

Finite Type Monge-Ampère Foliations

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Abstract

For plurisubharmonic solutions of the complex homogeneous Monge-Ampère equation whose level sets are hypersurfaces of finite type, in dimension 2, it is shown that the Monge-Ampère foliation is defined even at points of higher degeneracy. The result is applied to provide a positive answer to a question of Burns on homogeneous polynomials whose logarithm satisfies the complex Monge-Ampère equation and to generalize the work of P.M. Wong on the classification of complete weighted circular domains.

1 Introduction

Let M be a Stein manifold of dimension n and $\rho: M \rightarrow \mathbf{R}$ a continuous plurisubharmonic exhaustion of class C^∞ on $\{\rho > 0\}$ with $d\rho \neq 0$ on $\{\rho > 0\}$. Furthermore suppose that on $\{\rho > 0\}$ the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^n = 0. \tag{1}$$

On the open set, P , where $dd^c \rho > 0$ the complex gradient Z of ρ (with respect to the Kähler metric with potential ρ) is defined. In local coordinates Z is given by

$$Z = \sum_{\mu, \nu} \rho^{\mu\bar{\nu}} \rho_{\bar{\nu}} \frac{\partial}{\partial z^\mu} \tag{2}$$

where lower indices denote derivatives and $(\rho^{\mu\bar{\nu}})$ is defined by the relation

$$\sum_{\nu=1}^n \rho^{\mu\bar{\nu}} \rho_{\alpha\bar{\nu}} = \delta_{\alpha\mu}.$$

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In [6], it is shown that in case the function ρ is a real, homogeneous polynomial on \mathbf{C}^n with convex level sets, the vector field Z extends to all of \mathbf{C}^n and yields a holomorphic foliation by parabolic Riemann surfaces. Under the restrictive assumption of convexity we were able to give an answer to a question of Burns ([4]) and show that a plurisubharmonic, positive homogeneous polynomial, ρ on \mathbf{C}^n where $u = \log \rho$ satisfies $(dd^c u)^n = 0$ must be a (p, p) polynomial. The assumption of convexity allowed us to use the existence of complex geodesics for the Kobayashi metric to extend the Monge-Ampère foliation across the weakly pseudoconvex points and obtain a holomorphic foliation of \mathbf{C}^n by parabolic Riemann surfaces. In the general pseudoconvex case the results about complex geodesics for the Kobayashi metric are not available. Here we make the assumption that the level sets of our exhaustion are of finite type in order to extend the bundle $\text{Ann}(dd^c u)$ as an integrable complex line bundle on M . We are able to prove the existence of the extension in dimension $n = 2$. In this case we show that the extended bundle yields a holomorphic foliation on $\{\rho > 0\}$ and, under the assumption that $\{\rho > 0\}$ is connected, that the vector field Z extends as a holomorphic vector field and that the zero set of ρ is a point. Possibly under more restrictive assumptions, we believe that, the result holds for general n . Our methods, however, use formulas that are only valid in dimension 2, specifically the very simple form of the inverse of a 2×2 matrix. We believe that there should be a more natural approach that avoids these 2-dimensional calculations.

Our extension and holomorphicity results allow us to answer Burns's question in dimension 2 for any plurisubharmonic homogeneous polynomial. We are also able to extend the results of P.M. Wong ([10]) about classifying complete weighted circular domains to the finite type case, for dimension $n = 2$ assuming that the set $\{\rho > 0\}$ is connected.

2 Existence of the foliation

We start recalling the well known notion of hypersurface of finite type in the sense of Kohn [7] and Bloom-Graham [2]. Let S be a (real) hypersurface in a complex manifold M and $p \in S$. Let $\rho: U \rightarrow \mathbf{R}$ be a local defining function of S at p , i.e. $p \in U \cap S = \{z \in U \mid \rho(z) = 0\}$ and $d\rho \neq 0$ on $U \cap S$. Let \mathcal{L}^k denote the module over $C^\infty(U)$ functions generated by all vector fields W on U in the holomorphic tangent bundle of S (i.e. with $\partial\rho(W) = 0$ on S), their conjugates and their brackets of length at most k . Then \mathcal{L}^0 is the module of vector fields on U spanned by the vector fields on U holomorphically tangent to S and their conjugates and, for $k > 0$, \mathcal{L}^k is the module of vector fields on U spanned by brackets $[V, W]$ with $V \in \mathcal{L}^{k-1}$ and $W \in \mathcal{L}^0$. The point $p \in S$ is of *type m* if $\partial\rho(W) = 0$ for all $k = 0, \dots, m-1$ and $W \in \mathcal{L}^k$ and there exists $Y \in \mathcal{L}^m$ with $\partial\rho(Y) \neq 0$. We say that S is of *finite type* if for every $p \in S$ it is of type m at p ,

for some m which depends on p .

