MONGE-AMPÈRE FOLIATIONS FOR DEGENERATE SOLUTIONS

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ABSTRACT. We study the problem of the existence and the holomorphicity of the Monge-Ampère foliation associated to a plurisubharmonic solutions of the complex homogeneous Monge-Ampère equation even at points of arbitrary degeneracy. We obtain good results for real analytic unbounded solutions. As a consequence we also provide a positive answer to a question of Burns on homogeneous polynomials whose logarithm satisfies the complex Monge-Ampère equation and we obtain a generalization the work of P.M. Wong on the classification of complete weighted circular domains.

1. INTRODUCTION

Let M be a Stein manifold of dimension $n > 1$ and suppose that there is a continuous plurisubharmonic exhaustion $\rho \colon M \to [0, +\infty)$ of class C^∞ on $M_* = \{\rho > 0\}$ with $d\rho \neq 0$ on $\{\rho > 0\}$ such that on M_* the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^n = 0. \tag{1.1}$$

On the open set P of M_* where $dd^c \rho > 0$, it is well known that $\mathrm{Ann}(dd^c u)$ is an integrable distribution of complex rank 1 which is generated by the complex gradient Z of ρ (with respect to the Kähler metric with potential ρ)which, in local coordinates, is given by

$$Z = \sum_{\mu, \nu} \rho^{\mu\bar\nu} \rho_{\bar\nu} \frac{\partial}{\partial z^\mu}. \tag{1.2}$$

Here and throughtout the paper lower indices denote derivatives and $(\rho^{\mu\bar\nu})$ is the matrix defined by the relation $\sum_{\nu=1}^n \rho^{\mu\bar\nu} \rho_{\alpha\bar\nu} = \delta_{\alpha\mu}$.

It is natural to ask whether the vector field Z and therefore the foliation by complex curves defined by the distribution $\mathrm{Ann}(dd^c u)$ – known as the *Monge-Ampère foliation associated to u* – extends throughout the degeneracy set $M_* \setminus P$. This is the case in many natural examples. Let $\rho \colon \mathbf{C}^n \to [0, +\infty)$ be a continuous plurisuharmonic exhaustion, smooth on $\mathbf{C}^n_* = \{\rho > 0\}$ satisfying for suitable positive real numbers $c_1, \ldots, c_n$, the following "homogeneity" condition for $z = (z^1, \ldots, z^n) \in \mathbf{C}^n$:

$$\rho(e^{c_1 \lambda} z^1, \ldots, e^{c_n \lambda} z^n) = |e^\lambda|^2 \rho(z). \tag{1.3}$$

The sublevel set of such an exhaustion have been referred in [14] as *generalized weighted circular domains* a class of domains which includes the complete circular domains. Using (1.3) it is easy to show that $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation (1.1). In this case the complex gradient vector field extends on all $\mathbf{C}^n$ and in fact it may be computed for $z = (z^1, \ldots, z^n) \in \mathbf{C}^n$:

$$Z = \sum_\alpha c_\alpha z^\alpha \frac{\partial}{\partial z^\alpha}. \tag{1.4}$$

so that the leaves (i.e. the integral submanifolds of Z) are exactly the orbits of the action $z = (z^1, \ldots, z^n) \in \mathbf{C}^n \mapsto (e^{c_1 \lambda} z^1, \ldots, e^{c_n \lambda} z^n)$ (see again ([14]) for details).

In this paper we look for condition which grants the extendability of gradient vector field Z and consequently of the Monge-Ampère foliation. Indeed, if the exhaustion ρ is real analytic on M_* we are able to extend the bundle $\mathrm{Ann}(dd^c u)$ as an integrable complex line bundle on M_* generated by an extension of the complex gradient Z. Furthermore we prove that the extended vector field Z, and therefore the associated Monge-Ampère foliation, is holomorphic. The same result is shown to be true if the exhaustion ρ is of class C^∞ on M_* but the degeneracy is not too high. As a consequence, a repetition of the arguments in our previous work ([6],[7]) allows us to answer in full generality the question of Burns ([4]) regarding the characterizations of plurisuharmonic homogeneous polynomial and to extend the results of P.M. Wong ([14]) about classifying complete weighted circular domains assuming real analyticity of ρ and connectedness of the set M_*. In complex dimension $n = 2$, in [7] we were able to prove the same result if ρ is of class C^∞ on M_* and satisfies a finite type condition. For $n > 2$ it turns out that real analyticity is a crucial element of our arguments.

2. Extension and holomorphicity of the foliation on low-degeneracy set

Throughout this section we shall make the following assumptions:

(A1) M is a Stein manifold of dimension $n > 1$;

(A2) $\rho \colon M \to [0, +\infty)$ is continuous plurisubharmonic exhaustion of class C^ω with $d\rho \neq 0$ on $M_* = \{\rho > 0\}$;

(A3) on M_* the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^n = 0. \tag{2.1}$$

