) = \mathbb{C}\,Z$

$$\Rightarrow \begin{cases} J_{st}\,\mathcal{Z} = \mathcal{Z} \\ \mathcal{Z} = \operatorname{Ann} dd^c_{st}(\log \tau_0) \end{cases}$$

here $\tau_0(z) = \|z\|^2$

and $d^c = J^*_{st} \circ d \circ J^*_{st}$

Let $\quad \mathcal{H}_z = \operatorname{Ker} dd^c_{st}\,\tau_0(Z, \cdot)\big|_z$

"orthogonal complement" to $\mathcal{Z}_z$ w.r. to $dd^c_{st}\tau_0$

+ it is the holomorphic tg space to the sphere $S(\|z\|) = \{w \mid \tau_0(w) = \|z\|^2\}$

This coincidence is peculiar of J_{st}

w.r.to J_{st} on $\mathbb{B}^n \setminus \{0\}$

$$T_z^{\mathbb{C}} = \mathcal{Z}_z^{\mathbb{C}} \oplus \mathcal{H}_z^{\mathbb{C}} = \left(\mathcal{Z}_z^{1,0} \oplus \mathcal{Z}_z^{0,1}\right) \oplus \left(\mathcal{H}_z^{1,0} \oplus \mathcal{H}_z^{0,1}\right)$$

An almost complex structure J is determined prescribing the $(-i)$-eigenspace $(T_z \mathbb{B}^4)^{0,1}_J$ in $T^{\mathbb{C}}_z \mathbb{B}$ for $z \in \mathbb{B}^4 \setminus \{0\}$

Define it using deformation tensor(s).

$$(T_z \mathbb{B}^4)^{0,1}_J = (T^{0,1}_z \mathbb{B}^4)_{st} + \underbrace{\varphi\left((T^{0,1}_z \mathbb{B}^4)_{st}\right)}_{\subset (T^{1,0}_z \mathbb{B}^4)_{st}}$$

$$= \mathcal{Z}^{0,1}_z \oplus \varphi^{\mathcal{Z}}(\mathcal{Z}^{0,1}_z) \oplus \varphi^{\mathcal{Z},\mathcal{H}}(\mathcal{Z}_z)$$

$$+ \mathcal{H}^{0,1}_z \oplus \varphi^{\mathcal{H}}(\mathcal{H}^{0,1}_z) \oplus \varphi^{\mathcal{H},\mathcal{Z}}(\mathcal{H}^{0,1}_z)$$

here $\varphi = \varphi^{\mathcal{Z}} \oplus \varphi^{\mathcal{Z},\mathcal{H}} \oplus \varphi^{\mathcal{H}} \oplus \varphi^{\mathcal{H},\mathcal{Z}} \in \mathrm{Hom}(T^{0,1}\mathbb{B}^4, T^{1,0}\mathbb{B}^4)$

where

$\varphi^{\mathcal{Z}} \in \mathrm{Hom}(\mathcal{Z}^{0,1}, \mathcal{Z}^{1,0})$ $\quad$ $\varphi^{\mathcal{H}} \in \mathrm{Hom}(\mathcal{H}^{0,1}, \mathcal{H}^{1,0})$

$\varphi^{\mathcal{Z},\mathcal{H}} \in \mathrm{Hom}(\mathcal{Z}^{0,1}, \mathcal{H}^{1,0})$ $\quad$ $\varphi^{\mathcal{H},\mathcal{Z}} \in \mathrm{Hom}(\mathcal{H}^{0,1}, \mathcal{Z}^{1,0})$

Easy fact,

$\mathcal{Z}$ is J-invariant ($J\mathcal{Z} = \mathcal{Z}$) and $J|_{\mathcal{Z}} = J_{st}|_{\mathcal{Z}}$ $\iff$ $\varphi = \varphi^{\mathcal{H}} + \varphi^{\mathcal{H},\mathcal{Z}}$

This is condition (i) of the definition of L-Almost Compx Str.