Throughout this section we shall make the following assumptions:

- (i) M is a Stein manifold of dimension 2;
- (ii) $\rho: M \rightarrow \mathbf{R}$ is continuous plurisubharmonic exhaustion of class C^∞ on $\{\rho > 0\}$ with $d\rho \neq 0$ on $\{\rho > 0\}$;
- (iii) ρ is of finite type i.e. all the level hypersurfaces $\{\rho = c\}$, for $c > 0$ are of finite type (in the sense of Kohn [7] and Bloom-Graham [2]);
- (iv) the open set $M_* = \{\rho > 0\}$ is connected
- (v) on $\{\rho > 0\}$ the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^2 = 0. \quad (3)$$

Remark. If the exhaustion ρ is real analytic on $M_* = \{\rho > 0\}$, then (iii) holds. In fact, since the level sets of the exhaustion ρ are compact, all level hypersurfaces $\{\rho = c\}$, for $c > 0$ are of finite type if ρ is real analytic. Otherwise if p were a point of the real analytic hypersurface $S = \{\rho = c_0\}$ which is not of finite type, there would be a complex variety of positive dimension through p which lies on S . This cannot happen if S is compact (see [5])

Suppose the assumptions (i) to (v) hold for a complex manifold M . On the open set $P \subset M$ where $dd^c \rho > 0$, the complex gradient Z of ρ (with respect to the Kähler metric with potential ρ) is defined. In local coordinates Z is given by

$$Z = \rho^{\mu\bar{\nu}} \rho_{\bar{\nu}} \frac{\partial}{\partial z^\mu}. \quad (4)$$

It is well known (for instance see [4] or [10]) that, wherever it is defined, the vector field Z is a non vanishing section of the line bundle $\text{Ann}(dd^c u)$ and the Monge-Ampère equation (3) is equivalent to

$$Z(\rho) = \rho. \quad (5)$$

We shall start showing that $\text{Ann}(dd^c u)$ extends as a complex line bundle on M and it is an integrable distribution spanned by a suitable extension of Z .

In any coordinate system on an open set $U \subset \{\rho > 0\}$ the local vector field L defined by

$$L = \rho_2 \partial_1 - \rho_1 \partial_2, \quad (6)$$

where subscripts for functions denote derivatives, spans the holomorphic tangent space to level sets of ρ .

The module \mathcal{L}^k defined above is then the module over $C^\infty(U)$ functions generated by L and \bar{L} and their brackets of length at most k . Since the level sets of ρ are hypersurfaces of finite type, if $p \in U$, then there exists k such that $\mathcal{L}_{|p}^k$ is the full complex tangent space $T_p^{\mathbb{C}}(M)$.

On P , as $\partial\rho(Z) = Z(\rho) = \rho$, the complex gradient of ρ is transverse to holomorphic tangent bundle of any level set of ρ . We shall extend $\text{Ann}(dd^c u)$ by suitably choosing at the weakly pseudoconvex points the “missing direction” recovered by the finite type assumption. The following calculations, which will be useful later, suggest also that this is a reasonable approach.

Proposition 2.1 *If $U \subset \{\rho > 0\}$ is an open coordinate set and L is defined by (6), on $P \cap U$ we have*

$$[L, \bar{L}] = D(Z - \bar{Z}) \quad (7)$$

$$[L, Z] = \phi_1 L \quad (8)$$

$$[L, \bar{Z}] = \psi_1 L + \psi_2 \bar{L} \quad (9)$$

$$[Z, \bar{Z}] = \eta_1 L + \eta_2 \bar{L} \quad (10)$$

where $D = \det(\rho_{\mu\bar{\nu}})$ and ϕ_1, ϕ_2, ψ_2 are functions of class C^∞ .