We start recalling the notion of hypersurface of finite type in the sense of Kohn [8] and Bloom-Graham [2]. Let S be a (real) hypersurface in a complex manifold M and $p \in S$. Let $\rho \colon U \to \mathbf{R}$ be a local defining function of S at p, i.e. $p \in U \cap S = \{z \in U \mid \rho(z) = 0\}$ and $d\rho \neq 0$ on $U \cap S$. Let $\mathcal{L}^k$ denote the module over $C^\infty(U)$ functions generated by all vector fields W on U in the holomorphic tangent bundle of S (i.e. with $\partial\rho(W) = 0$ on S), their conjugates and their brackets of length at most k. Then $\mathcal{L}^0$ is the module of vector fields on U spanned by the vector fields on U holomorphically tangent to S and their conjugates and, for $k > 0$, $\mathcal{L}^k$ is the the module of vector fields on U spanned by brackets $[V, W]$ with $V \in \mathcal{L}^{k-1}$ and $W \in \mathcal{L}^0$. The point $p \in S$ is of *type* m if $\partial\rho(W_{|p}) = 0$ for any $W \in \mathcal{L}^k$, for all $k = 0, \ldots, m-1$, and and there exists $Y \in \mathcal{L}^m$ with $\partial\rho(Y_{|p}) \neq 0$. We say that S is of *finite type* if for every $p \in S$ it is of type m at p, where m may depend on p.

If the exhaustion ρ satisfies (A2), it is well known that for any $c > 0$, the level hypersurface $\{\rho = c\}$ is compact, real analytic, pseudoconvex and of finite type. In fact, $d\rho \neq 0$, ρ is plurisubharmonic and if, for some $c_0 > 0$, p were a point of the real analytic hypersurface $S = \{\rho = c_0\}$ which is not of finite type, there would be a complex variety of positive dimension through p lying on S. This cannot happen because is S compact (see [5]).

We shall use the following notations:

$$P = P_n = \{p \in M_* \mid (dd^c\rho)^n \neq 0\},$$

$$P_{n-1} = \{p \in M_* \mid (dd^c\rho)^{n-1} \neq 0\}.$$

We have the following:

Lemma 2.1. *Suppose that the assumptions* (A1), (A2) *and* (A3) *hold. Then* $P \subset P_{n-1}$ *and both are open dense sets in* M_* *(and hence also in M!).*

Proof. Both P and P_{n-1} are open since they have closed complements. If P_{n-1} were not a dense set, there would be an open set $A \subset M_*$ in its complement. But then for any $p \in A$ the Levi form at p restricted to the holomophic tangent space to the level set S of ρ through p would have at least one zero eigenvalue.

Hence $S \cap A$ would contain complex varieties of positive dimension which, as we observed above, cannot happen. On the other hand, since

$$dd^c u = \frac{dd^c \rho}{\rho} - \frac{d\rho \wedge d^c \rho}{\rho^2}, \tag{2.2}$$

then from (A3) it follows that on M_*

$$\rho(dd^c \rho)^n = n(dd^c \rho)^{n-1} \wedge d\rho \wedge d^c \rho. \tag{2.3}$$

Suppose there were an open set $B \subset M_*$ on which $(dd^c \rho)^n = 0$. Then let $p \in B$ and let S be the level set of ρ through p. If $V, Z \in T_p^{1,0}(M_*)$ with $d\rho_p(V) = 0$ and $d\rho_p(Z) \neq 0$, then

$$0 = \rho(dd^c \rho)^n(V, \bar{V}, \dots, V, \bar{V}, Z, \bar{Z}) = n(dd^c \rho)^{n-1}(V, \bar{V}, \dots, V, \bar{V})|d\rho(Z)|^2. \tag{2.4}$$

This allows us to argue as above and conclude that $S \cap B$ contains complex varieties, which cannot happen. ∎

On P, $dd^c \rho > 0$ and under the assumption (A3) it follows that

$$\text{Ann}(dd^c u) = \{W \in T^{1,0}(M_*) \mid dd^c u(W, \bar{V}) = 0 \;\; \forall v \in T^{1,0}(M_*)\} \tag{2.5}$$

is an integrable distribution of complex rank 1 which is generated by the complex gradient Z of ρ (with respect to the Kähler metric with potential ρ) (see for instance [13] or [4] or [14] for details). A computation in local coordinates shows that Z is given by

$$Z = \sum_{\mu, \nu} \rho^{\mu \bar{\nu}} \rho_{\bar{\nu}} \frac{\partial}{\partial z^\mu}. \tag{2.6}$$

It is well known (again see [13] or [4] or [14] for details) that, wherever it is defined, the vector field Z is a non vanishing section of the line bundle $\text{Ann}(dd^c u)$ and the Monge-Ampère equation (1.1) is equivalent to

$$Z(\rho) = \rho. \tag{2.7}$$

In this section we aim to extend the vector field Z and the distribution $\text{Ann}(dd^c u)$ to P_{n-1}. Consequently the open set P_{n-1} will be foliated by complex curves, the flow of the extended vector field Z. Using the unboundness of ρ in a crucial way, we shall also show that the corresponding foliation of P_{n-1} thus obtained is holomorphic.