For J on $\mathbb{B}^n$ 2 more useful conditions:

1. J is said <u>nice</u> if in addition

$$\varphi = \varphi^{\mathcal{H}}$$

Easy: $\iff$ $\mathcal{H}$ is J-invariant

2. J is said very nice if

$$\varphi = \varphi^{\mathcal{H}} \quad \text{and} \quad \mathcal{L}_{\mathcal{Z}^{0,1}}\, \varphi^{\mathcal{H}} = 0$$

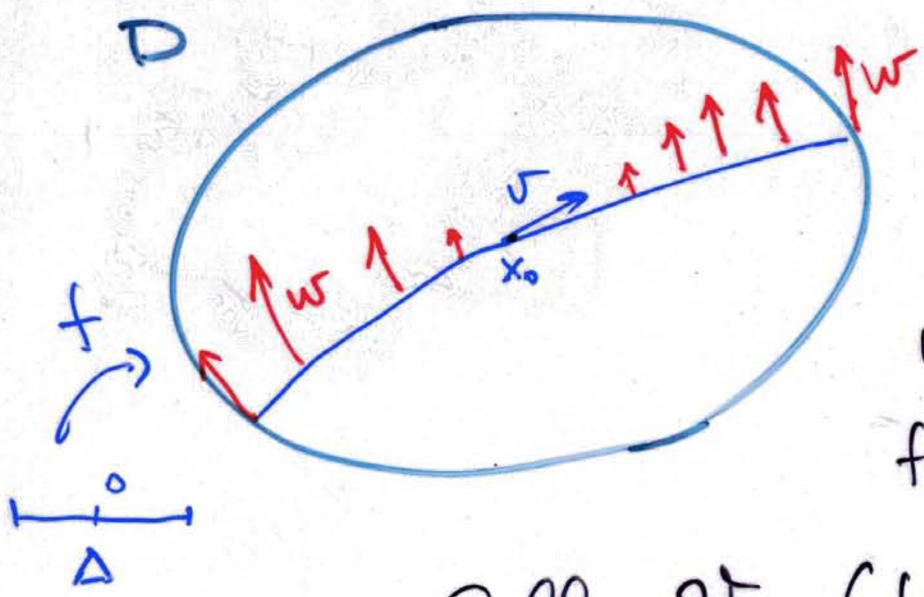
i.e $\iff$ J is nice and deformation is "holomorphic" along radial disks.

Geometric motivation for "very nice":
Relation between stationary and Kobayashi extremal disks in integrable case:

Idea probably goes back to Poletski and for sure to Lempert ('80) stationary disks satisfy Euler-Lagrange conditions for extremal disks ("critical")

Lempert ('80) The two notions agree for strictly convex domains in $\mathbb{C}^n$ (J_{st} !)

Recent Gaussier - Joo (Ann. Sc. Norm. Pisa 2010):
For almost complex domains the two notions do not agree in general (counterexamples and sufficient conditions for coincidence).



Fix $x_0 \in D = \{\rho < 0\} \subset (M, J)$
J-PSH

and $v \in T_{x_0} D$

want to apply "infinitesimal variation"
w to J-holom. disk f
fixing x_0, v and keeping $\partial\Delta \to \partial D$

Call $Var(f)$ all such variations
$Var(f)$ must be "big" and suitably "regular"

<u>Theorem</u> (Gaussier - Joo 2010) If $Var(f)$ contains a $(2n-2)$-dim J-invariant vector space and it's generated by $e_1, Je_1, \dots e_{n-1}, Je_{n-1}$ with $\zeta^{-1} e_j, \zeta^{-1} Je_j \in C^{\alpha,\varepsilon}(\bar{\Delta})$ then

f stationary $\Longleftrightarrow$ Kobayashi critical
(i.e. locally extremal).