Proof The proof of (7) consists of a straightforward calculation of the bracket, together with the standard formula for the inverse of a two by two matrix.

$$\begin{aligned} [L, \bar{L}] &= [\rho_2 \partial_1 - \rho_1 \partial_2, \rho_{\bar{2}} \partial_{\bar{1}} - \rho_{\bar{1}} \partial_{\bar{2}}] \\ &= (\rho_{\bar{1}} \rho_{2\bar{2}} - \rho_{\bar{2}} \rho_{2\bar{1}}) \partial_1 + (\rho_{\bar{2}} \rho_{1\bar{1}} - \rho_{\bar{1}} \rho_{1\bar{2}}) \partial_2 + \\ &\quad (\rho_2 \rho_{1\bar{2}} - \rho_1 \rho_{2\bar{2}}) \partial_{\bar{1}} + (\rho_1 \rho_{2\bar{1}} - \rho_2 \rho_{1\bar{1}}) \partial_{\bar{2}} \\ &= D(Z - \bar{Z}). \end{aligned}$$

In going to the last line we have used the identity:

$$\begin{pmatrix} \rho^{1\bar{1}} & \rho^{1\bar{2}} \\ \rho^{2\bar{1}} & \rho^{2\bar{2}} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \rho_{2\bar{2}} & -\rho_{1\bar{2}} \\ -\rho_{2\bar{1}} & \rho_{1\bar{1}} \end{pmatrix}.$$

As for (8), observe that using the Monge-Ampère equation $Z(\rho) = \bar{Z}(\rho) = \rho$ and $L(\rho) = 0$, we have $[L, Z](\rho) = LZ(\rho) - ZL(\rho) = 0$. Thus, the $(1, 0)$ vector field $[L, Z]$ is tangent to each level set $\{\rho = c > 0\}$ and must therefore be a multiple of L . The calculations of the brackets $[L, \bar{Z}]$ and $[Z, \bar{Z}]$ are entirely similar: both are tangent to the level sets, so must be linear combination of L and \bar{L} . ■

We can now prove

Theorem 2.2 *The complex gradient Z defined on P extends to a non zero smooth vector field on $\{\rho > 0\}$. The extension of the vector field Z generates an integrable complex line bundle \mathcal{A} on $\{\rho > 0\}$. Also the Monge-Ampère foliation defined by $\text{Ann}(dd^c u)$ extends to a foliation defined by the distribution \mathcal{A} .*

Proof Let $p \in \{\rho > 0\} \setminus P$ and suppose that L is the vector field defined by (6) on an open coordinate set $U \subset \{\rho > 0\}$ containing p . Since the boundary of each sublevel set of ρ is of finite type, for some positive integer m there exists a C^∞ nonzero vector field $Y \in \mathcal{L}^m$ with $\partial\rho(Y) \neq 0$ at p and hence in a neighborhood of p (which we may assume is U). Since \mathcal{L}^m is generated by L and \bar{L} and their brackets of length at most m , we can use (7),(8), (9), (10) to compute Y on $U \cap P$ and conclude

$$Y = \phi(Z - \bar{Z}) + AL + B\bar{L} \quad (11)$$

for suitable functions ϕ, A, B of class C^∞ on $U \cap P$. If we define

$$V = \frac{1}{2}(Y - iJY), \quad (12)$$

then on $U \cap P$ we have

$$V = \frac{1}{2}(Y - iJY) = \phi Z + AL. \quad (13)$$

We need now to compute and study the functions ϕ and A . On U the function $\partial\rho(V)$ is smooth and non zero. On the other hand on $U \cap P$, using (5) and (13)

$$\partial\rho(V) = \phi\partial\rho(\rho) = \phi\rho \quad (14)$$

so that

$$\phi = \frac{\partial\rho(V)}{\rho} \quad (15)$$

extends as a non zero function of class C^∞ on all U .

If, as before, $D = \det(\rho_{\mu\bar{\nu}})$ denotes the determinant of the Levi form of ρ , we define on $U \cap P$ the form

$$\Omega = \frac{\rho dd^c u}{D} + \frac{d\rho \wedge d^c \rho}{\rho}. \quad (16)$$

The form Ω itself does not extend on all U but it is a sort of a “singular” metric which allows to compute and study the function A . First of all, using the equality

$$dd^c u = \frac{dd^c \rho}{\rho} - \frac{d\rho \wedge d^c \rho}{\rho^2}, \quad (17)$$

observe that on $U \cap P$

$$\begin{aligned}\Omega^2 &= \left(\frac{\rho dd^c u}{D} + \frac{d\rho \wedge d^c \rho}{\rho} \right)^2 = \frac{2}{D} dd^c u \wedge d\rho \wedge d^c \rho \\ &= \frac{2}{D} \left(\frac{dd^c \rho}{\rho} - \frac{d\rho \wedge d^c \rho}{\rho^2} \right) \wedge d\rho \wedge d^c \rho = \frac{2}{D} \frac{dd^c \rho \wedge d\rho \wedge d^c \rho}{\rho}.\end{aligned}$$