To prove that $\text{Ann}(dd^c u)$ extends as a complex line bundle on P_{n-1} and it is an integrable distribution spanned by a suitable extension of Z, we first need the following remark:

Lemma 2.2. *Suppose that (A1), (A2) and (A3) hold. For any $p \in P_{n-1}$ there exist a coordinate neighborhood $U \subset \{\rho > 0\}$ such that:*
(i) there are real functions $\lambda_1, \dots, \lambda_{n-1}$ on U such that at every point of $q \in U$ the values $\lambda_1(q), \dots, \lambda_{n-1}(q)$ are the eigenvalues of the restriction of the Levi form of $\log \rho$ to $\text{Ker} \partial \rho(q)$. The coordinates on U may be chosen so that if $\lambda_j(q), \lambda_k(q) \neq 0$

for $q \in U$ and $j \neq k$ then $\lambda_j(q) \neq \lambda_k(q)$ and the functions $\lambda_1, \ldots, \lambda_{n-1}$ are of class C^ω;

(ii) there are linearly independent vector fields $L_1, \ldots, L_{n-1} \in \text{Ker}\partial\rho$ on U which are linear analytic and form a diagonalizing basis for the hermitian form defined by the restriction of the Levi form of $\log \rho$ to $\text{Ker}\partial\rho(q)$ at every point $q \in U$ so that for each j the vector $L_j(q)$ is the eigenvector corresponding to $\lambda_j(q)$.

Proof. At $p \in P_{n-1}$, and hence for every q in a neighborhood U of p, with respect to any coordinate system, at most one eigenvalue of $\rho_{\mu\bar{\nu}}$ is zero. Therefore the restriction of the Levi form of ρ to the holomorphic tangent space $\text{Ker}\partial\rho(q)$ to the level set $S_q = \{\rho = \rho(q)\} \cap U$ through q has at most one zero eigenvalue The same happens for the restriction of the Levi form of any other defining function of S_q at q, thus, in particular for the restriction of the Levi form of $\log \rho$. Thus, by suitably rescaling coordinates, it may be assumed that the eigenvalues of the restriction of the Levi form of $\log \rho$ are pairwise distinct and therefore (i) follows. If (i) holds, then (ii) is straightforward. ∎

In a coordinate neighborhood $U \subset \{\rho > 0\}$ as in Lemma 2.1, the module $\mathcal{L}^k$ is generated over C^∞ functions by $L_1, \ldots, L_{n-1}$ and $\bar{L}_1, \ldots, \bar{L}_{n-1}$ and their brackets of length at most k. Since the level sets of ρ are hypersurfaces of finite type, if $p \in U$, then there exists k such that $\mathcal{L}^k_{|p}$ is the full complex tangent space $T_p^{\mathbf{C}}(M)$.

Recall that on P we have that $\partial\rho(Z) = Z(\rho) = \rho$ and hence the complex gradient of ρ is transverse to holomorphic tangent bundle of any level set of ρ. We shall extend the bundle $\text{Ann}(dd^c u)$ to all P_{n-1} by suitably choosing the "missing direction" recovered by the finite type property at the weakly pseudoconvex points in $P_{n-1} \setminus P$.

We can prove

Proposition 2.3. *Suppose that the assumptions $(A1)$, $(A2)$ and $(A3)$ hold. The complex gradient Z defined on P extends to a non-zero C^∞ vector field on P_{n-1}. The extension of the vector field Z generates an integrable complex line bundle $\mathcal{A}$. In particular, the Monge-Ampère foliation defined by $\text{Ann}(dd^c u)$ on P extends to a foliation defined by the distribution $\mathcal{A}$ on P_{n-1}.*

Proof. Let $p \in P_{n-1} \setminus P$ and $\lambda_1, \ldots, \lambda_{n-1}$ be the functions which provide the eigenvalues of the Levi form of $\log \rho$ on $\text{Ker}\partial\rho$. Let also $L_1, \ldots, L_{n-1} \in \text{Ker}\partial\rho$ be the vector fields of class C^ω defined in Lemma 2.2 on an open coordinate set $U \subset \{\rho > 0\}$ containing p.

Since the boundary of each sublevel set of ρ is of finite type, for some positive integer m there exists a C^∞ nonzero vector field $Y \in \mathcal{L}^m$ with $\partial\rho(Y) \neq 0$ at p and hence in a neighborhood of p (which we may assume is U). Since $(Z - \bar{Z})_{|q}$, $L_{1|q}, \ldots, L_{n-1|q}$ and $\bar{L}_{1|q}, \ldots, \bar{L}_{n-1|q}$ span the tangent space to the level set of ρ

through any $q \in U \cap P$, it follows that

$$Y = \phi(Z - \bar{Z}) + A_1 L_1 + \cdots + A_{n-1} L_{n-1} + B_1 \bar{L}_1 + \cdots + B_{n-1} \bar{L}_{n-1} \qquad (2.8)$$

for suitable functions $\phi, A_1, \ldots, A_{n-1}, B_1, \ldots, B_{n-1}$ of class C^∞ on $U \cap P$. If we define

$$V = \frac{1}{2}(Y - iJY), \qquad (2.9)$$

then on $U \cap P$ we have

$$V = \frac{1}{2}(Y - iJY) = \phi Z + A_1 L_1 + \cdots + A_{n-1} L_{n-1}. \qquad (2.10)$$

We now compute and study the functions ϕ and $A_1, \ldots, A_{n-1}$. On U the function $\partial \rho(V)$ is smooth and non zero. On the other hand on $U \cap P$, using (2.7) and (2.10)

$$\partial \rho(V) = \phi \partial \rho(Z) = \phi \rho \qquad (2.11)$$

so that

$$\phi = \frac{\partial \rho(V)}{\rho} \qquad (2.12)$$

extends as a non zero function of class C^∞ on all U.