Application in our case: $(\mathbb{B}^n, J)$ is an almost complex domain of circular type in normal form (i.e. J is an L-Alm. Compx Str.) May construct variations of radial disks deforming along the directions of $\mathcal{H}$ (J_{st} holom. tg bundle to spheres)

Get a class of variations $\tilde{\mathcal{V}} \subset \mathrm{Var}(f)$ $\forall$ radial f

Fact: $J\tilde{\mathcal{V}} \subset \mathrm{Var}(f) \iff \mathcal{L}_{Z^{1,0}} J = 0$

moreover need $\mathcal{H}$ J-invariant (i.e. J is nice)

Conclusion:

Proposition: If J is very nice then the radial disks of an almost complex domain of circular type in normal form $(\mathbb{B}^n, J)$ are not only stationary but also Kobayashi critical (extremal).

Plurisubharmonic functions/pseudoconvex manifolds

(M, J) almost cmplx manifold, $\Omega^k(M)$, $k \geq 0$, k-forms of M.

Denote by $d^c : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ the classical d^c-operator

$$d^c\alpha = (-1)^k (J^* \circ d \circ J^*)(\alpha) \ ,$$

where J^* denotes the usual action of J on k-forms, i.e.

$$J^*\beta(v_1, \dots, v_k) \stackrel{\mathrm{def}}{=} (-1)^k \beta(Jv_1, \dots, Jv_k)$$

When J is integrable:

$$d^c = i(\overline{\partial} - \partial) \ , \quad \partial\overline{\partial} = \frac{1}{2i} dd^c \ , \quad dd^c = -d^c d$$

and $dd^c u$ is a J-Hermitian 2-form for any $\mathcal{C}^2$-function u.

when J is not integrable, $d^c d \neq -dd^c$ and the 2-form $dd^c u$, for $u \in \mathcal{C}^2(M)$, is usually not J-Hermitian. In fact:

$$dd^c u(JX_1, X_2) + dd^c u(X_1, JX_2) = 4N_{X_1 X_2}(u) \ , \tag{1}$$

where $N_{X_1X_2}$ is the Nijenhuis tensor evaluated on X_1, X_2 and is – of course – in general non zero. This fact suggests the following definition.

Definition 0.1. Let $u : \mathcal{U} \subset M \longrightarrow \mathbb{R}$ be of class $\mathcal{C}^2$. The *J-Hessian of u at x* is the symmetric form $\mathcal{H}ess(u)_x \in S^2 T_x M$, whose associated quadratic form is $\mathcal{L}(u)_x(v) = dd^c u(v, Jv)_x$. By polarization formula and (1), one has that, for any v, $w \in T_x M$,

$$\begin{aligned} \mathcal{H}ess(u)_x(v, w) = \frac{1}{2}\left(dd^c u(v, Jw) + dd^c u(w, Jv)\right)\Big|_x = \\ = dd^c u(v, Jw)_x - 2N_{vw}(u) \ . \end{aligned} \tag{2}$$

Remark $\mathcal{H}ess(u)_x$ is also J-Hermitian, i.e.

$$\mathcal{H}ess(u)_x(Jv, Jw) = \mathcal{H}ess(u)_x(v, w) \text{ for any } v, w.$$

and it is associated with the Hermitian antisymmetric tensor

$$\begin{aligned}\mathcal{H}ess(u)(J\cdot, \cdot) &= \frac{1}{2}\left(dd^c u(\cdot, \cdot) + dd^c u(J\cdot, J\cdot)\right) \\ &= \frac{1}{2}\left(dd^c u + J^* dd^c u\right).\end{aligned}$$

The *Levi form of* u *at* x is the quadratic form

$$\mathcal{L}(u)_x(v) = dd^c u(v, Jv)|_x$$

and it is related with the notion of J-plurisubharmonicity.