On the other hand, as u satisfies the Monge-Ampère equation, we have

$$0 = \rho^3 (dd^c u)^2 = \rho (dd^c \rho)^2 - 2dd^c \rho \wedge d\rho \wedge d^c \rho$$

so that

$$\rho (dd^c \rho)^2 = 2dd^c \rho \wedge d\rho \wedge d^c \rho.$$

We may conclude:

$$\begin{aligned}\Omega^2 &= \frac{2}{D} \frac{dd^c \rho \wedge d\rho \wedge d^c \rho}{\rho} = \frac{\rho (dd^c \rho)^2}{D\rho} = \frac{(dd^c \rho)^2}{D} \\ &= \left(\frac{i}{2\pi} \right)^2 dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2\end{aligned}$$

and therefore Ω^2 is defined and of class C^∞ on all U . On $U \cap P$ we have also

$$\Omega(Z, \bar{Z}) = \frac{\rho dd^c u(Z, \bar{Z})}{D} + \frac{d\rho \wedge d^c \rho(Z, \bar{Z})}{\rho} = \frac{\rho^2}{\rho} = \rho, \quad (18)$$

$$\Omega(Z, \bar{L}) = \frac{\rho dd^c u(Z, \bar{L})}{D} + \frac{d\rho \wedge d^c \rho(Z, \bar{L})}{\rho} = 0, \quad (19)$$

where we have used the fact that $dd^c u(Z, \bar{W}) = 0$ for any vector field W of type $(1, 0)$ and that $d\rho \wedge d^c \rho(Z, \bar{L}) = 0$ holds, since $L \in \mathbf{Ker} \partial \rho$. Using definitions and $Z(\rho) = \rho$, a direct computation gives $dd^c \rho(L, \bar{L}) = D\rho$. Therefore, using the fact that $L \in \mathbf{Ker} \partial \rho$, we have

$$\Omega(L, \bar{L}) = \frac{\rho dd^c u(L, \bar{L})}{D} + \frac{d\rho \wedge d^c \rho(L, \bar{L})}{\rho} = \frac{\rho dd^c u(L, \bar{L})}{D} = \rho. \quad (20)$$

The computations (18), (19), (20) show that $\Omega(Z, \bar{Z})$, $\Omega(Z, \bar{L})$ and $\Omega(L, \bar{L})$ extend of class C^∞ throughout U . Thus $\Omega^2(V, \bar{L}, L, \bar{L})$ is defined and of class C^∞ throughout U and on $U \cap P$ we have following expression:

$$\begin{aligned}\Omega^2(V, \bar{L}, L, \bar{L}) &= \Omega^2(\phi Z + AL, \bar{L}, L, \bar{L}) = A\Omega^2(L, \bar{L}, L, \bar{L}) \\ &= A[\Omega(L, \bar{L})]^2 = A\rho^2\end{aligned}$$

from which it follows that A extends as a C^∞ functions throughout U .

Therefore the complex gradient Z extends as a C^∞ vector field at the weakly pseudoconvex point p by setting $Z = \frac{1}{\phi}(V - AL)$. The rest of the statement is now obvious. \blacksquare

Definition We call the foliation defined by \mathcal{A} the *extended Monge-Ampère foliation*.

Remark Let ℓ be a leaf of the extended Monge-Ampère foliation. At each point $q \in \ell$ the tangent space $T_q\ell$ is the complex subspace of T_qM spanned by $Z(q)$ and hence the leaf ℓ is a Riemann surface. The restriction $u|_\ell$ of the function $u = \log \rho$ to ℓ is harmonic. To see this, note that for $q \in M_*$, if ℓ is the leaf through q and ζ is a holomorphic coordinate along ℓ in a coordinate neighborhood of q , then for some smooth function ψ one has $\frac{\partial}{\partial \zeta} = \psi Z$. Hence the claim is equivalent to

$$dd^c u\left(\frac{\partial}{\partial \zeta}, \frac{\partial}{\partial \bar{\zeta}}\right) = |\psi|^2 dd^c u(Z, \bar{Z}) = 0$$

where equality $dd^c u(Z, \bar{Z}) = 0$ holds on M^* as it is obvious at the points of the dense set P where $dd^c \rho > 0$.

