MONGE-AMPÈRE EQUATIONS AND MODULI SPACES OF MANIFOLDS OF CIRCULAR TYPE

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Abstract. A *(bounded) manifold of circular type* is a complex manifold M of dimension n admitting a (bounded) exhaustive real function u, defined on M minus a point x_o, so that: a) it is a smooth solution on $M \setminus \{x_o\}$ to the Monge-Ampère equation $(dd^c u)^n = 0$; b) x_o is a singular point for u of logarithmic type and e^u extends smoothly on the blow up of M at x_o; c) $dd^c(e^u) > 0$ at any point of $M \setminus \{x_o\}$. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$.

The moduli spaces of bounded manifolds of circular type are studied. In particular, for each biholomorphic equivalence class of them it is proved the existence of an essentially unique manifold in normal form. It is also shown that the class of normalizing maps for an n-dimensional manifold M is a new holomorphic invariant with the following property: it is parameterized by the points of a finite dimensional real manifold of dimension n^2 when M is a (non-convex) circular domain while it is of dimension $n^2 + 2n$ when M is a strictly convex domain. New characterizations of the circular domains and of the unit ball are also obtained.

1. Introduction

In this paper we analyze the moduli spaces of a family of complex manifolds, which includes the smoothly bounded strictly linearly convex domains and the smoothly bounded strictly pseudoconvex circular domains in $\mathbb{C}^n$.

More precisely, we consider a larger class of manifolds, called *(bounded) manifolds of circular type*, which naturally includes the previous two families of domains and are characterized by the property of admitting a (bounded) exhaustive real function u, defined on M minus a point x_o, so that: a) it is a smooth solution on $M \setminus \{x_o\}$ to the Monge-Ampère equation $(dd^c u)^n = 0$; b) x_o is a singular point for u of logarithmic type and e^u extends smoothly on the blow up of M at x_o; c) $dd^c(e^u) > 0$ at any point of $M \setminus \{x_o\}$.

In any biholomorphic equivalence class of such domains, we prove the existence of an essentially unique *manifold in normal form*, consisting of the unit ball B^n together with a non-standard complex structure J, which satisfies some suitable conditions: One of them consists on requiring that the non-standard CR structures induced on the spheres $S^{2n-1}(r) = \{\ |z| = r\ \}$, $0 < r < 1$, have the same real distribution of the standard ones as underlying distribution of J-invariant real subspaces. The other conditions on J imply that J is uniquely determined by only one of such CR structures. This CR structure is completely determined by a sequence $\{\phi_k\}_{k=0}^{\infty}$, of $(1,1)$-tensor fields on $\mathbb{C}P^{n-1} \simeq S^{2n-1}(r_o)/S^1$, obtained by expanding in Fourier

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series the tensor field ϕ that gives the complex structure of the CR structure as a deformation of the standard one. As applications, we use these results to obtain new characterizations of the circular domains in $\mathbb{C}^n$ and of the unit ball.

The normal forms considered in this paper are essentially the same of the normal forms constructed in [5, 7] by Bland and Duchamp only for domains that are small deformations of the unit ball. The major improvement consists in showing that *such normal forms exist for any bounded manifold of circular type*, i.e. for any complex manifold which admits a solution to the described Monge-Ampère differential problem. Such result has been obtained by methods and techniques that are substantially different from those of [5] and [7].

We should also mention that our normal forms can be also considered as versions "without distinguished point" of the normal forms constructed by Lempert and Bland and Duchamp in [16, 6], where equivalence classes of pointed strictly convex domains were studied.

In many regards, the properties of normal forms of manifolds of circular type recall those of the well-known Chern-Moser normal forms for Levi non-degenerate real hypersurfaces ([10]). For instance, if D is a domain of circular type in a Stein manifold, it turns out that the class $\mathcal{N}(D)$ of the diffeomorphisms $f : D \to B^n$, which map D into a normal form (B^n, J), is naturally parameterized by a subset of the automorphism group $\mathrm{Aut}(B^n, J_o)$ of the standard unit ball (B^n, J_o). As for Chern-Moser normal forms, this fact determines a natural embedding of $\mathrm{Aut}(D)$ as a subgroup of $\mathrm{Aut}(B^n, J_o)$.

On the other hand, in contrast with what occurs for Chern-Moser normal forms, the parameter set for the class $\mathcal{N}(D)$ is *not* independent of D and it represents an important biholomorphic invariant for the domain (we recall that, on the contrary, the Chern-Moser normalizing maps are always parameterized by the isotropy $\mathrm{Aut}_{x_o}(B^n, J_o)$ of a fixed boundary point $x_o \in \partial B^n$).

For example, if D is a smoothly bounded, strictly convex domain in $\mathbb{C}^n$, then $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$, while if D is a generic (non convex) strongly pseudoconvex circular domain, then $\mathcal{N}(D) \simeq U_n \subsetneq \mathrm{Aut}(B_n, J_o)$. These facts motivate the following question:

Is it true that a domain of circular type D is biholomorphic to a smoothly bounded, strictly convex domains in $\mathbb{C}^n$ if and only if $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$?

We conjecture that the answer is "yes", at least when $n \geq 3$ and some boundary regularity conditions are assumed. However, besides such conjecture, there is another reason of interest for the domains of circular type for which $\mathcal{N}(D) \simeq \mathrm{Aut}(B^n, J_o)$, namely the abundance of solutions to the quoted Monge-Ampère problem that there exists on any such domain (there is at least one such solution for any point $x_o \in D$ - see remarks after Thm. 6.5). In addition, any such domain is endowed with a biholomorphically invariant complex Finsler metric: In case of a strictly convex domain of $\mathbb{C}^n$, this metric is the Kobayashi metric of the domain (e.g. [12, 20, 2, 27]). A detailed discussion of this and related questions on such domains will be the object of a future paper.

Some applications of the theory of normal forms we developed here can be found in the last section. We obtain f.i. the following result (Thm. 6.9): *a bounded manifold of circular type M, with $u : M \setminus \{x_o\} \to [0, r^2)$ satisfying a), b), c), is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if there exists at least two* *subdomains* $M_{<c} = \{ u < c \}$, $M_{<c'} = \{ u < c' \}$, $0 < c < c' < r^2$, *which are biholomorphic one to the other by a map fixing* x_o. This theorem represents a generalization to any manifold of circular type of results in [17] and [2], originally proved only for smoothly bounded, strictly convex domains in $\mathbb{C}^n$ or complex Finsler manifolds. Such generalization is obtained by an approach which is quite different from the ones in [17] and [2] . By a result in [23], our Thm. 6.9 has also an immediate corollary which gives a new characterization of the unit ball (Cor. 6.11).

The structure of the paper is as follows: After §2, devoted to preliminaries, in §3 we introduce the *circular representation* of a manifold of circular type, a map which naturally generalizes the standard circular representation of strictly convex domains (see [23, 6]); In §4, the normal forms of domains of circular type is defined and the existence of normalizing maps is proved; In §5, it is shown that the complex structure of a manifold in normal form is completely determined by the associated "deformation tensor" ϕ; In the same section, the Bland and Duchamp invariants $\phi^{(k)}$, determined by Fourier series expansion of ϕ, are defined; In §6 we give a geometrical interpretation of the Bland and Duchamp invariant $\phi^{(0)}$, we establish the parameterization by elements of $\mathrm{Aut}(B^n)$ of the family of normalizing maps of a manifold of circular type and we prove the mentioned characterizations of circular domains and of the unit ball.

2. Preliminaries

2.1. **Notation, first definitions and some basic properties.**

2.1.1. *Complex, CR and contact structures.*

In all the following, we will be denoted by J_o the standard complex structure of $\mathbb{C}^n$ and by B^n and $S^{2n-1} = \partial B^n$ the unit ball and the unit sphere, respectively, centered at $0 \in \mathbb{C}^n$. We will also indicate by $\Delta = B^1$ the unit disc in $\mathbb{C}$.

For any two complex manifolds (M, J) and (M', J'), a map $f : M \to M'$ is called (J, J')*-holomorphic* if $f_* \circ J = J' \circ f_*$. However, anytime it will be clear what are the considered complex structures, we will just write "holomorphic" in place of "(J, J')-holomorphic" .

We recall that a CR structure on a manifold M is a subbundle $\mathcal{D}^{1,0} \subset T^{\mathbb{C}}M$ of the complexified tangent bundle of M so that $\mathcal{D}^{1,0} \cap \overline{\mathcal{D}^{1,0}} = \{0\}$ and $[\mathcal{D}^{1,0}, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}$. For any given CR structure, we call *underlying real distribution* the subbundle $\mathcal{D} \subset TM$ determined by the subspaces $\mathcal{D}_x = \mathrm{Re}(\mathcal{D}^{1,0}_x) \subset T_xM$; We also call *underlying complex structure* the smooth family of complex structures $J_x : \mathcal{D}_x \to \mathcal{D}_x$ defined by $J_x(\mathrm{Re}(v)) = \mathrm{Re}(i \cdot v)$ for any $v \in \mathcal{D}^{1,0}_x$. In the following, a CR structure $\mathcal{D}^{1,0}$ will be often indicated by the associated pair $(\mathcal{D}, J)$, of underlying distribution and complex structure. Notice that any CR distribution $\mathcal{D}^{1,0}$ can be completely recovered from its associated pair $(\mathcal{D}, J)$: In fact, at any $x \in M$, the subspace $\mathcal{D}^{1,0}_x \subset T^{\mathbb{C}}M$ coincides with the J_x-eigenspace in $\mathcal{D}^{\mathbb{C}}_x$ of eigenvalue $+i$.