Fact: This dd^c is the same as N. Pali's (Manuscripta 2005) used to study positivity and by Pliś to study the inhomogeneous complex Monge-Ampère equation (ArXiv June 2011)

An upper semicontinuous function $u : \mathcal{U} \subset M \longrightarrow \mathbb{R}$ is called *J-plurisubharmonic* if $u \circ f : \Delta \longrightarrow \mathbb{R}$ is subharmonic for any J-holomorphic disk $f : \Delta \longrightarrow \mathcal{U} \subset M$. As for complex manifolds for $u \in \mathcal{C}^2(\mathcal{U})$

$$u \text{ is } J\text{-plurisubharmonic}$$

$$\Longleftrightarrow$$

$$\mathcal{L}(u)_x(v) = \mathcal{H}ess(u)_x(v, v) \geq 0 \text{ for any } x \in \mathcal{U} \text{ and } v \in T_x M.$$

$u \in \mathrm{Psh}(\mathcal{U}) \cap \mathcal{C}^2(\mathcal{U})$ is said strictly plurisubharmonic if and only if $\mathcal{H}ess(u)_x$ is positive definite at any $x \in \mathcal{U}$.

The almost complex manifold (M, J) is called *strongly pseudoconvex (or Stein) manifold* if it admits a $\mathcal{C}^2$ strictly plurisubharmonic exhaustion $\tau : M \longrightarrow]-\infty, \infty[$

Maximal plurisubharmonic functions

J-plurisubharmonic functions share most basic properties of classical plurisubharmonic functions. In particular for any open domain $\mathcal{U} \subset M$, , as for domains in complex manifolds, the class $\mathrm{Psh}(\mathcal{U})$ is a convex cone and if $u_i \in \mathrm{Psh}(\mathcal{U})$ also $u' = \max\{ u_1, \ldots, u_n \}$ are in $\mathrm{Psh}(\mathcal{U})$. Thus it is natural to consider the notion of "maximal" J-plurisubharmonic functions.

Definition Let D domain in a strongly pseudoconvex almost complex manifold (M, J). $u \in \mathrm{Psh}(D)$ is called *maximal* if for any open $\mathcal{U} \subset\subset D$ and $h \in \mathrm{Psh}(\mathcal{U})$

$$\limsup_{z \to x} h(z) \leq u(x) \text{ for all } x \in \partial\mathcal{U} \implies \qquad h \leq u|_{\mathcal{U}} \tag{3}$$

The characterization of maximal plurisubharmonic functions "nails down" the right candidate for almost complex Monge-Ampère operator:

Theorem *Let $D \subset M$ be a domain of a strongly pseudo-convex almost complex manifold (M, J) of dimension $2n$. A function $u \in \mathrm{Psh}(D) \cap \mathcal{C}^2(D)$ is maximal if and only if it satisfies*

$$(dd^c u + J^*(dd^c u))^n = 0 \; . \tag{4}$$

Proof. Let $\tau : M \longrightarrow]-\infty, +\infty[$ be a $\mathcal{C}^2$ strictly plurisubharmonic exhaustion for M and assume that u satisfies (4). Let $h \in \mathrm{Psh}(\mathcal{U})$ with

$$\limsup_{z \to x} h(z) \leq u(x) \text{ for all } x \in \partial\mathcal{U} \; .$$

Let $x_o \in \mathcal{U}$ so that $u(x_o) < h(x_o)$. Let $\lambda > 0$ so small that

$$h(x_o) + \lambda\left(\tau(x_o) - M\right) > u(x_o)\ , \qquad M = \max_{y \in \overline{\mathcal{U}}} \tau(y)\ ,$$

and define $\qquad \widehat{h} \overset{\text{def}}{=} h + \lambda(\tau - M)\big|_{\mathcal{U}}$. By construction,

$$\widehat{h} \in \mathrm{Psh}(\mathcal{U}) \qquad \limsup_{z \to x \in \partial\mathcal{U}} h(z) \leq u(x), \qquad (\widehat{h} - u)(x_o) > 0.$$