On $U \cap P$, consider the form

$$\Omega = \frac{(dd^c u)^{n-1} \wedge d\rho \wedge d^c \rho}{\lambda_1 \ldots \lambda_{n-1}}. \qquad (2.13)$$

The form Ω is a volume form on U which allows to compute and study the functions $A_1, \ldots, A_{n-1}$. First of all notice that if W is any vector field on U such that $L_1, \ldots, L_{n-1}, W$ span $T_q^{1,0}(M)$ at every point $q \in U$ then necessarily $d\rho(W) \neq 0$. If $\theta_1, \ldots, \theta_n$ is the coframe dual to $L_1, \ldots, L_{n-1}, W$ on U, then

$$\Omega(L_1, \bar{L}_1, \ldots, L_{n-1} \bar{L}_{n-1}, W, \bar{W}) = \frac{i}{2\pi} \frac{\prod_{j=1}^{n-1} dd^c u(L_j, \bar{L}_j)}{\lambda_1 \ldots \lambda_{n-1}} |d\rho(W)|^2$$

$$= \frac{i}{2\pi} |d\rho(W)|^2.$$

As a consequence

$$\Omega = \frac{i}{2\pi} |d\rho(W)|^2 \theta_1 \wedge \bar{\theta}_1 \wedge \cdots \wedge \theta_n \wedge \bar{\theta}_n \qquad (2.14)$$

is a C^∞ non–zero volume form on U. On the other hand we can compute for each $j = 1, \ldots, n-1$:

$$\Omega(L_1, \bar{L}_1, \ldots, V, \bar{L}_j \ldots, L_{n-1} \bar{L}_{n-1}, W, \bar{W}) = \frac{i}{2\pi} |d\rho(W)|^2 A_j \qquad (2.15)$$

from which it follows that A_j extends as a C^∞ functions throughout U.

Therefore the complex gradient Z extends as a C^∞ vector field everywhere on U by setting $Z = \frac{1}{\phi}(V - A_1 L_1 - \cdots - A_{n-1} L_{n-1})$. The rest of the statement is now obvious. ∎

Definition. We call the foliation defined by $\mathcal{A}$ the *extended Monge-Ampère foliation.*

Before moving on, we need more precise information about the extended Monge-Ampère foliation. In particular we need to show that a leaf starting at a point in the open set P lays entirely in P. In this step we use the use the hypothesis real analyticity.

We start recalling a few known facts proved by Bedford an Burns ([1]). Let $p \in P$ and let ℓ_p be the leaf through p. According to [1] the leaf ℓ_p extends also through the set where $dd^c u$ has rank strictly less than $n-1$ to yield a "complete" leaf that we still call ℓ_p. The vector field Z, originally defined on the points of $\ell_p \cap P$, extends on all ℓ_p and, by continuity satisfies $d\rho(Z) = \rho$. Hence in particular it is non zero. As a consequence the vector field Z is defined on the union of all the leaves passing through points of P. Furthermore we recall that in [1] it is shown that if $p \in P$ and ℓ_p is the extended leaf through p thus obtained, the set $\ell_p \setminus P$ is discrete in ℓ_p. We start with a useful computation:

Lemma 2.4. *Let us denote by L_Z and $L_{\overline{Z}}$ the Lie derivatives with respect to the vector fields Z and $\overline{Z}$. At all points where Z is defined*

$$L_Z(dd^c\rho)^n = n(dd^c\rho)^n = L_{\overline{Z}}(dd^c\rho)^n \tag{2.16}$$

Proof. It is enough to prove the assertion on $P = \{z \in M \mid dd^c\rho(z) > 0\}$, the claim will follow by continuity. For any differential form ω, we recall that $L_Z\omega = di_Z\omega + i_Zd\omega$ where ι_Z denotes interior multiplication. Since $(dd^c\rho)^n$ is closed, $d\rho(Z) = id^c\rho(Z) = \rho$ and $\iota_Z dd^c u = 0$, using (2.2), one computes

$$\begin{aligned}
L_Z(dd^c\rho)^n &= n\,(dd^c\rho)^{n-1} \wedge L_Z(dd^c\rho) \\
&= n\,(dd^c\rho)^{n-1} \wedge d\iota_Z(dd^c\rho) \\
&= n\,(dd^c\rho)^{n-1} \wedge d\iota_Z(\rho dd^c u + \frac{d\rho \wedge d^c\rho}{\rho}) \\
&= n\,(dd^c\rho)^{n-1} \wedge d\iota_Z(\frac{d\rho \wedge d^c\rho}{\rho}) \\
&= n\,(dd^c\rho)^{n-1} \wedge d\left(\iota_Z\left(\frac{d\rho}{\rho}\right)d^c\rho - \iota_Z\left(\frac{d^c\rho}{\rho}\right)d\rho\right) \\
&= n(dd^c\rho)^n.
\end{aligned}$$