Furthermore $u|_\ell$ is unbounded above on ℓ . Suppose in fact that $u|_\ell$ were bounded above and let $r = \sup_\ell \rho > 0$. Let $p \in \bar{\ell}$ with $\rho(p) = r$ and let ℓ_p be the leaf of the Monge-Ampère foliation passing through p . Then necessarily $\ell_p \subset \{z \in M \mid \rho(z) \geq r\}$ otherwise the leaf ℓ would extend past $\{z \in M \mid \rho(z) = r\}$. But then p would be a local maximum for the harmonic function $u|_{\ell_p}$ which is impossible since $u|_{\ell_p}$ is not constant.

Theorem 2.3 *The extended Monge-Ampère foliation on $M_* = \{\rho > 0\}$ is holomorphic.*

Proof We have shown that the complex gradient Z extends as a C^∞ -smooth vector field on $M_* = \{z \in M \mid \rho(z) > 0\}$. We denote again Z this extension. The extended vector field Z is holomorphic along any leaf of the Monge-Ampère foliation. To see this, let us cover M_* by coordinate neighborhoods U with C^∞ -smooth coordinates z_1, z_2 such that the intersection of a leaf of the Monge-Ampère foliation with U is given by $z_2 = c_2$ for suitable constants c_2 so that z_1 is a holomorphic coordinate along each leaf. Since from (5) it follows that the leaves of the Monge-Ampère foliation are transverse to the level set of ρ and hence of u , in this coordinates, we have $du \wedge dz_2 \neq 0$ so that, if $du = u_1 dz_1 + u_2 dz_2$ then necessarily, with respect these coordinates, on U we have $u_1 \neq 0$. Furthermore since $\frac{\partial}{\partial z_1}$ is tangent to the Monge-Ampère foliation, on $U' = U \cap \{z \in M_* \mid dd^c \rho(z) > 0\}$ we have $Z = \varphi \frac{\partial}{\partial z_1}$ for some C^∞ -smooth function φ . On the other hand on U' we have $\rho = Z(\rho) = \varphi \frac{\partial \rho}{\partial z_1}$ from which it follows that on U' we get

$$Z = \frac{1}{u_1} \frac{\partial}{\partial z_1}. \quad (21)$$

Thus (21) provides an expression of the C^∞ -smooth extension of Z on U . Since u is harmonic along the leaves, we have that also the extended vector field Z is

holomorphic along the intersection of U with any leaf. Since M_* is covered by such open coordinate neighborhoods, we got the claim.

By continuity notice that we have also on all M_* :

$$Z(\rho) = \rho \tag{22}$$

Let $X = \frac{1}{2}(Z + \bar{Z})$ and $Y = \frac{1}{2i}(Z - \bar{Z})$ be respectively the real and the imaginary part of the vector field Z . Then, from (22) we conclude that

$$X(\rho) = \frac{1}{4}\rho \quad \text{and} \quad Y(\rho) = 0. \tag{23}$$

Since $\rho: M_* \rightarrow (0, +\infty)$ is proper and the level sets of ρ are compact, it follows from (23) that X and Y are complete i.e. the flows ϕ and ψ of X and Y respectively, are both defined on \mathbf{R} . Fixed any point $p \in M_*$, if l_p is the leaf through p then the map $f: \mathbf{C} \rightarrow l_p$ defined by

$$f(t + is) = \phi(t, (\psi(s, p)))$$

is holomorphic since $f'(t + is) = Z(f(t + is))$ and Z is holomorphic along the leaf and f is non degenerated as $Z \neq 0$ on M_* . Therefore the leaf l_p is parabolic. Since this is true for all leaves, Burns's Theorem 3.2 in [4] applies and therefore the Monge-Ampère foliation is holomorphic. ■

Under the assumptions assumptions (i) to (v) stated at the beginning of the section, the main conclusion is the following:

Theorem 2.4 *The vector field Z extends as a holomorphic vector field on M and the minimal set $\{z \in M \mid \rho(z) = 0\}$ reduces to a point.*