Thus $\widehat{h} - u$ has maximum at some inner point $y_o \in \mathcal{U}$.
Let $0 \neq v_o \in T_{y_o}M$ with

$$\left(dd^c u + J^*(dd^c u)\right)_x (v_o, Jv_o) = \mathcal{H}ess_x(u)(v_o, v_o) = 0\ .$$

and $f : \Delta \longrightarrow M$ a J-holomorphic disk with $f(0) = y_o$ and

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_0\right) = v_o\ , \quad f_*\left(\left.\frac{\partial}{\partial y}\right|_0\right) = f_*\left(J_{\mathrm{st}}\left.\frac{\partial}{\partial x}\right|_0\right) = Jv_o\ .$$

But then he function $G \stackrel{\text{def}}{=} (\widehat{h}-u)\circ f = (h+(\lambda\tau-\lambda M-u))\circ f$ is subharmonic on some disk$\Delta_r = \{|\zeta| < r\}$. In fact τ is $\mathcal{C}^2$, strictly plurisubharmonic and $\mathcal{H}ess(u)_{y_o}(v_o, v_o) = 0$, so that

$$0 < \mathcal{H}ess((\lambda\tau-\lambda M-u))_{y_o}(v_o, v_o) = 2i\,\partial\overline{\partial}((\lambda\tau - \lambda M - u)\circ f)\Big|_0 .$$

By continuity, there exists $r > 0$ so that

$$0 < 2i\,\partial\overline{\partial}((\lambda\tau - \lambda M - u)\circ f)\Big|_\zeta \qquad \text{for any} \quad \zeta \in \overline{\Delta_r} .$$

It follows that $(\lambda\tau-\lambda M-u)\circ f|_{\Delta_r}$ is strictly subharmonic and that $G|_{\Delta_r}$ is subharmonic, being sum of subharmonic functions.
Since y_o is a point of maximum for $\widehat{h}-u$ on $f(\Delta)\subset\mathcal{U}$, then G has a maximum in thew interior ofΔ_r and hence it is constant and that $h\circ f|_{\Delta_r}$ is $\mathcal{C}^2$ with $2i\,\partial\overline{\partial}(h\circ f)\Big|_{\Delta_r} < 0$, contradicting the subharmonicity of $h\circ f$.

Conversely, let $u \in \mathcal{C}^2(D) \cap \operatorname{Psh}(D)$ be maximal with $\mathcal{H}ess_{y_o}(u)(v,v) > 0$ for some $y_o \in D$ and all $0 \neq v \in T_{y_o}M$.

Known Fact: $\exists$ a rel. cmpct neigh. $\mathcal{U}$ of y_o (J, J')-biholom. to (B^n, J'), with J' arbitra. $\mathcal{C}^2$-close to the stand cmplx struc.

$\Rightarrow$ Pulling back the squared norm, may assume that $\exists$ $\mathcal{C}^2$ strictly J-PSH exhaustion τ on $\mathcal{U}$, with $\tau \to 1$ at $\partial\mathcal{U}$.
$\exists\, c > 0$ s. t. for $x \in \mathcal{U}$ and $v \in T_xM \simeq \mathbb{R}^{2n}$ with $|v| = 1$

$$\mathcal{H}ess_x(u + c(1-\tau))(v,v) \geq 0$$

$\Rightarrow$ $\widehat{h} \stackrel{\text{def}}{=} (u + c(1-\tau))|_{B_{y_o}(r)}$ is $\mathcal{C}^2(\mathcal{U}) \cap \operatorname{Psh}(\mathcal{U})$, and it is dominated by u on $\partial\mathcal{U}$ so that by maximality satisfies $\widehat{h} \leq u$ on $\mathcal{U}$. But for $\epsilon > 0$ $\emptyset \neq \tau^{-1}([0, 1-\epsilon[) \subsetneq \mathcal{U}$ and hence such that, on this subset, $\widehat{h} \geq u + c\epsilon > u$, contradicting the maximality of u. $\square$

Green functions of nice circular domains

Definition Let D be a domain in a strongly pseudoconvex, almost complex manifold (M, J). We call *almost pluricomplex Green function with pole at* $x_o \in D$ an exhaustion $u : \overline{D} \longrightarrow [-\infty, 0]$ such that

i) $u|_{\partial D} = 0$ and $u(x) \simeq \log \|x - x_o\|$ when $x \to x_o$, for some Euclidean metric $\| \cdot \|$ on a neighborhood of x_o;
ii) it is J-plurisubharmonic;
iii) it is a solution of the generalized Monge-Ampere equation $(dd^c u + J^*(dd^c u))^n = 0$ on $D \setminus \{x_o\}$.