The same calculation shows that

$$L_{\overline{Z}}(dd^c\rho)^n = n(dd^c\rho)^n$$

■

Since $Z(\rho) = \overline{Z}(\rho)$ the vector field $\Theta = Z - \overline{Z}$ is tangent to the level set of ρ. Furthermore, by Lemma 2.4,

$$L_\Theta(dd^c\rho)^n = L_{(Z-\overline{Z})}(dd^c\rho)^n = 0. \tag{2.17}$$

From this we get the following key fact:

Proposition 2.5. *Let $x \in P_{n-1} \setminus P$. Then $(dd^c\rho)^n = 0$ on the orbit of x in the set $\{\rho = a\}$ under the vector field Θ. As a consequence any (extended) leaf passing through a point in P lies entirely in P and any leaf passing through a point in $P_{n-1} \setminus P$ lies entirely in $P_{n-1} \setminus P$. In particular P and $P_{n-1} \setminus P$ are foliated by leaves of the Monge-Ampère foliation.*

Proof. With respect to local coordinates local coordinates $z^1, \ldots, z^n$ we have $(dd^c\rho)^n = D dz^1 \wedge \ldots d\overline{z}^n$, where $D = \det(\rho_{i\overline{j}})$. Therefore the Lie derivative equation (2.17) says that

$$(L_\Theta D) dz^1 \wedge \ldots d\overline{z}^n + D L_\Theta(dz^1 \wedge \ldots d\overline{z}^n) = 0$$

Since $L_\Theta dz^1 \wedge \ldots d\overline{z}^n$ is an (n,n) form, it is equal to $A dz^1 \wedge \ldots d\overline{z}^n$ for some smooth function A and hence D satisfies the first order linear differential equation

$$L_\Theta D + AD = 0 \tag{2.18}$$

So along an integral curve $\gamma(t)$ of Θ, equation (2.18) implies that $\tilde{D}(t) =_{def} D(\gamma(t))$ satisfies

$$\frac{d\tilde{D}}{dt} = -A_{|\gamma(t)}\tilde{D}$$

and hence $\tilde{D}(t) \equiv 0$ if $\tilde{D}(0) = 0$. The claim then follows. ∎

Now we are ready to prove the main result of the section:

Proposition 2.6. *Suppose that the assumptions (A1), (A2) and (A3) hold. Then the extended Monge-Ampère foliation P_{n-1} is holomorphic.*

Proof. Let ℓ be a leaf of the extended Monge-Ampère foliation. At each point $q \in \ell$ the tangent space $T_q\ell$ is the complex subspace of T_qM spanned by $Z(q)$ and hence the leaf ℓ is a Riemann surface. The restriction $u_{|\ell}$ of the function $u = \log\rho$ to ℓ is harmonic. To see this, note that for $q \in M_*$, if ℓ is the leaf through q and ζ is a holomorphic coordinate along ℓ in a coordinate neighborhood of q, then for some smooth function ψ one has $\frac{\partial}{\partial\zeta} = \psi Z$. Hence the claim is equivalent to

$$dd^c u\left(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\overline{\zeta}}\right) = |\psi|^2 dd^c u(Z, \overline{Z}) = 0$$

where equality $dd^c u(Z, \overline{Z}) = 0$ holds on M_* as it is obvious at the points of the dense set P where $dd^c\rho > 0$.

Furthermore we can see that $u_{|\ell}$ is unbounded above on ℓ. Suppose in fact that $u_{|\ell}$ were bounded above and let $r = \sup_\ell \rho > 0$. Let $p \in \overline{\ell}$ with $\rho(p) = r$ and let ℓ_p be the leaf of the Monge-Ampère foliation passing through p. Then necessarily $\ell_p \subset \{z \in M \mid \rho(z) \geq r\}$ otherwise the leaf ℓ would extend past

$\{z \in M \mid \rho(z) = r\}$. But then p would be a local maximum for the harmonic function $u_{|\ell_p}$ which is impossible since $u_{|\ell_p}$ is not constant.

It follows from the previous theorem that the complex gradient Z extends as a real analytic vector field on M_* which we denote by Z. The extended vector field Z is holomorphic along any leaf of the Monge-Ampère foliation. To see this, let us cover M_* by coordinate neighborhoods U with C^∞-smooth coordinates $z^1, \ldots, z^n$ such that the intersection of a leaf of the Monge-Ampère foliation with U is given by $z^2 = c_2, \ldots, z^n = c_n$ for suitable constants $c_2 \ldots, c_n$ so that z^1 is a holomorphic coordinate along each leaf. Since from (2.7) it follows that the leaves of the Monge-Ampère foliation are transverse to the level set of ρ and hence of u, on U we have $du \wedge dz^j \neq 0$ for $j \neq 1$ and $du = u_1 dz^1 + \cdots + u_n dz^n$ with $u_1 \neq 0$. Furthermore since $\frac{\partial}{\partial z^1}$ is tangent to the Monge-Ampère foliation, on $U' = U \cap \{z \in M_* \mid dd^c\rho(z) > 0\}$ we have $Z = \varphi \frac{\partial}{\partial z^1}$ for some C^∞-smooth function φ. On the other hand on U' we have $\rho = Z(\rho) = \varphi \frac{\partial\rho}{\partial z^1}$ from which it follows that on U' we get

$$Z = \frac{1}{u_1} \frac{\partial}{\partial z^1}. \tag{2.19}$$

Thus (2.19) provides an expression of the C^∞-smooth extension of Z on U. Since u is harmonic along the leaves, we also have that the extended vector field Z is holomorphic along the intersection of U with any leaf. Since M_* is covered by such open coordinate neighborhoods, we got the claim.