Proof Let $p \in M_* = \{z \in M \mid \rho(z) > 0\}$. Since the extended Monge-Ampère foliation is holomorphic, there exists a coordinate neighborhood U of p with holomorphic coordinates z_1, z_2 such that the intersection of a leaf of the Monge-Ampère foliation with U is given by $z_2 = c_2$ for suitable constants c_2 and z_1 is a holomorphic coordinate along each leaf. Then, as we have seen above, with respect to these coordinates we get

$$Z = \frac{1}{u_1} \frac{\partial}{\partial z_1}. \tag{24}$$

Thus, to show that Z is holomorphic on U , we need only to prove that u_1 is holomorphic. This is a simple consequence of the fact u is solution of the Monge-Ampère equation. In fact since the vector field Z generates at every point the distribution $\mathcal{A} = \text{Ann}(dd^c u)$, then for every vector field $X = \sum X_\alpha \frac{\partial}{\partial z_\alpha}$ of

type $(1, 0)$ on U , it must be that $dd^c u(Z, \bar{X}) = 0$. In local coordinates, using the expression (24), this is equivalent to

$$u_{1\bar{1}} \frac{1}{u_1} \bar{X}^1 + u_{1\bar{2}} \frac{1}{u_1} \bar{X}^2 = 0$$

which, since X is arbitrary and $u_1 \neq 0$, implies that $u_{1\bar{1}} = u_{1\bar{2}} = 0$ and therefore u_1 , and hence Z , is holomorphic on U . Since p is arbitrary in M_* , then Z is holomorphic on M_* . Since M is Stein, if M_* is connected and $\{z \in M \mid \rho(z) = 0\}$ is compact, then the vector field Z extends holomorphically to all M .

Finally we like to show that the minimal set $\{z \in M \mid \rho(z) = 0\}$ of ρ reduces to one point. Firstly observe that it cannot be empty as otherwise $u = \log \rho$ would be a smooth exhaustion which solves the complex homogeneous Monge-Ampère equation on a Stein manifold. But such function does not exist (Theorem 1.1 of [8]). The holomorphic flow $\Psi: \mathbf{C} \times M_* \rightarrow M_*$ on M_* of the vector field Z extends to a holomorphic flow $\Psi: \mathbf{C} \times M \rightarrow M$ as $\{z \in M \mid \rho(z) = 0\}$ is compact. On the other hand for any $p \in \{z \in M \mid \rho(z) = 0\}$, since the flow of any point in M_* is contained in M_* we must have $\Psi(\mathbf{C} \times \{p\}) \subset \{z \in M \mid \rho(z) = 0\}$, which, by Liouville theorem implies that $\Psi(\mathbf{C} \times \{p\}) = \{p\}$. Thus $Z(p) = 0$. But then $\{z \in M \mid \rho(z) = 0\} = \{z \in M \mid Z(z) = 0\}$ is a compact analytic set, i.e. a finite set of points. To finish our proof it is enough to show that the minimal set of ρ is connected. This is achieved repeating verbatim the argument at page 357 of [4]. We give here an outline of the proof. Suppose that for some compact, non empty disjoint compact subset $K_1, K_2 \subset M$ we have $\{z \in M \mid \rho(z) = 0\} = K_1 \cup K_2$. Let V_1, V_2 disjoint open sets with $K_i \subset V_i$ for $i = 1, 2$. For $r > 0$ sufficiently small we have $S_i = \rho^{-1}(r) \cap V_i \neq \emptyset$ for $i = 1, 2$. If $G: \mathbf{R} \times M_* \rightarrow M_*$ is the flow of the real part X of the vector field Z , then $G(\mathbf{R} \times \rho^{-1}(r)) = M_*$. On the other hand if $U_i = G(\mathbf{R} \times S_i)$ for $i = 1, 2$, then U_1, U_2 are open disjoint subsets with $U_1 \cup U_2 = M_*$ contradicting the fact that M_* is connected. ■

3 Burns's Problem

Taking advantage of the results of section 2 and using the final arguments of [6], we can provide a complete solution to the problem proposed by D. Burns [4] in dimension 2. Namely we can show the following:

Theorem 3.1 *Let ρ be a positive homogeneous polynomial on \mathbf{C}^2 such that $u = \log \rho$ is plurisubharmonic and satisfies*

$$(dd^c u)^2 = 0. \tag{25}$$

Then ρ is a homogeneous polynomial of bidegree (k, k) .