Notice that, if a Green function with pole x_o exists, by a direct consequence of property of maximality it is unique.

For an almost complex domain D of circular type in (M, J) with center x_o with Riemann map $\exp : \widetilde{B^n} \longrightarrow \widetilde{D}$ the *standard exhaustion of* D $\tau_{(x_o)} : D \longrightarrow [0, 1[$ is defined by

$$\tau(x) = \begin{cases} |\exp^{-1}(x)|^2 & \text{if } x \neq 0 \ , \\ \\ 0 & \text{if } x = x_o \ . \end{cases}$$

so that if D is in normal form, i.e. $D = (B^n, J)$ with J almost L-complex structure, its standard exhaustion is just $\tau_o(z) = |z|^2$.

Proposition *Let* D *be a domain of circular type in* (M, J) *with center* x_o *and standard exhaustion* $\tau_{(x_o)}$*. If* $u = \log \tau_{(x_o)}$ *is* J*-plurisubharmonic, then* u *is an almost pluricomplex Green function with pole at* x_o*.*

Proof. With no loss of generality, we may assume that the domain is in normal form, i.e. $D = (B^n, J)$ and $\tau_{(x_o)}(z) = \tau_o(x) = |x|^2$. Since τ_o is smooth on $B^n \setminus \{0\}$ and $u = \log \tau_o$ is J-plurisubharmonic, we have that $\mathcal{H}ess(u)_x \geq 0$ for any $x \neq 0$. On the other hand, for any straight disk $f : \Delta \longrightarrow B^n$ of the form $f(\zeta) = v \cdot \zeta$, we have that $u \circ f$ is harmonic and $\mathcal{H}ess(u)_{f(\zeta)}(v, v) = 0$ for any $\zeta \neq 0$. This means that $\mathcal{H}ess(u)_x \geq 0$ has at least one vanishing eigenvalue at any point of $B^n \setminus \{0\}$ so that u satisfies the Monge-Ampère equation. The ther conditions can be checked directly from definitions. $\square$

When J is integrable, *the standard exhaustion $u = \log \tau_{(x_o)}$ of the normal form of a domain of circular type is automatically plurisubharmonic* .

In the almost complex case, this is no longer true!

EXAMPLE

On the blow up of the unit 2-ball $\widetilde{B}^2$ consider vector fields

$$Z, J_{\mathrm{st}}Z, E, J_{\mathrm{st}}E$$

where Z is the lift of $\Re\left(z^i \frac{\partial}{\partial z^i}\right)$, and E is any vector field in the distribution $\mathcal{H}$ that satisfies the conditions

$$[Z, E] = [J_{\mathrm{st}}Z, E] = 0 \ , \quad [E, J_{\mathrm{st}}E] = -J_{\mathrm{st}}Z. \qquad (*)$$

The standard holomorphic bundle $T^{10}\widetilde{B}^2$ is generated at all points by the complex vector fields $Z^{10} = Z - iJ_{\mathrm{st}}Z$ (which generates the "radial" distribution) and $E^{10} = E - iJ_{\mathrm{st}}E$ (which generates the holomorphic tangent bundle to "spheres").

Denote by $(E^{10*}, E^{01*}, Z^{10*}, Z^{01*})$ the field of complex coframes, dual to $(E^{10}, E^{01} = \overline{E^{10}}, Z^{10}, Z^{01} = \overline{Z^{10}})$ at all points.