A *CR-equivalence* between two CR manifolds $(M, \mathcal{D}^{1,0})$ and $(M', \mathcal{D}'^{1,0})$ is a diffeomorphism $\varphi : M \to M'$ such that $\phi_*(\mathcal{D}^{1,0}) = \mathcal{D}'^{1,0}$. If we consider the pairs $(\mathcal{D}, J)$ and $(\mathcal{D}', J')$ associated with $\mathcal{D}^{1,0}$ and $\mathcal{D}'^{1,0}$, respectively, it follows immediately from definition that a diffeomorphism $\varphi : M \to M'$ is a CR-equivalence if and if $\phi_*(\mathcal{D}) = \mathcal{D}'$ and $\phi_*(J) = J'$

Let $(\mathcal{D}, J)$ be a CR structure of hypersurface type (i.e. with $\operatorname{codim} \mathcal{D} = 1$) on a manifold M with $\dim M = 2n-1$. Such CR structure is said to be *Levi non-degenerate* if the underlying real distribution $\mathcal{D}$ is a *contact distribution*. This corresponds to say that any 1-form θ, satisfying $\operatorname{Ker} \theta|_x = \mathcal{D}_x$ at any x, is a contact form, i.e. so that $\theta \wedge (d\theta)^n \equiv 0$ or, equivalently, so that $d\theta_x|_{\mathcal{D}_x \times \mathcal{D}_x}$ is a non-degenerate 2-form on $\mathcal{D}_x$. We would like to point out that this definition is completely equivalent to the classical definition of "Levi non-degeneracy".

For any real hypersurface $S \subset \mathbb{C}^n$, the CR structure of hypersurface type induced on S from the standard complex structure J_o will be denoted by $(\mathcal{D}_o, J_o)$ and it will be called *standard CR structure of* S. When S is the smooth boundary of a strongly pseudoconvex domain, the distribution $\mathcal{D}_o$ is a a contact distribution and will be called *the standard contact distribution of* S.

2.1.2. *Circular domains and Minkowski functions.*

For any $\zeta \in \mathbb{C}$, let us denote by $\zeta \cdot (\ \ ) : \mathbb{C}^n \to \mathbb{C}^n$ the holomorphic transformation

$$\zeta \cdot (z^1, \dots, z^{n+1}) \overset{\text{def}}{=} (\zeta \cdot z^1, \dots, \zeta \cdot z^{n+1}) \ . \tag{2.1}$$

The circle $S^1 = \partial\Delta$ will be often tacitly identified with the compact 1-parameter group of holomorphic transformations $S^1 = \{ \ e^{it} \cdot (\ ) \ \}$. We recall that a domain $D \subset \mathbb{C}^{n+1}$ is called *circular* if it is invariant under any transformation of S^1, while it is called *circular and complete* if it is invariant under all transformations $\zeta \cdot (\ \ )$ with $\zeta \in \bar{\Delta}$.

For any complete circular domain $D \subset \mathbb{C}^n$, the associated *Minkowski function* is the map $\mu_D : \mathbb{C}^{n+1} \to \mathbb{R}_{\geq 0}$ defined by

$$\mu_D(z) = \frac{1}{t_{\,z}} \ , \qquad \text{where } t_z = \sup\{ \ s \in \mathbb{R} \ : \ s \cdot z \in D \ \} \ . \tag{2.2}$$

Notice that, $\mu_D(\zeta \cdot z) = |\zeta| \mu_D(z)$ and that $D = \{ \ z \in \mathbb{C}^{n+1} \ : \ \mu_D(z) < 1 \ \}$. In particular, if D is smoothly bounded, the function $\tau = \mu_D^2 - 1$ is a defining functions for D which is smooth on $\mathbb{C}^{n+1} \setminus \{0\}$. In this case, D is strictly linearly convex if and only if $\tau = \mu_D^2 - 1$ has strictly positive Hessian at any $x \neq 0$.

We conclude with the following definition. If $D \subset \mathbb{C}^n$ is a complete circular domain with Minkowski function μ_D, for any $v \in T_0\mathbb{C}^n = \mathbb{C}^n$ we call *standard radial disc of* D *tangent to* v the holomorphic map

$$f_v^{(\mu)} : \bar{\Delta} \to D \ , \qquad f_v^{(\mu)}(\zeta) = \zeta \cdot \frac{v}{\mu(v)} \ .$$

For instance, in case $D = B^n$ the "standard radial discs" are the holomorphic discs of the form $f_v(\zeta) = \zeta \cdot \frac{v}{|v|}$ for some $v \in \mathbb{C}^n$. We remark that any standard radial disc of a smoothly bounded complete circular domain D is indeed a *stationary disc for* $S = \partial D$, according to the well-known definition of Lempert ([14]; for a definition of stationary discs of an hypersurface see also [30]); for a proof of such property we refer e.g. [21], Lemma 3.34.

2.1.3. *Blow ups of $\mathbb{C}^n$ and lifts of standard radial discs: some elementary facts.*

In all this paper, we will denote by $\widetilde{\mathbb{C}}^n$ the blow up at the origin of $\mathbb{C}^n$, by $\widetilde{\pi} : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ the standard projection and, for any domain $D \subset \mathbb{C}^n$ containing the origin, we will indicate by $\widetilde{D}$ the blow up of D at 0, considered as a subset of $\widetilde{\mathbb{C}}^n$. In other words, $\widetilde{D} \overset{\text{def}}{=} \widetilde{\pi}^{-1}(D) \subset \widetilde{\mathbb{C}}^n$.

We recall that $\widetilde{\mathbb{C}}^n$ coincides with the tautological line bundle $\pi : \widetilde{\mathbb{C}}^n = E \longrightarrow \mathbb{C}P^{n-1}$, i.e. with the bundle E over $\mathbb{C}P^n$ given by the pairs $([v], z) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n$ such that $z \in [v]$, and that the exceptional divisor of $\widetilde{\mathbb{C}}^n = E$ coincides with the image of the zero section of $\pi : E \to \mathbb{C}P^n$.

Given a system of affine coordinates $v = (v^1, \ldots, v^{n-1}) : \mathcal{U} \subset \mathbb{C}P^{n-1} \to \mathbb{C}^{n-1}$ for some open subset of $\mathbb{C}P^{n-1}$ we call *associated system of coordinates* the system of coordinates $(v, \zeta) = (v^1, \ldots, v^{n-1}, \zeta) : \widetilde{\mathcal{U}} = \pi^{-1}(\mathcal{U}) \subset E \to \mathbb{C}^n$, which maps any point $([v], z) \in \widetilde{\mathcal{U}}$ into the n-tuple, whose first $n-1$ components are the affine coordinates of $[v]$, while the last is the unique complex number so that

$$([v], z) = \left([v^1, \ldots, \underset{\text{i-th place}}{1}, \ldots v^{n-1}], \zeta \cdot v^1, \ldots, \underset{\text{i-th place}}{\zeta}, \ldots, \zeta \cdot v^{n-1}) \right) . \tag{2.5}$$

We conclude this short subsection noticing that any standard radial disc $f_v : \Delta \to B^n$ admits a unique lifted map

$$\widetilde{f}_v : \Delta \to \overline{\widetilde{B}^n} , \tag{2.6}$$

which projects down to f_v. It can be checked that any map $\widetilde{f}_v$ is a stationary disc for the sphere S^{2n-1}, considered this time as a real hypersurface of $\widetilde{\mathbb{C}}^n$. Moreover, the image of $\widetilde{f}_v$ coincides with one of the fibers of $\pi_{B^n} : \widetilde{B}^n \to \mathbb{C}P^{n-1}$. In the following, we will shortly refer to the discs $\widetilde{f}_v$ as the *standard radial discs of* $\widetilde{B}^n$: Notice that the images of such standard radial discs determines a holomorphic foliation for $\widetilde{B}^n$ (they are the fibers of the bundle structure of $\widetilde{B}^n$ over $\mathbb{C}P^{n-1}$).

Analogous conclusions hold for the lifts in $\widetilde{\mathbb{C}}^n$ of the the radial discs $f_v^{(\mu)}$ of a complete circular domain D.

2.2. **Manifolds of circular type, indicatrices and Monge-Ampère foliations.**

In this section, we give the definition of "manifolds of circular type", a notion introduced by the first author in [24], and we define a few other related concepts that will reveal to be essential in the study of the moduli spaces of such manifolds. In particular, we are going to show in the next section that *any manifold of circular type admits a "circular representation"* (see §3 for definition) extending a property usually stated only for strictly convex domains or circular domains in $\mathbb{C}^n$.