Proof From the results of Section 2 it follows that the complex gradient Z , defined on P , using as usual the summation convention, by

$$Z = Z^\mu \frac{\partial}{\partial z^\mu} = \rho^{\mu\bar{\nu}} \rho_{\bar{\nu}} \frac{\partial}{\partial z^\mu}, \quad (26)$$

extends holomorphically to all \mathbf{C}^2 . On the other hand, (26) shows that Z is homogeneous of degree one on a dense subset of \mathbf{C}^2 , so that Z is in fact linear. We can get an explicit expression for Z using a bidegree argument. Suppose that

$$\rho = \sum_{l+m=2k} \rho^{l,m}$$

is the decomposition of ρ in homogeneous polynomial of bidegree (l, m) . The Monge-Ampère equation (25) is equivalent on P to

$$Z(\rho) = \rho^{\mu\bar{\nu}} \rho_{\bar{\nu}} \rho_\mu = \rho = \sum_{l+m=2k} \rho^{l,m}.$$

Differentiating this equation with respect to \bar{z}^α , we get

$$\sum_{\substack{l+m=2k \\ m \geq 1}} \rho_{\bar{\alpha}}^{l,m} = \rho_{\bar{\alpha}} = (Z^\mu \rho_\mu)_{\bar{\alpha}} = \sum_{\substack{l+m=2k \\ l, m \geq 1}} Z^\mu \rho_{\mu\bar{\alpha}}^{l,m}.$$

Comparing bidegrees, we conclude:

$$0 = \rho_{\bar{\alpha}}^{0,2k} = \rho_{\bar{\alpha}}^{2k,0}$$

and, for every l, m with $l + m = 2k, l, m \geq 1$,

$$\rho_{\bar{\alpha}}^{l,m} = Z^\mu \rho_{\mu\bar{\alpha}}^{l,m}$$

where, as usual we use the summation convention. Let $w = (w^1, \dots, w^n)$ be such that $dd^c \rho_w > 0$; then $dd^c \rho_w^{k,k} > 0$ (see for instance Lemma 2.1 in [6]) and, if $(\rho^{k,k})^{\bar{\alpha}\mu}$ is defined by the relation

$$\sum_{\nu=1}^n (\rho^{k,k})^{\mu\bar{\nu}} (\rho^{k,k})_{\alpha\bar{\nu}} = \delta_{\alpha\mu},$$

then

$$Z_w^\mu = (\rho^{k,k})^{\bar{\alpha}\mu}(w) \rho_{\bar{\alpha}}^{k,k}(w) = \frac{1}{k} w^k \quad (27)$$

where the last equality is immediate from (5) as $\rho_{\bar{\alpha}}^{k,k}$ is homogeneous of degree $(k, k-1)$ and hence $\rho_{\mu\bar{\alpha}}^{k,k}(W) w^\mu = k \rho_{\bar{\alpha}}^{k,k}$. By continuity (27) holds on all \mathbf{C}^2 .

As consequence the (extended) Monge-Ampère foliation associated to $u = \log \rho$ is given by the foliation of $\mathbf{C}^2 \setminus \{0\}$ in lines through the origin. It follows that the restriction of $\log \rho$ to any complex line through the origin is harmonic with a logarithmic singularity of weight $2k$ at the origin and therefore for any $0 \neq z \in \mathbf{C}^2$ and $\lambda \in \mathbf{C}$

$$\log \rho(\lambda z) = 2k \log |\lambda| + O(1).$$

Hence the restriction of ρ to any complex line through the origin is homogeneous of bidegree (k, k) and therefore ρ is a homogeneous polynomial of bidegree (k, k) on \mathbf{C}^2 . ■

4 Classification of solutions

Given what we have already proven, we see that the argument of P.M. Wong in ([10]) goes over and we are able to prove the following:

Theorem 4.1 *Let M be a Stein manifold of dimension 2 equipped with a continuous plurisubharmonic exhaustion $\rho: M \rightarrow \mathbf{R}$ of class C^∞ on $\{\rho > 0\}$ with $d\rho \neq 0$ on $\{\rho > 0\}$ such that*

- (i) ρ is of finite type i.e. all the level hypersurfaces $\{\rho = c\}$, for $c > 0$ are of finite type (in the sense of Kohn [7] and Bloom-Graham [2]),
- (ii) the open set $M_* = \{\rho > 0\}$ is connected,
- (iii) on $\{\rho > 0\}$ the function $u = \log \rho$ satisfies the complex homogeneous Monge-Ampère equation,

$$(dd^c u)^2 = 0. \tag{28}$$

Then there exists a biholomorphic map $\Phi: \mathbf{C}^2 \rightarrow M$, such that, for suitable positive real numbers c_1, c_2 , the pull back $\rho_0 = \rho \circ \Phi$ of the exhaustion ρ satisfies

$$\rho_0(e^{c_1 \lambda} z_1, e^{c_2 \lambda} z_2) = |e^\lambda|^2 \rho_0(z). \tag{29}$$

so that the sublevelsets of ρ are biholomorphic to generalized weighted circular domains.