For a smooth real valued function $h : \widetilde{B}^n \longrightarrow \mathbb{R}$, with

– h constant on spheres $S_c = \{\ \tau_o(z) = c\ \}$
– $h \equiv 0$ on an open neighborhood of $\pi^{-1}(0) = \mathbb{C}P^1$.

Define a deformation tensor $\phi \in \mathrm{Hom}(\mathcal{H}^{01}, \mathcal{Z}^{10} + \mathcal{H}^{10})$ by

$$\phi_z = h(z) Z^{10}_z \otimes E^{01*}_z$$

and let J be almost complex structure determined by the deformation tensor ϕ i.e. such that

$$T^{10}_{Jz}\widetilde{B}^n = \mathbb{C}Z^{10}_z \oplus \mathbb{C}\widetilde{E}^{10}_z \quad \widetilde{E}^{10}_z \stackrel{\text{def}}{=} E^{10}_z + h(z) Z^{01}_z$$

It is not hard (by direct inspection) to prove that such J is an almost L-complex structure and $(\widetilde{B}^n, J)$ is an almost complex domain of circular type in normal form.

Fact: *if $h \not\equiv 0$, the function $u = \log \tau_o$ is not J-plurisubharmonic.*

Using the definition of J and $[Z,E]=[J_{\mathrm{st}}Z,E]=0$, $[E,J_{\mathrm{st}}E]=-J_{\mathrm{st}}Z$, one computes

$$\mathcal{H}ess(E^{10},\widetilde{E}^{10})=2(1+2hh_Z)$$

$$\mathcal{H}ess(E^{10},Z^{10})=2h_Z \qquad \mathcal{H}ess(Z^{10},Z^{10})=0\ ,$$

(here, $(\cdot)_Z \overset{\mathrm{def}}{=} Z(\cdot)$ is the derivation along Z) so that the matrix H of the components of $\mathcal{H}ess(u)$ with respect to the frame $\{E^{10},Z^{10}\}$, is

$$H=2\begin{pmatrix}1+2hh_Z & h_Z\\ h_Z & 0\end{pmatrix}\ .$$

Since the eigenvalues of H are

$$\lambda_\pm=2\frac{(1+2hh_Z)\pm\sqrt{(1+2hh_Z)^2+4h_Z^2}}{2},$$

u is J-plurisubharmonic if and only if $h_Z\equiv 0$, i.e. if and only if $h\equiv 0$ (h vanishes in a neighborhood of 0!).

Notice that in the example the complex stucture J is arbitrarily close to an integrable complex structure. The key is that one should restrict to the class of to small deformations of integrable structures and **nice**:

Theorem 0.2. *Let D be a nice circular domain with standard exhaustion $\tau_{(x_o)}$ and normal form (B^n, J). If J is a sufficiently small $\mathcal{C}^1$-deformation of J_{st}, then $u = \log \tau_{(x_o)}$ is the Green function with pole at x_o.*

Proof. Need only to show that $u = \log |z|^2$ is J-PSH on $B^n \setminus \{0\}$. If (B^n, J) is nice, then $\mathcal{H}ess(u)(\mathcal{Z}, \mathcal{H}) = 0$ at any $z \neq 0$. Since spheres are J-stongly pscx for J close to the standard structure, then the plurisubharmonicity of u follows easily computing along "orthogonal" directions.$\square$

CONCLUSION

Putting together all results one gets:

THEOREM *Let D be an almost complex domain of circular type with center x_o in (M, J) strongly pseudoconvex. If the normal form (B^n, J') of (D, J) is very nice with J' sufficiently close to J_{st}, then*

a) *the stationary foliation $\mathcal{F}^{(x_o)}$ consists of extremal disks w.r.t. Kobayashi metric;*

b) *the function $u = \log \tau_{(x_o)}$ is the almost pluricomplex Green function of D with pole x_o;*

c) *the distribution $\mathcal{Z}_z = \ker \mathcal{H}ess(u)_z$ is integrable and the closures of its integral leaves are the disks in $\mathcal{F}^{(x_o)}$.*