2.2.1. *Manifolds and domains of circular type.*

In the next phrases, for any function $\tau : M \to \mathbb{R}$, we will denote $M_{\tau=0} = \tau^{-1}(0)$ and $M_{\tau\neq 0} = M \setminus M_{\tau=0}$.

Definition 2.1. Let M be a non-compact complex manifold of complex dimension n. We say that M is a *manifold of circular type* if it admits an exhaustion function $\tau : M \longrightarrow [0, r^2)$, for some $r^2 \in (0, \infty]$, such that

a) $\tau \in \mathcal{C}^0(M) \cap \mathcal{C}^\infty(M_{\tau\neq 0})$;
b) on $M_{\tau\neq 0}$, τ is so that $dd^c\tau > 0$ and $dd^c \log\tau \geq 0$;
c) on $M_{\tau\neq 0}$, $(dd^c \log\tau)^n \equiv 0$;
d) there exists a point $x_o \in M_{\tau=0}$ so that, if $\pi : \widetilde{M} \to M$ is the blow up of M at x_o, then $\tau \circ \pi : \widetilde{M} \to \mathbb{R}$ is smooth and there exist two positive constant C_1, C_2 so that $C_1\|x - x_o\|^2 \leq \tau(x) \leq C_2\|x - x_o\|^2$ at all points of a neighborhood of x_o (here "$\|\cdot\|$" is the Euclidean norm in some coordinate neighborhood centered at x_o).

It is easy to see that $\{\tau = 0\} = \{x_o\}$ ([24]). Such point x_o is called *center of M w.r.t. τ* and $\tau : M \longrightarrow [0, r^2)$ is called *parabolic exhaustive function*. If $r < \infty$ (i.e. τ is bounded), we call M *bounded*.

A smoothly bounded, relatively compact domain D in a complex manifold (M, J) is called *domain of circular type* if (D, J) is a bounded manifold of circular type admitting at least one parabolic exhaustive function τ, which admits a smooth extension up to the boundary and satisfies b) also at the points of ∂D. For domains of circular type, we shall consider only exhaustions that are smooth and satisfying b) up the boundary. For such domains, the expression *parabolic exhaustion* will always mean such a function.

Lemma 2.2.

a) *Any manifold of circular type is Stein;*
b) *if (M, J, τ) is a domain of circular type and τ, τ' are two parabolic exhaustive functions of M (smooth up to the boundary) having the same center x_o, then $\tau' = k\tau$ for some suitable positive constant k.*

Proof. Claim (a) is immediate (see f. i. [19]). Claim (b) is a consequence of Thm. 4.3 in [11]. In fact, M is a hyperconvex domain according to Def. 2.1 in [11] and hence, for the cited theorem, there exists a unique plurisubharmonic function u so that $u|_{\partial M} = 0$, $(dd^c u)^n \equiv 0$ and $u(z) \sim \log|z - x_o|$ for $z \to x_o$. This implies that $\log\tau = u + c$ and $\log\tau' = u + c'$ where c and c' are the constant values $c = \log\tau|_{\partial M}$, $c' = \log\tau|_{\partial M}$. From this the claim follows immediately. □

Any complete circular domain of $\mathbb{C}^n$ is a domain of circular type. In fact, for any such domain D, if we denote by $\mu_D : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ the Minkowski functional of D, then the map $\tau = \mu_D^2$ is a parabolic exhaustive function with center $x_o = 0$. Notice that, *generically, the origin* 0 *is the only center of a complete circular domain.*

Other examples of domains of circular type are given by all bounded, strictly convex domain $D \subset \mathbb{C}^n$ with smooth boundary. In fact, using the properties of the Kobayashi pseudo-distance of such domains (see [14, 15, 24]), we have that for any point $x_o \in D$, the function

$$\tau_{x_o} : D \longrightarrow \mathbb{R}_+ \ , \qquad \tau_{x_o}(x) = \left(\frac{e^{\kappa_D(x,x_o)-1}}{e^{\kappa_D(x,x_o)+1}}\right)^2 \ ,$$

(κ_D = Kobayashi pseudo-distance of D) is a parabolic exhaustive function for D which extends smoothly up to the boundary. In particular, *any point x_o of a smoothly bounded, strictly convex domain D in $\mathbb{C}^n$ is a center for D w.r.t. some parabolic exhaustive function.*

2.2.2. *Monge-Ampère foliations and normal distributions.*

Let (M, J, τ) be a bounded manifold of circular type and $x_o \in M$ be the center for M w.r.t. τ. By multiplication of τ by positive constant, we may assume that $\tau : M \to [0, 1)$. We will also denote by $\widetilde{M}$ the blow up of M at x_o.

First of all, we want recall a basic identity that follows from property (c) of Definition 2.1 (see e.g. [29]). Using just the definitions, one finds immediately that

$$\tau^2 dd^c \log\tau = \tau dd^c\tau - d\tau \wedge d^c\tau \ , \tag{2.7}$$

$$\tau^{k+1}(dd^c \log\tau)^k = \tau(dd^c\tau)^k - kd\tau \wedge d^c\tau \wedge (dd^c\tau)^{k-1} \ . \tag{2.8}$$

So, condition (c) of Definition 2.1 is equivalent to the equality

$$\tau(dd^c\tau)^n = nd\tau \wedge d^c\tau \wedge (dd^c\tau)^{n-1} \tag{2.9}$$

that has to be satisfied at all points of $M \setminus \{x_o\}$.

Consider now the vector field Z on $M \setminus \{x_o\}$ defined by the condition

$$dd^c\tau(Z, JX) = X(\tau) \qquad \text{for any} \quad X \in T\left(M \setminus \{x_o\}\right) \ . \tag{2.10}$$

Let also denote by $\mathcal{Z}$ and $\mathcal{H}$ the J-invariant distributions on $M \setminus \{x_o\}$ defined as the family of spaces

$$\mathcal{Z}_x \overset{\text{def}}{=} \operatorname{span}_{\mathbb{R}}\{\ Z_x\ ,\ JZ_x\ \}\ , \tag{2.11}$$

$$\mathcal{H}_x = (\mathcal{Z}_x)^\perp \overset{\text{def}}{=} \{\ X \in T_xM\ :\ dd^c\tau(Z, X) = dd^c\tau(JZ, X) = 0\ \}\ . \tag{2.12}$$

and notice that $T_xM = \mathcal{Z}_x \oplus \mathcal{H}_x$. Using (2.10) and (2.9), it follows that

$$dd^c\tau(Z, JZ)|_x = d\tau(Z)|_x = \tau_x \ . \tag{2.13}$$

From (2.13) and (2.7), it can be checked that $\mathcal{Z}$ coincides with the family of subspaces

$$\mathcal{Z}_x = \ker\, dd^c \log\tau|_x \tag{2.14}$$

and hence *is an integrable distribution*. On the other hand, from (2.12) and (2.10), it turns out that any space $\mathcal{H}_x \subset \mathcal{H}$ is tangent to the real hypersurface $S_x = \{\ y\ :\ \tau(y) = \tau(x)\ \}$. By the fact that $\mathcal{H}_x$ is J-invariant and of real dimension $2(n-1)$, it follows that *for any $S_c \overset{def}{=} \{\ \tau = c\ \}$, the distribution $\mathcal{H}|_{S_c}$ is the real distribution underlying the CR structure of S_c.*

The distributions $\mathcal{Z}$ and $\mathcal{H}$ are defined just on $M \setminus \{x_o\}$ and they do not extend to a regular distribution over the whole M. But they do extend on $\widetilde{M}$:

Lemma 2.3. *The distributions $\mathcal{Z}$ and $\mathcal{H}$ extend in a unique way as smooth J-invariant distributions on the whole blow up $\widetilde{M}$. The extension of $\mathcal{Z}$ is integrable everywhere, while the extension of $\mathcal{H}$ is integrable only when restricted on $\pi^{-1}(x_o)$. More precisely, after identification of a neighborhood of $\pi^{-1}(x_o) \subset \widetilde{M}$ with an open neighborhood of $\mathbb{C}P^{n-1}$ in $E = \widetilde{\mathbb{C}}^n$, then $\mathcal{Z}|_{\mathbb{C}P^{n-1}}$ coincides with the restriction to $\mathbb{C}P^{n-1}$ of the vertical distribution of $\pi : E \to \mathbb{C}P^n$, while $\mathcal{H}|_{\mathbb{C}P^{n-1}} = T\mathbb{C}P^{n-1}$.*

Proof. For what concerns the extendibility of $\mathcal{Z}$, it suffices to re-write formulae (2.7) and (2.10) of [24] using a system of coordinates $(v^1, \dots, v^{n-1}, \zeta)$ as described in (2.5). From those formulae it is immediate to realize that the vector fields Z and JZ extend smoothly at $\mathbb{C}P^{n-1} = \pi^{-1}(x_o) \subset \widetilde{M}$ setting $Z_{([v],0)} = \operatorname{Re}\left(\partial/\partial\zeta|_{([v],0)}\right)$, $JZ_{([v],0)} = \operatorname{Im}\left(\partial/\partial\zeta|_{([v],0)}\right)$, and that the extension of $\mathcal{Z}$ is integrable.