By continuity we notice that $Z(\rho) = \rho$ on all M_*. Then, if we denote by $X = \frac{1}{2}(Z + \bar{Z})$ and $Y = \frac{1}{2i}(Z - \bar{Z})$ the real and the imaginary part of Z, we conclude that on M_*

$$X(\rho) = \frac{1}{4}\rho \qquad \text{and} \qquad Y(\rho) = 0. \tag{2.20}$$

Since $\rho\colon M_* \to (0, +\infty)$ is proper and the level sets of ρ are compact, it follows from (2.20) that X and Y are complete i.e. the flows ϕ and ψ of X and Y respectively, are both defined on $\mathbf{R}$. If l_p is the leaf through any fixed point $p \in M_*$, then the map $f\colon \mathbf{C} \to l_p$ defined by

$$f(t + is) = \phi(t, (\psi(s, p))$$

is holomorphic since $f'(t + is) = Z(f(t + is))$, Z is holomorphic along the leaf and f is non degenerated as $Z \neq 0$ on M_*. Therefore the leaf l_p is a parabolic Riemann surface. Since this is true for all leaves, Burns's Theorem 3.2 in [4] applies on the set P and implies that the vector field Z is holomorphic on P. As Z is smooth on P_{n-1}, the extended Monge-Ampère foliation is holomorphic. ∎

3. EXTENDING THE FOLIATION ON HIGHER DEGENERACY SET

In this section we shall prove that indeed the Monge-Ampère foliation extends to the set where the rank of the form $dd^c\rho$ is less than $n - 1$. To this purpose we use a remark that, again, we can prove only assuming real analyticity:

Lemma 3.1. *Suppose that the assumptions* $(A1)$, $(A2)$ *and* $(A3)$ *hold. The set of "weak points"*

$$W = \{p \in M_* \mid \mathrm{rank} dd^c \rho \leq n - 2\}$$

does not contain divisors (i.e. complex hypersurfaces)

Proof. Suppose that $S = \{\rho = c > 0\}$ is a generic level set of ρ which intersects W. Then $S \cap W$ is a real analytic subvariety of S. For a (local) real analytic subvariety V of S, let us denote $holdim(V)$ the *holomorphic dimension* of V as defined in [5], Definition 1, pg 372 (see also [9] where the notion was originally defined). We claim that

$$holdim(S \cap W) \geq 1.$$

This fact is consequence of the following observation: at a point $z \in S \cap W$ the Levi form of ρ has eigenvalue 0 with multiplicity at least 2 and hence there is at least 1 eigenvector with eigenvalue 0 which is in the intersection of the holomorphic tangent space to S and the complex tangent space to W at z. Then by Theorem 3 of [5] there is a complex curve (i.e. complex submanifold of dimension $\geq$ 1) contained in S. But this cannot happen since S is a compact real analytic hypersurface (see Theorem 4 of [5]). ∎

Using Lemma 3.1 we have the desired extension result for the vector field Z and the Monge-Ampère foliation:

Proposition 3.2. *The vector field* Z *and the Monge-Ampère foliation extend holomophically to the set of "weak points"* W.

Proof. The vector field Z is holomorphic on $M_* \setminus W$. Since W does not contain a complex hypersurface, it follows that W is a removable set for the vector field Z (see [11], Theorem 2.30, Chapter VI, and its proof). The extended vector field, which we still denote Z, is holomorphic and nonzero on M_* since by continuity $Z(\rho) = \rho$. Furthermore, again by continuity, we have $dd^c\rho(Z, \overline{Z}) = 0$ also on W. Therefore we get the extension and the holomorphicity of the Monge-Ampère foliation on all M_*. ∎

Our main result can now be stated:

Theorem 3.3. *Let* M *be a Stein manifold of dimension* n *and* $\rho\colon M \to [0, +\infty)$ *be a continuous plurisubharmonic exhaustion of class* C^ω *with* $d\rho \neq 0$ *on* $M_* = \{\rho > 0\}$ *and such that*
(i) the open set M_* *is connected,*
(ii) on M_* *the function* $u = \log \rho$ *satisfies the complex homogeneous Monge-Ampère equation*

$$(dd^c u)^n = 0, \tag{3.1}$$

Then the complex gradient vector field Z *of* ρ *defined by* (2.6) *on* $P = \{p \in M_* \mid (dd^c\rho)^n \neq 0\}$, *extends as a holomorphic vector field on* M *and the minimal set* $M_0 = \{z \in M \mid \rho(z) = 0\}$ *reduces to one point.*