Proof Because of the results of Section 2, our hypotheses imply that the Monge-Ampère foliation associated to $u = \log \rho$ extends to a holomorphic foliation throughout $M_* = \{\rho > 0\}$.

Furthermore the complex gradient vector field Z , locally defined by

$$Z = \rho^{\mu\bar{\nu}} \rho_{\bar{\nu}} \frac{\partial}{\partial z^\mu}, \tag{30}$$

on the set where $dd^c\rho > 0$, extends holomorphically to all M so that the extension, which we shall keep denoting Z , is tangent to the leaves of the extended Monge-Ampère foliation on M_* .

The equality $Z(\rho) = \rho$, which is equivalent to the Monge-Ampère equation (28) on the set where $dd^c\rho > 0$, holds, by continuity on all M .

Finally we have that the minimal set of the function ρ reduces to a point: $\{\rho = 0\} = \{O\}$. We shall call the point O the *center* of M . The proof rests on the ground that the argument in ([10]) uses only these facts and no other consequence of strict pseudoconvexity. We outline the main steps of the rest of the proof there to underline the minor variations needed under our weaker assumptions.

Step 1: For any $r_1, r_2 > 0$ the level sets $M < r_i \geq \{\rho = r_i\}$ for $i = 1, 2$ are CR isomorphic and the sublevel sets $M(r_i) = \{\rho < r_i\}$ for $i = 1, 2$ are biholomorphic.

Step 1 is consequence of a Morse Theory type of argument. The flow of the real part $X = Z + \bar{Z}$ of the (extended) complex gradient vector field maps level defines a local group of biholomorphisms when Z is holomorphic and maps level sets of ρ into other level sets. Details of the argument can be found on page 24 of ([9]). There it was assumed ρ to be strictly plurisubharmonic merely to ensure that the vector field Z is defined everywhere which in our case we prove by other means. The claim then follows from a standard application of Bochner-Hartogs extension theorem.

Step 2: There exists a coordinate neighborhood U of the center O , with coordinates z_1, z_2 centered at O , such that for suitable positive real numbers $c_1, c_2 > 0$, on U , the vector field Z has the following expression:

$$Z = c_1 z_1 \frac{\partial}{\partial z_1} + c_2 z_2 \frac{\partial}{\partial z_2}. \quad (31)$$

Here we can repeat the proof on page 248 of [10] which goes as follows: the equation $Z(\rho) = \rho$ implies that the one parameter group associated with imaginary part of Z preserves the level sets of ρ and therefore have compact closure in the automorphism group of each sublevel set. A theorem in Bochner-Martin ([3], Chapter III) then implies the given form of vector field Z .

Step 3: There exist suitable positive real numbers c_1, c_2 so that for $z = (z_1, z_2) \in U$ and $\lambda \in \mathbf{C}$ so that $(e^{c_1\lambda}z_1, e^{c_2\lambda}z_2) \in U$, we have:

$$\rho(e^{c_1\lambda}z_1, e^{c_2\lambda}z_2) = |e^\lambda|^2 \rho(z). \quad (32)$$

The proof is as in page 249 of [10] and consists of integrating the vector field Z explicitly. Again, one uses only the fact that $Z(\rho) = \rho$ in this integration.

What is left to prove is that there is a global biholomorphism of \mathbf{C}^2 onto M . This follows from the fact that we can define the biholomorphism on a neighborhood of the origin to a neighborhood of the center of M and then flow in each manifold with the real part of Z . More specifically, from Step 3 we may conclude for $\epsilon > 0$ small enough the sublevel set $M(\epsilon)$ of ρ is biholomorphic to a (fixed) weighted circular domain

$$G(\epsilon) = \{z \in \mathbf{C}^2 \mid \rho_0(z) < \epsilon\}$$

for some plurisubharmonic exhaustion ρ_0 of \mathbf{C}^2 which satisfies (32). If $\varphi : M(\epsilon) \rightarrow G(\epsilon)$ is the biholomorphism of Step 3, then $\rho_0 = \rho \circ \varphi$ on $G(\epsilon)$, it is defined on all \mathbf{C}^2 and it satisfies (i), (ii), (iii). By Step 1 it follows that the flows of the real parts of the complex gradients of ρ and ρ_0 are biholomorphisms of sublevel sets. The required biholomorphic map $\Phi : \mathbf{C}^2 \rightarrow M$ can be defined by composition of φ and the flows of the real parts of the complex gradients. ■

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