About the extendibility of $\mathcal{H}$, we recall that (see e.g. [31])

$$\mathcal{L}_Z dd^c\tau = d\left(\imath_Z dd^c\tau\right) \overset{(2.19)}{=}$$

$$= d\left(\frac{1}{\tau}\imath_Z d\tau \wedge d^c\tau + \tau \cdot \imath_Z dd^c \log\tau\right) \overset{(2.12),(2.8),(2.11)}{=} dd^c\tau \ . \tag{2.15}$$

From (2.15) and (2.12), it follows that $\mathcal{H}$ is preserved by the flow Φ^Z_t of Z on $M \setminus \{x_o\}$[1] and hence it is natural to extend $\mathcal{H}$ by setting, for any $y \in \mathbb{C}P^{n-1} = \pi^{-1}(x_o) \subset \widetilde{M}$,

$$\mathcal{H}_y = \Phi^Z_{-t*}(\mathcal{H}_{\Phi^Z_t(y)}) \qquad \text{for some sufficiently small} \quad t > 0 \ . \tag{2.16}$$

It is quite direct to check that (2.16) does not depend on t and that in this way $\mathcal{H}$ is extended smoothly. Moreover, by Lemma 2.1 in [24], there exists a smooth positive map h such that, in a system of coordinates (2.5), τ is of the form

$$\tau(v,\zeta) = |\zeta|^2 h^2(v) + o(|\zeta|^3) \tag{2.17}$$

(see (2.18) below). So, for any $y = ([v],0) \in \mathbb{C}P^{n-1}$, $\mathcal{H}_y = T_y\mathbb{C}P^{n-1}$. $\square$

The integrable distribution $\mathcal{Z}$ and its associated foliation, usually called *Monge-Ampère foliation*, were considered for the first time by Bedford and Kalka in [9]. We will therefore call the complementary distribution $\mathcal{H}$ the *normal distribution of the Monge-Ampère foliation*.

Lemma 2.4. *Let (M, J, τ) be a manifold of circular type and τ, τ' two parabolic exhaustive functions of M with the same associated center x_o. If there exists $c, c' > 0$ such that $\emptyset \neq \{\ \tau = c\ \} = \{\ \tau' = c'\ \}$, then $\tau' = k\tau$ for some positive constant k.*

Proof. By Lemma 2.2, if we denote by $D = \{\ \tau < c\ \} = \{\ \tau' < c'\ \}$, then $\tau|_D = k\tau'|_D$ for some $k > 0$. Replacing τ' by $k \cdot \tau'$, it remains to show that if $\tau|_D = \tau'|_D$ then $\tau = \tau'$. For this, let us denote by $\widetilde{\tau} = \tau \circ \pi$ and $\widetilde{\tau}' = \tau' \circ \pi$ the lifts of τ, τ' at the blow up $\pi : \widetilde{M} \to M$. Let also $\widetilde{D} = \pi^{-1}(D)$. Consider a leaf S of the Monge-Ampère foliation of $\widetilde{M}$, determined by τ, and observe that $S \cap \widetilde{D}$ coincides with an the open subset $S' \cap \widetilde{D}$ of some leaf S' of the Monge-Ampère foliation determined by τ'. Being S and S' analytic submanifolds of $\widetilde{M}$, $S = S'$ and the Monge-Ampère foliations of τ, τ' coincide. Now, let Z and Z' be the vector fields defined via (2.10) by τ and τ', respectively, and consider the complex vector fields $Z^{1,0} = \frac{1}{2}(Z - iJZ)$, $Z'^{1,0} = \frac{1}{2}(Z' - iJZ')$. The restrictions $Z^{1,0}|_S$ and $Z'^{1,0}|_S$ on a leaf S of the Monge-Ampère foliation are holomorphic vector fields for S. Since they coincide on $S \cap \widetilde{D}$, we get that $Z^{1,0}|_S \equiv Z'^{1,0}|_S$. By arbitrariness of S, $Z = Z'$ and hence $\tau = \tau'$ by (2.13). $\square$

[1]The same arguments imply that $\mathcal{H}$ is preserved also by the flow of JZ.

We conclude this section with the following technical lemma, which will be needed in the following and shows that $\mathcal{H}$ is uniquely determined by the center x_o and the vector field (2.10).

Let (M, J, τ) be a bounded manifold of circular type and $\pi : \widetilde{M} \to M$ the usual blow up at the center x_o. Let also (J', τ') another pair so that also (M, J', τ') is a domain of circular type with the same center x_o and let $\pi : \widetilde{M}' \to M$ the new blow up at x_o. Then, we have the following:

Lemma 2.5. *If (J, τ) and (J', τ') determine the same vector field Z via (2.10), then $\widetilde{M} = \widetilde{M}'$ (as real manifolds) , $\tau = \tau'$ and $\mathcal{H} = \mathcal{H}'$, i.e. the normal distributions of the Monge-Ampère foliations of (M, J, τ) and (M, J', τ') are the same.*

In particular, the CR structures induced by J and J' on any hypersurface $S_c = \{\tau = \tau' = c\}$ have the same underlying real distributions.

Proof. Consider the identity map between $\widetilde{M} \setminus \pi^{-1}(x_o) = M \setminus \{x_o\} = \widetilde{M}' \setminus \pi'^{-1}(x_o)$ and extend it continuously along each integral curve of Z. From Lemma 2.3, such map is unique and smooth relatively to the real manifold structures of $\widetilde{M}$ and $\widetilde{M}'$ and implies that we may consider $\widetilde{M} = \widetilde{M}'$ as real differentiable manifolds. To see that $\tau = \tau'$, it suffices to observe that, by (2.17) and (2.14), $\tau|_{\pi^{-1}(x_o)} = \tau'|_{\pi^{-1}(x_o)} = 0$ and, for any x of the form $x = \Phi^Z_t(y)$, $t > 0$, for some $y \in \pi^{-1}(x_o)$, we have that $\tau(x) = e^t = \tau'(x)$. Finally, since by Lemma 2.3 $\mathcal{H}_y = \mathcal{H}'_y = T_y\mathbb{C}P^{n-1}$ for any point $y \in \mathbb{C}P^{n-1} = \pi^{-1}(x_o)$, by the invariance of $\mathcal{H}$ and $\mathcal{H}'$ under the flow of Z, the same argument of before implies that $\mathcal{H}_x = \mathcal{H}'_x$ at any $x \in M$.□

2.2.3. *Indicatrices of a manifold of circular type.*

Let (M, J, τ) be a bounded manifold of circular type and $x_o \in M$ the center for M w.r.t. τ. As before, we assume that $\tau : M \to [0, 1)$.

Consider a system of complex coordinates $(z^1, \dots, z^n)$ on a neighborhood of x_o with $z^j(x_o) = 0$, $j = 1, \dots, n$. By Lemma 2.1 in [24], there exists a smooth map $h : S^{2n-1} \to \mathbb{R}_{>0}$ such that

$$\sqrt{\tau(z)} = |z| h\left(\frac{z}{|z|}\right) + o(|z|^2) \ . \tag{2.18}$$

We set

$$\kappa : T_{x_o}M \simeq \mathbb{C}^n \to \mathbb{R}_{\geq 0} \ , \qquad \kappa(v) = \begin{cases} |v| \cdot h\left(\frac{v}{|v|}\right) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0 \end{cases} \ . \tag{2.19}$$

It can be immediately checked that the value $\kappa(v)$ coincides with

$$\kappa(v) = \lim_{t_o \to 0} \frac{d}{dt}\sqrt{\tau(\gamma_t)}\bigg|_{t=t_o} \ ,$$

for any smooth curve γ_t with $\gamma_0 = x_o$ and $\dot{\gamma}_0 = v$. In particular, it does not depend on the choice of the coordinate system. Furthermore we have that

$$\kappa(\lambda v) = |\lambda|\kappa(v) \qquad \text{for any } \lambda \in \mathbb{C}$$

and that κ is a smooth function on $T_{x_o}M \setminus \{0\}$.

Definition 2.6. The *indicatrix of M at the center x_o determined by τ* is the smoothly bounded, complete circular domain $I_{x_o} \subset T_{x_o}M$ defined by

$$I_{x_o} = \{ \ v \in T_{x_o}M \ : \ \kappa(v) < 1 \ \} \ . \tag{2.20}$$

Remark 2.7. Notice that I_{x_o} and κ^2 coincide with the domain $G(r)$ and the function σ defined in (4.3) and (4.1) of [24], respectively. In particular, by the proof of Prop. 4.1 of that paper, $dd^c\kappa^2 > 0$ at all points of $\overline{I}_{x_o} \setminus \{0\}$ and hence I_{x_o} is strongly pseudoconvex. Observe also that if M is a strictly convex domain in $\mathbb{C}^n$ and τ is as in (1.1), then κ coincides with the infinitesimal Kobayashi metric of M at x_o and I_{x_o} is the Kobayashi indicatrix of M at x_o.