Proof. We only need to show that the minimal set V of ρ reduces to one point. We observe that M_0 cannot be empty as otherwise $u = \log \rho$ would be a smooth exhaustion which solves the complex homogeneous Monge-Ampère equation on a Stein manifold. But such function do not exist (Theorem 1.1 of [10]). The holomorphic flow $\Psi \colon \mathbf{C} \times M_* \to M_*$ on M_* of the vector field Z extends to a holomorphic flow $\Psi \colon \mathbf{C} \times M \to M$ as M_0 is compact. On the other hand for any $p \in M_0$, since the flow of any point in M_* is contained in M_* we must have $\Psi(\mathbf{C} \times \{p\}) \subset M_0$, which, by Liouville theorem implies that $\Psi(\mathbf{C} \times \{p\}) = \{p\}$. Thus $Z(p) = 0$. But then $M_0 = \{z \in M \mid Z(z) = 0\}$ is a compact analytic set, i.e. a finite set of points. To finish our proof it is enough to show that the minimal set of ρ is connected. This can be done by repeating verbatim the argument at page 357 of [4]. We give here an outline of the proof. Suppose that for some non empty disjoint compact subsets $K_1, K_2 \subset M$ we have $M_0 = K_1 \cup K_2$. Let V_1, V_2 disjoint open sets with $K_i \subset V_i$ for $i = 1, 2$. For $r > 0$ sufficiently small we have $S_i = \rho^{-1}(r) \cap V_i \neq \emptyset$ for $i = 1, 2$. If $G \colon \mathbf{R} \times M_* \to M_*$ is the flow of the real part X of the vector field Z, then $G(\mathbf{R} \times \rho^{-1}(r)) = M_*$. On the other hand if $U_i = G(\mathbf{R} \times S_i)$ for $i = 1, 2$, then U_1, U_2 are open disjoint subsets with $U_1 \cup U_2 = M_*$ contradicting the fact that M_* is connected. $\blacksquare$

4. APPLICATIONS AND FINAL REMARKS

As announced, Theorem 3.3 can be used to give a full answer to a question of D. Burns ([4]):

Theorem 4.1. *Let ρ be a positive homogeneous polynomial on $\mathbf{C}^n$ such that $u = \log \rho$ is plurisubharmonic and satisfies*

$$(dd^c u)^n = 0. \tag{4.1}$$

Then ρ is a homogeneous polynomial of bidegree (k, k).

Proof. Using Theorem 3.3, the proof is a repetition of the elementary argument used to prove Theorem 3.1 of [7]. We give here a very short outline of the key steps of the proof for completenss referring the reader to [7] for the details. Theorem 3.3 implies that the complex gradient Z, defined on P by

$$Z = \sum_\mu Z^\mu \frac{\partial}{\partial z^\mu} = \sum_{\mu,\nu} Z^\mu \rho^{\mu\bar\nu} \rho_{\bar\nu} \frac{\partial}{\partial z^\mu}, \tag{4.2}$$

extends holomorphically to all $\mathbf{C}^n$. On the other hand, (4.2) shows that Z is homogeneous of degree one on a dense subset of $\mathbf{C}^n$, so that Z is in fact linear. If $\rho = \sum_{l+m=2k} \rho^{l,m}$ is the decomposition of ρ in homogeneous polynomial of bidegree (l, m), a bidegree argument using the fact that on P one has $Z(\rho) = \rho$, shows that

$$0 = \rho_{\bar\alpha}^{0,2k} = \rho_{\bar\alpha}^{2k,0} \qquad \text{and} \qquad \rho_{\bar\alpha}^{l,m} = \sum_\mu Z^\mu \rho_{\mu\bar\alpha}^{l,m}$$

for every l, m with $l+m = 2k, l, m \geq 1$. If $w = (w^1, \ldots, w^n) \in P$, then $dd^c \rho^{k,k}_{|w} > 0$ and, again by bidegree reasons and the fact that $\rho^{k,k}$ is homogeneous of bidegree (k, k), then

$$Z^\mu_{|w} = (\rho^{k,k})^{\bar{\alpha}\mu}(w)\rho^{k,k}_{\bar{\alpha}}(w) = \frac{1}{k}w^\mu. \tag{4.3}$$

By continuity (4.3) holds on all $\mathbf{C}^n$. As consequence the (extended) Monge-Ampère foliation associated to $u = \log \rho$ is given by the foliation of $\mathbf{C}^n \setminus \{0\}$ in lines through the origin and the restriction of $\log \rho$ to any complex line through the origin is harmonic with a logarithmic singularity of weight $2k$ at the origin. Thus, for any $0 \neq z \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$,

$$\log \rho(\lambda z) = 2k \log |\lambda| + O(1).$$

Hence the restriction of ρ to any complex line through the origin is homogeneous of bidegree (k, k) and therefore ρ is a homogeneous polynomial of bidegree (k, k) on $\mathbf{C}^n$.

∎

Again using Theorem 3.3 and, with the obvious formal changes, the arguments outlined in section 4 of [7], we have the following classification result:

Theorem 4.2. *Let M be a Stein manifold of dimension n equipped with a continuous plurisubharmonic exhaustion $\rho \colon M \to [0, +\infty)$ of class C^ω with $d\rho \neq 0$ on $M_* = \{\rho > 0\}$ and such that*
(i) the open set $M_* = \{\rho > 0\}$ is connected,
(ii) on $\{\rho > 0\}$ the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^n = 0. \tag{4.4}$$

Then there exists a biholomorphic map $\Phi \colon \mathbf{C}^n \to M$, such that, for suitable positive real numbers $c_1, \ldots, c_n$, the pull back $\rho_0 = \rho \circ \Phi$ of the exhaustion ρ satisfies

$$\rho_0(e^{c_1\lambda}z^1, \ldots, e^{c_n\lambda}z^n) = |e^\lambda|^2 \rho_0(z). \tag{4.5}$$

so that the sublevelsets of ρ are biholomorphic to generalized weighted circular domains.