3. Circular representations

In this section, (M, J, τ) is a bounded manifold of circular type, τ takes values in $[0, 1)$, $x_o \in M$ is the center associated with τ and $I = I_{x_o} \subset T_{x_o}M$ the corresponding indicatrix at x_o. Keep in mind that, under an identification $T_{x_o}M \simeq \mathbb{C}^n$, I can be considered as a circular domain, whose Minkowski function is the map κ defined in (2.19). We will also denote by $\widetilde{M}$ the blow up of M at x_o and we systematically insert the tilde " $\widetilde{}$ " on top of symbols of manifolds, domains or maps, whenever we want to indicate a blow up or a lift of a map at such blow up. In particular, $\widetilde{\tau} = \tau \circ \pi : \widetilde{M} \to [0, 1)$, $\widetilde{I}$ and $\widetilde{T_{x_o}M}$ are the blow ups of I and $T_{x_o}M \simeq \mathbb{C}^n$ at 0 and, for any $v \in \partial I$, we set $\widetilde{f}_v : \Delta \to \widetilde{I}$ to be the the radial disc of $\partial\widetilde{I}$, obtained by lifting the radial disc $f^{(\kappa)}_v$ and defined in (2.6) and following.

The purpose of this section is to recall a few well known properties, usually stated for strictly convex domains (see e.g. [6, 8, 14, 16, 21, 23]) and which actually hold for all domains of circular type. We collect such properties in the next proposition, whose proof is a direct application of the results in [24]. Since the notation and terminology used [24] is substantially different from the one of this paper, we give also an outline of the proof.

Proposition 3.1. *Let (M, J, τ) be a bounded domain of circular type with $\tau : M \to [0, 1)$, and let $I \subset T_{x_o}M$ be the indicatrix determined by τ. Then there exists a unique diffeomorphism $\Psi : \widetilde{I} \to \widetilde{M}$ with the following properties:*

i) *$\Psi|_{\pi^{-1}(0)} = Id_{\pi^{-1}(0)}$, provided that we naturally identify the exceptional divisor $\pi^{-1}(0)$ of $\widetilde{T_{x_o}M}$ with the exceptional divisor $\pi^{-1}(x_o)$ of $\widetilde{M}$;*
ii) *For any $t \in]0, 1[$, consider the map*
$$\Phi^{(t)} : \partial\widetilde{I} \to \widetilde{M} \ , \qquad \Phi^{(t)}([v], z) = \Psi([v], tz) \ ; \tag{3.1}$$
Then $\Phi^{(t)}|_{\partial\widetilde{I}}$ is a diffeomorphism between $\partial\widetilde{I}$ and the level hypersurface $S^{(t)} = \{ \ \tau = t^2 \ \} \subset \widetilde{M}$;
iii) *for any $t \in]0, 1[$, $\Phi^{(t)}|_{\partial\widetilde{I}}$ maps the real distribution of the CR structure of $\partial\widetilde{I}$ onto the real distribution of the CR structure of $S^{(t)}$;*
iv) *for any $([v], z) \in \partial\widetilde{I}$ and $t \in]0, 1[$, the map*
$$\widetilde{f}^{(t)}_{([v],z)} : \Delta \to \widetilde{M} \ , \qquad \widetilde{f}^{(t)}_{([v],z)}(\zeta) = \Psi([v], t\zeta z) \tag{3.2}$$

is holomorphic, injective, with $\widetilde{f}^{(t)}_{([v],z)}(\partial\Delta) \subset S^{(t)}$ *and with* $\widetilde{f}^{(t)}_{([v],z)}(\Delta)$*equal to an integral leaf of the Monge-Ampère foliation of* $\widetilde{M}_{<t^2} = \{\ \widetilde{\tau} < t^2\ \}$.

In case M is a domain of circular type and τ is smooth up to the boundary, then Ψ extends smoothly to the boundary with $\Psi(\partial\widetilde{I}) \subset \partial\widetilde{M}$ and (iii), (iv) are valid also for $t = 1$.

Proof. Pick a complex basis for $T_{x_o}M$ and use it to identify $T_{x_o}M \simeq \mathbb{C}^n$ in the rest of the proof. Then, consider the map

$$\Psi : I \setminus \{0\} \longrightarrow M \setminus \{x_o\}\ , \qquad \Psi(v) = F\left(\frac{v}{|v|}, \kappa(v)\right) \tag{3.3}$$

where $F : S^{2n-1} \times \Delta \to M$ is the smooth map of Thm. 3.4 of [24], with S^{2n-1} unit sphere in $\mathbb{C}^n = T_{x_o}M$ and Δ unit disc in $\mathbb{C}$. In [24] it is proved that:

a) for any $v \in S^{2n-1}$, the map $f^{(v)} : \Delta \to M$, defined by $f^{(v)}(\zeta) = F(v, \zeta)$ is proper, holomorphic and injective, with $f^{(v)}(0) = x_o$ and such that $f^{(v)}(\Delta \setminus \{0\})$ is a leaf of the Monge-Ampère foliation;

b) if (ρ, θ) are the standard polar coordinates of $\mathbb{C}$, the map in (a) is so that

$$f^{(v)}_*\left(\left.\frac{\partial}{\partial\rho}\right|_{\zeta}\right) = 2\sqrt{\tau}\cdot Z\big|_{F(v,\zeta)}\ , \qquad \text{for any } \zeta \in \Delta \setminus \{0\} \tag{3.4}$$

$$f^{(v)}_*\left(\left.\frac{\partial}{\partial\rho}\right|_{0}\right) = \frac{v}{\kappa(v)} \in T_{x_o}M\ \ ; \tag{3.5}$$

c) $F(e^{i\theta}\cdot y, \zeta) = F(y, e^{i\theta}\zeta)$ for any $\theta \in \mathbb{R}$;

d) $\tau(F(v,\zeta)) = |\zeta|^2$ for any $v \in S^{2n-1}$.

Now, identify the exceptional divisor of $\widetilde{T_{x_o}M}$ with the exceptional divisor of $\widetilde{M}$ and extend Ψ to a map $\Psi : \widetilde{I} \to \widetilde{M}$ by setting $\Psi|_{\mathbb{C}P^{n-1}} = Id_{\mathbb{C}P^{n-1}}$. Using definitions, Thm. 3.4 and Lemma 2.5 of [24], it is possible to check that such extended map is a diffeomorphism.

Claim (ii) follows directly from d) and the definition of Ψ. To check (iii), notice that $\Phi^{(t)} = \Psi \circ \delta^{(t)}$ where $\delta^{(t)}$ is the contraction

$$\delta^{(t)} : \partial\widetilde{I} \to t\cdot\partial\widetilde{I} \overset{\text{def}}{=} \{\ ([v], z) \in \widetilde{T_{x_o}M}\ :\ |z| = t\ \}\ ,\quad \delta^{(t)}([v], z) = ([v], tz)\ . \tag{3.6}$$

Since $\delta^{(t)}$ is a CR map, we just need to show that Ψ maps the real distribution of the CR structure of $t\cdot\partial\widetilde{I}$ onto the real distribution of the CR structure of $S^{(t)}$. For this, let us first observe that d) implies $\Psi^*(\tau) = \tau_o$ with $\tau_o([v], z) \overset{\text{def}}{=} |z|^2$. Then, let us denote by J the original complex structure of $\widetilde{I} \subset \widetilde{T_{x_o}M}$ and by J' the complex structure $J' = \Psi_*^{-1}(J)$: The proof reduces now to check that the CR structures induced by J and J' on $t\cdot\partial\widetilde{I} = \{\ x \in \widetilde{I}\ :\ \tau_o(x) = t^2\ \}$ have the same underlying real distribution. But this follows from Lemma 2.5 applied to the pairs (J, τ_o) and (J', τ_o).

Claim (iv) is an immediate consequence of the definition of the map $\widetilde{f}^{(t)}_{([v],z)}$, of properties (a) and (d) of F and the fact that $\widetilde{f}^{(t)}_{([v],z)}$ is holomorphic over the whole Δ, because of boundedness and holomorphicity of $\widetilde{f}^{(t)}_{([v],z)}$ on $\Delta \setminus \{0\}$,.

To check the last claim, observe that by definition the map $F : S^{2n-1} \times \Delta \to M$ of Thm. 3.4 in [24] extends smoothly to a map $F : S^{2n-1} \times \overline{\Delta} \to \overline{M}$ (see also §4 in

[25]) and, being the maps in a) proper, $F(S^{2n-1} \times \partial\Delta) \subset \partial M$. From this we infer that Ψ extends smoothly up to the boundary with $\Psi(\partial\widetilde{I}) = \partial\widetilde{M}$. Properties (iii) and (iv) when $t = 1$ are obtained with the same arguments used for $t < 1$. $\square$

The diffeomorphism $\Psi : \widetilde{I} \to \widetilde{M}$ of the previous proposition will be called *circular representation of M associated with τ* and it can be considered as the lift at blow ups level of the circular representation defined in [24]. When M is a smoothly bounded, strictly convex domain in $\mathbb{C}^n$, Ψ coincides with the circular representation considered in [8].