Proof. Also in this case Theorem 3.3 provides the necessary tools to adapt at this situation the arguments of ([14]). In fact Theorem 3.3 shows that the Monge-Ampère foliation associated to $u = \log \rho$ extends to a holomorphic foliation throughout $M_* = \{\rho > 0\}$ and the complex gradient vector field Z, locally defined by

$$Z = \rho^{\mu\bar{\nu}}\rho_{\bar{\nu}}\frac{\partial}{\partial z^\mu}, \tag{4.6}$$

on the set $P = \{dd^c \rho > 0\}$, extends holomorphically to all M so that the extension, which we shall keep denoting Z, is tangent to the leaves of the extended

Monge-Ampère foliation on M_* and the equality $Z(\rho) = \rho$, which is equivalent to the Monge-Ampère equation on the set where P, holds, by continuity on all M. Finally we have that the minimal set of the function ρ reduces to a single point: $\{\rho = 0\} = \{O\}$. We shall call the point O the *center* of M.

The rest of the proof in ([14]) uses only these facts and no other consequence of strict pseudoconvexity. Exactly as for the proof Theorem 3.1 in [7], it is enough to outline the main steps of the rest of the proof there to underline the minor variations needed under our weaker assumptions.

Step 1: For any $r_1, r_2 > 0$ the level sets $\{\rho = r_1\}$ and $\{\rho = r_2\}$ are CR isomorphic and the sublevel sets $\{\rho < r_1\}$ and $\{\rho < r_2\}$ are biholomorphic.

This follows from a Morse Theory type of argument. The flow of the real part $X = Z + \bar{Z}$ of the (extended) complex gradient vector field defines a local group of biholomorphims when Z is holomorphic and maps level sets of ρ into other level sets. Details of the argument can be found on page 24 of ([12]). There it was assumed ρ to be strictly plurisubharmonic merely to ensure that the vector field Z is defined everywhere which in our case we prove by other means. The claim is an application of Bochner-Hartogs extension theorem.

Step 2: There exists a coordinate neighborhood U of the center O, with coordinates $z^1, \ldots z^n$ centered at O, and positive real numbers $c_1, \ldots, c_n > 0$, on U, the vector field Z has the following expression:

$$Z = c_1 z^1 \frac{\partial}{\partial z^1} + \ldots c_n z^n \frac{\partial}{\partial z^n}. \tag{4.7}$$

The expression for the vector field Z can be proved as on on page 248 of [14] as a consequence of a standard linearization theorem (see [3], Chapter III, for instance) since the equation $Z(\rho) = \rho$ implies that the one parameter group associated with imaginary part of Z preserves the level sets of ρ and therefore has compact closure in the automorphism group of each sublevel set.

Step 3: There exist suitable positive real numbers $c_1, \ldots, c_n$ so that for $z = (z^1, \ldots, z^n) \in U$ and $\lambda \in \mathbf{C}$ so that $(e^{c_1 \lambda} z^1, \ldots, e^{c_n \lambda} z^n) \in U$, we have:

$$\rho(e^{c_1 \lambda} z^1, \ldots, e^{c_n \lambda} z^n) = |e^\lambda|^2 \rho(z). \tag{4.8}$$

As in page 249 of [14] one integrates vector field Z explicitly. The only relevant fact needed here is that $Z(\rho) = \rho$.

What is left to prove is that there is a global biholomorphism of $\mathbf{C}^n$ onto M. Step 3 implies that for $\epsilon > 0$ small enough there exists a biholomorphism $\varphi : M(\epsilon) \to G(\epsilon)$ of the sublevel set $M(\epsilon)$ of ρ into a (fixed) weighted circular domain $G(\epsilon) = \{z \in \mathbf{C}^n \mid \rho_0(z) < \epsilon\}$ Then $\rho_0 = \rho \circ \varphi$ on $G(\epsilon)$ is defined on all

$\mathbf{C}^n$ and it satisfies (i), (ii), (iii). By Step 1 it follows that the flows of the real parts of the complex gradients of ρ and ρ_0 are biholomorhisms of sublevel sets. The required biholomorphic map $\Phi \colon \mathbf{C}^n \to M$ can be defined by composition of φ and the flows of the real parts of the complex gradients.

■

A final remark is in order. In complex dimension $n = 2$ we were able to obtain the main results of this paper under the hypothesis that the exhaustion is smooth and satisfies a finite type condition. This was possible because of results which hold only for dimension $n = 2$ – in particular the fact that foliation with parabolic leaves are always holomorphic in that case – and the fact that, simply for dimensional reasons, the Levi form of the solution of the Monge-Ampère restricted to the holomorphic tangent space to a level set may have at most one zero eigendirection. In higher dimension, of course, higher degeneracies are possible and we are able to overcome this difficulty only using in an essential way results which need real analyticity due to Burns and Bedford ([1]) and Diederich and Fornaess ([5]). Whether or not it is possible to replace real analyticity with assumptions such as finite type, or the like, remains open.

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