4. Normal forms for manifolds of circular type

As in the previous sections, we will systematically insert the tilde " $\widetilde{\ }$ " on top of symbols of manifolds, domains or maps, whenever we want to indicate a blow up or a lift of a map at such blow up. In particular, $\widetilde{B}^n$ is the blow of the unit ball at the origin. We will also denote by J_o the standard complex structure of $\mathbb{C}^n$ and of $\widetilde{\mathbb{C}^n}$ and by $\tau_o : \mathbb{C}^n \to [0, \infty)$ the standard parabolic exhaustion $\tau_o(z) = |z|^2$.

4.1. Complex structures of Lempert type and manifolds in normal form.

Definition 4.1. Let $I \subset \mathbb{C}^n$ be a complete circular domain with Minkowski function μ and let $\pi : \widetilde{I} \to I$ the blow up of I at 0 and $\tilde{\mu} = \mu \circ \pi$. A complex structure J on $\widetilde{I}$ is called *of Lempert type* if

a) on any hypersurface $S(c) = \{ z \in \widetilde{I} : \widetilde{\mu}(z) = c \}$, $0 < c < 1$, the distribution $\mathcal{D}$ underlying the CR structure induced by J coincides with the distribution $\mathcal{D}_o$ underlying the CR structure induced by J_o ;
b) the projection $\pi : \widetilde{I} \to I$ induces a complex manifold structure on $I \setminus \{0\}$, whose charts are smoothly overlapping with the charts of the standard manifold structure of $\mathbb{C}^n \setminus \{0\}$, i.e. the projected complex structure is given by a smooth tensor field J of type $(1, 1)$ on $I \setminus \{0\}$;
c) the restriction of J on any tangent space of a standard radial disc of I coincides with the standard complex structure J_o.

Notice that, from a), b) and c), the function $\widetilde{\tau} \overset{\text{def}}{=} \mu^2 \circ \pi$ is plurisubharmonic on $\widetilde{I}$ and strictly plurisubharmonic on the complement of the exceptional set. By Narasimhan's result in [19], if I is endowed with a suitable complex manifold structure, this implies that $\widetilde{I}$ is a proper modification of I. Such complex structure coincides with the one described in b) on $I \setminus \{0\}$, but *need not* to smoothly overlap with the standard complex structure at 0.

We call such complex structure the *projected complex structure of Lempert type on I* and it will be indicated by the associated tensor field J, even if such tensor is a smooth tensor w.r.t. the standard coordinates of $\mathbb{C}^n$ only at the points of $I \setminus \{0\}$.

Two complex structures J and J' of Lempert type are called *Lempert isotopic* (or, shortly, *L-isotopic*) if there exists a smooth family J_t, $t \in [0, 1]$, of complex structures of Lempert type on $\widetilde{I}$, such that $J_0 = J$ and $J_1 = J'$.

Theorem 4.2. *Any complex structure J on $\widetilde{B}^n$, which is of Lempert type and L-isotopic to J_o, projects onto a non-standard complex manifold structure J on B^n which makes (B^n, J, τ_o) a bounded manifold of circular type.*

Proof. Since (B^n, J_o, τ_o) is a domain of circular type, we only need to prove that conditions b) and c) of Definition 2.1 are still true after replacing J_o with J. Let $\mathcal{Z}_o$ and $\mathcal{H}_o$ be the tangent and normal distributions of the Monge-Ampère foliation of $\widetilde{B}^n$ determined by (J_o, τ_o) and Z the vector field defined in (2.10). Recall that, by remarks after (2.14) and property (a) of Definition 4.1, the distribution $\mathcal{H}_o$ is J-invariant. Recall also that the operator $d^{c'}$, obtained from d^c by replacement of J_o with J, is

$$d^{c'} = -\frac{1}{4\pi} J^{-1} \circ d \circ J \tag{4.1}$$

where for any p-form α, $J(\alpha)$ denotes the p-form defined by $(J\alpha)(X_1, \dots, X_p) = (-1)^p \alpha(JX_1, \dots, JX_p)$ (see e.g. [3], p. 68). From this we get that, for any vector field X in $\mathcal{H}_o$,

$$dd^{c'}\tau_o(Z, X) = \frac{1}{4\pi}\left\{Z(JX(\tau_o)) - X(JZ(\tau_o)) - J([X, Z])(\tau_o)\right\} = 0\ . \tag{4.2}$$

Here we used the fact that $\mathcal{H}_o$ is preserved by the flow of Z and that, by (b) of Definition 4.1, $JZ(= J_oZ)$ and JX are both tangent to the hypersurfaces $\{\ \tau_o = \mathit{const.}\ \}$.

Formulae (2.8) and (4.2) show that it suffices to check only condition b) of Definition 2.1, because this would automatically imply also c) of that definition. Now, $dd^{c'}\tau_o|_{\mathcal{Z}_o \times \mathcal{Z}_o} = dd^c\tau_o|_{\mathcal{Z}_o \times \mathcal{Z}_o} > 0$, because (B^n, J_o, τ_o) satisfies b) of Definition 2.1 and $J|_{\mathcal{Z}_o} = J_o|_{\mathcal{Z}_o}$. On the other hand, recall that $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o}$ coincides with the Levi form (w.r.t. to J) of the hypersurfaces $\{\ \tau_o = \mathit{const.}\ \}$. Since $\mathcal{H}_o$ is also the real distribution underlying the CR structure induced by J_o and the hypersurfaces $\{\ \tau_o = \mathit{const.}\ \}$ are strongly pseudo-convex (they are spheres), $\mathcal{H}_o$ is a contact distribution over each such hypersurface (see §2.1). This implies that, at any point, $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o}$ is a *non-degenerate* J-Hermitian form. The same claim is true for all complex structures J_t of an L-isotopy between J and J_o. A trivial continuity argument implies that $dd^{c'}\tau_o|_{\mathcal{H}_o \times \mathcal{H}_o} > 0$. From this, (4.8) and (2.7), $dd^{c'}\tau_o > 0$ and $dd^{c'}\log\tau_o \geq 0$, i.e. b) of Definition 2.1 is true also when J_o is replaced by J. □

Definition 4.3. We call *manifold of circular type in normal form* any bounded manifold of circular type of the form (B^n, J, τ_o), where τ_o is the standard exhaustive function $\tau_o = |\cdot|^2$ and J is a complex structure of Lempert type that is L-isotopic to J_o.

Let (M, J, τ) be a bounded manifold of circular type. We call *normalizing map for M relative to τ and its center x_o* any (J, J')-biholomorphism $\Phi : \widetilde{M} \to \widetilde{B}^n$ between the blow up $(\widetilde{M}, J)$ at x_o and the blow up $(\widetilde{B}^n, J')$ at 0 of a manifold in normal form (B^n, J', τ_o), so that:

a) Φ induces a diffeomorphism between the exceptional divisors;
b) $\widetilde{\tau} = \widetilde{\tau}_o \circ \Phi$, where $\widetilde{\tau}$ and $\widetilde{\tau}_o$ are the lifts of τ and τ_o at the blow ups.

4.2. **Existence and uniqueness of normalizing maps.**

Theorem 4.4. *Let (M, J, τ) be a bounded manifold of circular type, with x_o center of M associated with τ and blow up $\pi : \widetilde{M} \to M$ at x_o. Then, there exists at least one normalizing map $\Phi : (\widetilde{M}, J) \to (\widetilde{B}, J')$ relative to τ and x_o. Moreover, any* *two normalizing maps Φ and $\hat{\Phi}$, both relative to τ and x_o, are equal if and only if $(\Phi' \circ \Phi^{-1})|_{\pi^{-1}(x_o)} = Id$ and $(\Phi' \circ \Phi^{-1})_*$ induces the identity map on the tangent spaces of the leaves of the Monge-Ampère foliation at the points of $\pi^{-1}(x_o)$.*

If (M, J, τ) is a domain of circular type and τ extends up to the boundary, then there exists a normalizing map which extends smoothly up to the boundary.

Proof. Fix a complex basis $(e_0, \dots, e_{n-1})$ for $T_{x_o}M$ and consider the unique isomorphism of complex vector spaces $\imath : T_{x_o}M \to \mathbb{C}^n$ which maps each vector e_i into the corresponding vector of the standard basis $e_i^o = \imath(e_i)$ of $\mathbb{C}^n$. In what follows, we constantly identify $T_{x_o}M$ with $\mathbb{C}^n$ by means of such isomorphism. In particular, use such isomorphism in order to identify the indicatrix $I \subset T_{x_o}M$ associated with τ with the corresponding circular domain $I \subset \mathbb{C}^n$ with Minkowski function κ.

Let $\Psi : \widetilde{I} \subset \widetilde{\mathbb{C}}^n \to \widetilde{M}$ be the circular representation associated with τ and consider the complex structure $J_M = \Psi_*^{-1}(J)$ on $\widetilde{I}$. Keep in mind that, by (i) of Proposition 3.1 and definition of J_M, the blow up at the origin of (I, J_M) is precisely the blow up at the origin of (I, J_o) and that $J_M|_{\mathbb{C}P^n} = J_o|_{\mathbb{C}P^n}$. Moreover, by construction of Ψ (see formula (3.3)) and property d) in the proof of Proposition 3.1, it follows immediately that $\tau \circ \Psi = \kappa^2$ and hence that (I, J_M, κ^2) is a bounded manifold of circular type. Moreover, using the proof of (iii) in Proposition 3.1 and by (iv) of the same proposition, it is quite direct to check that J_M satisfies all three conditions for being a complex structure of Lempert type on I. We claim that J_M is also L-isotopic to J_o. For this, it suffices to consider the diffeomorphisms

$$\Psi^{(t)} : \widetilde{I} \to \widetilde{M}_{<t} = \{ \tau < t^2 \} , \qquad \Psi^{(t)}([v], z) = \Psi([v], tz) , \qquad 0 < t < 1$$

and the complex structures $J^{(t)} = \Psi_*^{(t)-1}(J)$ on $\widetilde{I}$. The same arguments of before show that each complex structure $J^{(t)}$ is of Lempert type. If we set $J^{(0)} = J_o$ and $J^{(1)} = J_M$, using explicit coordinate expressions for the maps Ψ and $\Psi^{(t)}$, it can be checked that $J^{(t)}$, $t \in [0, 1]$, is a smooth family of complex structures even at $t = 0$ and $t = 1$, proving that J_M is L-isotopic to J_o.

From these remarks, the proof can be done assuming that M is a smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$, the parabolic exhaustive function τ is equal to the $\tau = \mu^2$ where μ is the Minkowski function of I, and J is a complex structure of Lempert type on $M = I$, which is L-isotopic to the standard one and so that the blow up of (I, J) at the origin coincides with the blow up of (I, J_o) and with $J|_{\mathbb{C}P^n} = J_o|_{\mathbb{C}P^n}$.

Now, notice that if we replace J by J_o, then (I, J_o, μ^2) remains a manifold of circular type. We claim that if $\Phi : (\widetilde{I}, J_o) \to (\widetilde{B}^n, J')$ is a normalizing map for (I, J_o, μ^2) relative to μ^2 and center $x_o = 0$, then it is also a normalizing map also for (I, J, μ^2). In fact, it is enough to consider the complex structure $J'' = \Phi_*(J_o)$ on $\widetilde{B}^n$ and observe that:

1) Φ is (J, J'')-biholomorphic map by construction;
2) $\widetilde{\tau}_o \circ \Phi = \mu^2 \circ \pi$ because Φ is a normalizing map for (I, J_o, μ^2);
3) J'' is of Lempert type because J is of Lempert type on I;
4) J'' is L-isotopic to J' (because J is L-isotopic to J_o) and J' is L-isotopic to J_o; from this it follows that J'' is L-isotopic to J_o.

So, if we show the existence of a normalizing map for any smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$, relative to $\tau = \mu^2$ and $x_o = 0$, we automatically prove the existence of normalizing maps for any other manifold of circular type.

This is done using the lemma that follows. In order to state it, we have to fix some notation. As usual, the blow-up of $\mathbb{C}^n$ at the origin is identified with the tautological line bundle $\pi : \widetilde{\mathbb{C}}^n = E \longrightarrow \mathbb{C}P^{n-1}$ and we set $E_* = E \setminus \{\text{zero section}\} = \mathbb{C}^n \setminus \{0\}$. We remark that E_* is a holomorphic principal $\mathbb{C}_*$-bundle over $\mathbb{C}P^{n-1}$.

Let now $\mu : \mathbb{C}^n \to \mathbb{R}_{\geq 0}$ be the Minkowski function of a smoothly bounded, complete circular domain $D \subset \mathbb{C}^n$ and let $\widetilde{\mu} = \mu \circ \pi : E_* \to \mathbb{R}_{\geq 0}$. It is quite direct to realize that $\tilde{\mu}^2$ is the quadratic form of an Hermitian metric $h^{(\mu)}$ on $\pi : E \to \mathbb{C}P^{n-1}$ and that the distribution $\mathcal{H}$, defined by

$$\mathcal{H}_u = \{ \ v \in T_u E^* \ : \ d\widetilde{\mu}_u(v) = d\widetilde{\mu}_u(J_o v) = 0 \ \} \ , \tag{4.3}$$

is a connection on the principal $\mathbb{C}_*$-bundle E_*. The associated curvature 2-form $\tilde{\omega}^{(\mu)}$ is a $\mathbb{C}_*$-invariant, horizontal 2-form on E_* and projects down onto a closed, 2-form $\omega^{(\mu)}$ on $\mathbb{C}P^{n-1}$. A direct computation shows that, for any two Minkowski functions μ, μ', the associated 2-forms $\omega^{(\mu)}$ and $\omega^{(\mu')}$ are cohomologous (see e.g. [26], Thm. 3.3).

From the definition, it is clear that, for any hypersurface $S(c) = \{ \ \mu = cost. \ \}$, the restriction $\mathcal{H}|_{S(c)}$ coincides with the real distribution underlying the CR structure of $S(c)$. When D is strongly pseudo-convex, the function $\tau = \mu^2$ is a parabolic exhaustive function for D and hence $\mathcal{H}|_D$ coincides with the normal distribution of the Monge-Ampère foliation of D determined by τ. Moreover, being each hypersurface $S(c)$ strongly pseudo-convex, it is simple to check that the associated 2-form $\omega^{(\mu)}$ is a Kähler form.

In the following, μ_o is the Minkowski function of B^n, i.e. $\mu_o = |\cdot|$, $\widetilde{\mu}_o = \mu_o \circ \pi$ and $\mathcal{H}_o$ is the corresponding connection on E_* as defined in (4.3).

Lemma 4.5. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded, strongly pseudoconvex circular domain with Minkowski function μ. Set $\widetilde{\mu} = \mu \circ \pi$ and let $\mathcal{H}$ be the corresponding connection on E_* defined in (4.3). Then, there exists a diffeomorphism $\phi : \widetilde{\mathbb{C}}^n \longrightarrow \widetilde{\mathbb{C}}^n$ with the following properties:*

- i) *it is a fiber preserving map for the bundle $\pi : \widetilde{\mathbb{C}}^n = E \longrightarrow \mathbb{C}P^{n-1}$ which is holomorphic on any fiber;*
- ii) $\widetilde{\mu}_o = \widetilde{\mu} \circ \phi$*;*
- iii) $\phi_*(\mathcal{H}_o) = \mathcal{H}$*.*

Moreover, if $\{\mu^{(t)} \ , \ 0 \leq t \leq 1\}$ is a smooth 1-parameter family of Minkowski functions of smoothly bounded, strongly pseudoconvex circular domains, then it is possible to choose a family of diffeomorphisms $\phi^{(t)} : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n$, satisfying i) - iii) for $\mu = \mu^{(t)}$, which depends smoothly on t.

Proof. Let ω_o and ω be the Kähler forms on $\mathbb{C}P^n$ determined by the curvatures on E_* of the connections $\mathcal{H}_o$ and $\mathcal{H}$, respectively. Since ω_o and ω are cohomologous, by Moser's theorem [18] there exists a diffeomorphism $\psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ such that

$$\psi^* \omega = \omega_o \ . \tag{4.4}$$

Indeed, by the proof of Moser's theorem, there exists a smooth 1-parameter family of diffeomorphisms ψ_t, $t \in [0,1]$, such that $\psi_0 = Id_{\mathbb{C}P^{n-1}}$, $\psi_1 = \psi$. Such family of diffeomorphism is obtained by integrating a family of vector fields $X_t = \dot{\psi}_t$ satisfying a particular system of differential equations. Consider the vector fields $\hat{X}_t$ on $E_* \simeq \mathbb{C}^n \setminus \{0\}$, which are horizontal w.r.t. $\mathcal{H}_o$ and project onto the vector fields X_t. Since any space of the distribution $\mathcal{H}_o$ is tangent to some spheres in $E_* = \mathbb{C}^n \setminus \{0\}$, it is possible to integrate such vector fields and obtain another 1-parameter family of diffeomorphisms $\hat{\psi}_t : E_* \to E_*$ so that $\hat{X}_t = \dot{\hat{\psi}}_t$. We then define

$$\tilde{\psi}_t : E \to E \ , \qquad \tilde{\psi}_t\,([v], z) = \begin{cases} (\psi_t([v]), \hat{\psi}_t(z)) & \text{if } z \neq 0 \\ \\ (\psi_t([v]), 0) & \text{if } z = 0 \end{cases} \ ,$$

which can be easily checked to be a family of diffeomorphisms and we set $\widetilde{\psi} \overset{\text{def}}{=} \hat{\psi}_1$. By construction, $\widetilde{\psi} : E \to E$ commutes with the action of $\mathbb{C}$ on E and restricts to ψ on $\mathbb{C}P^{n-1}$. Consider now a diffeomorphism $\phi : E \to E$ of the form

$$\phi([v], z) \overset{\text{def}}{=} \left(\psi([v]), e^{i\lambda([v])} \frac{|z| \cdot \hat{\psi}(z)}{\mu(\hat{\psi}(z))} \right) \tag{4.5}$$

where we denote by $\lambda : \mathbb{C}P^{n-1} \to \mathbb{R}$ a smooth function to be fixed later. It is clear that a map of the form (4.5) satisfies (i). But it satisfies also (ii), since

$$\widetilde{\mu}\,(\phi([v], z)) = \mu \left(e^{i\lambda([v])} \frac{|z| \cdot \hat{\psi}(z)}{\mu(\hat{\psi}(z))} \right) = |z| = \widetilde{\mu}_o([v], z) \ ,$$

and we claim that there exists a λ so that it satisfies also (iii). To check this, observe that, since $\phi|_{E_*}$ commutes with the action of $\mathbb{C}_*$, it maps the connection $\mathcal{H}_o$ into a connection $\mathcal{H}'$ on E_* whose curvature 2-forms projects down on $\mathbb{C}P^{n-1}$ onto the $\mathbb{C}$-valued 2-form

$$\omega' = \psi_* \omega_o = \omega \ . \tag{4.6}$$

Moreover, by (ii), all spaces of $\mathcal{H}'$ are tangent to the hypersurfaces $\{\ \mu = cost.\ \}$.

By standard arguments of theory of connections, we obtain that, for any $u \in E_*$ there exists a linear map

$$\nu_u : T_{\pi(u)} \mathbb{C}P^{n-1} \longrightarrow \mathbb{R} \tag{4.7}$$

so that any space $\mathcal{H}'_u$ is of the form

$$\mathcal{H}'_u = \left\{ \ w = v + \nu_u(\pi_*(v)) \cdot \left(i \left.\frac{\partial}{\partial \zeta}\right|_u - i \left.\frac{\partial}{\partial \bar{\zeta}}\right|_u \right) \ \text{for some } v \in \mathcal{H}_u \ \right\} \tag{4.8}$$

(here we denoted by ζ the standard coordinate of $\mathbb{C}$ and by $\frac{\partial}{\partial \zeta}$ the vertical holomorphic vector field on E induced by the holomorphic action of $\mathbb{C}$ on E). Equivalently, we may say that if $\varpi : TE_* \to \mathbb{C}_*$ is the connection form for $\mathcal{H}$, then the connection form ϖ' of $\mathcal{H}'$ is

$$\varpi' = \varpi - i\nu \ . \tag{4.9}$$

By the invariance of $\mathcal{H}'$ and $\mathcal{H}$ under the $\mathbb{C}_*$-action, the linear map ν_u depends only on the point $x = \pi(u) \in \mathbb{C}P^{n-1}$ and we may consider ν as a 1-form on $\mathbb{C}P^n$. Computing the curvature, we get from (4.9) and (4.6) that

$$d\nu = i(\omega' - \omega) = 0 \ . \tag{4.10}$$

Since $H^1(\mathbb{C}P^{n-1}) = 0$, there exists a smooth function $\widetilde{\lambda} : \mathbb{C}^n \to \mathbb{R}$ such that $\nu = d\widetilde{\lambda}$. Now, let us replace the function λ in (4.5) with the function $\lambda - \widetilde{\lambda} \circ \psi^{-1}$.

By construction, in (4.8) the function ν_u has to be replaced by the function $\widetilde{\nu}_u = \nu_u - d\widetilde{\lambda}\Big|_{\pi(u)} = 0$ and the new map ϕ satisfies (iii).

It remains to prove the final part of the statement. First of all, notice that by the first part of the proof, the function $\lambda : \mathbb{C}P^{n-1} \to \mathbb{R}$ is uniquely determined (up to a constant) by the diffeomorphism $\psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ which satisfies (4.4). By choosing some suitable normalization condition for λ, we may assume that the map ϕ is uniquely determined by ψ. If $\mu^{(t)}$ is a smooth family of Minkowski functions of strongly pseudoconvex, complete circular domains, then also the corresponding Kähler forms $\omega^{(t)}$ are smoothly depending on t (it suffices to see the explicit expression of $\omega^{(t)}$ in term of $\mu^{(t)}$ - see e.g. [26], p.27). Now, the proof of Moser's theorem in [18] shows that there exists a smooth family of diffeomorphisms $\psi^{(t)}$, $t \in [0,1]$, each of them satisfying $\psi^{(t)*}\omega^{(t)} = \omega_o$. This automatically implies the existence of a smooth family $\phi^{(t)}$ satisfying (i) - (iii) for any t. □

We may now conclude the proof of the theorem. Given a smoothly bounded, strongly pseudoconvex circular domain $I \subset \mathbb{C}^n$ with Minkowski function μ, let $\Phi = \phi^{-1}|_{\widetilde{I}} : \widetilde{I} \to \widetilde{B}^n$, where ϕ is diffeomorphism of the previous lemma, and let $J' = \Phi_*(J_o)$. From (i) - (iii) of the lemma, it follows immediately that J is of Lempert type and that $\widetilde{\tau} = \widetilde{\tau}_o \circ \Phi$ with $\tau = \mu^2$. Moreover, we may consider the smooth 1-parameter family of Minkowski functions $\mu^{(t)} = (1-t)\mu + t\mu_o$ with $t \in [0,1]$, a corresponding family of diffeomorphisms $\Phi_t = \phi_t^{-1}$ with ϕ_t associated by the lemma with $\mu^{(t)}$ and smoothly depending on t, and the 1-parameter family of complex functions $J_t \overset{\text{def}}{=} \Phi_{t*}(J_o)$. By construction, J_t is an L-isotopy between J' and J_o and we may conclude that Φ is a normalizing map for $(I, J_o, \tau = \mu^2)$, as needed.

Assume now that $\Phi, \hat{\Phi} : \widetilde{M} \to \widetilde{B}^n$ are two normalizing maps so that $(\Phi'^{-1} \circ \Phi)|_{\pi^{-1}} = Id_{\pi^{-1}}$ or, equivalently, that $(\Phi' \circ \Phi^{-1})|_{\mathbb{C}P^{n-1}} = Id|_{\mathbb{C}P^{n-1}}$. Now, Φ and Φ' map the leaves of the Monge-Ampère foliation of (M, J, τ) into the leaves of the Monge-Ampère foliation of $(B^n, \Phi_*(J), \tau_o)$ and of $(B^n, \Phi'_*(J), \tau_o)$, which are in both cases the images of the standard radial discs. This means that $\Phi' \circ \Phi^{-1}$ maps biholomorphically any standard radial disc into itself, since $(\Phi' \circ \Phi^{-1})|_{\mathbb{C}P^n} = Id|_{\mathbb{C}P^{n-1}}$. If in addition $(\Phi'^{-1} \circ \Phi)_*$ induces the identity map on any tangent space of a leaf of the Monge-Ampère foliation at the points of $\pi^{-1}(x_o)$, we get that $\Phi' \circ \Phi^{-1}$ maps any radial disc into itself by a biholomorphism which fixes the origin and with derivative equal to 1 at 0. By Schwarz lemma, $\Phi' \circ \Phi^{-1} = Id$ on any radial disc and hence on the whole B^n.

It remains to check the smooth extendibility up the boundary of Φ if τ is smoothly extendible. But this is a consequence of the fact that Φ is obtained by composing the inverse of the circular representation (which is smoothly extendible to the boundary because of Proposition 3.1) and the diffeomorphism between $\widetilde{I} \subset \widetilde{T}_{x_o}M = \widetilde{\mathbb{C}}^n$ and $\widetilde{B}^n$, which is given in Lemma 4.5 and which is trivially smooth up to the boundary. □

We remark that one can prove a stronger statement about the uniqueness of normalizing maps. We will come back on this topic in §6.

5. Bland and Duchamp's invariants

In this section, we consider only manifolds of circular type in normal form, i.e. of the form (B^n, J, τ_o) with $\tau_o = |\cdot|^2$ and J complex structure on $\widetilde{B}^n$ of Lempert type and L-isotopic to the standard one. The distribution in $\widetilde{B}^n$, which is normal to the standard Monge-Ampère foliation, will be denoted $\mathcal{H}$. We recall that, for any sphere $S(c) = \{ \tau_o = c^2 \}$, the restriction $\mathcal{H}|_{S(c)}$ is the distribution underlying the standard CR structure of $S(c)$ and that, for any $x \in \widetilde{B}^n$,

$$T_x\widetilde{B}^n = \mathcal{Z}_x \oplus \mathcal{H}_x$$

where $\mathcal{Z}$ is the distribution tangent to the standard radial disc. We also denote by Z the vector field defined in (2.10): in the standard coordinates of $\mathbb{C}^n$, the corresponding holomorphic and anti-holomorphic parts of Z are

$$Z^{1,0} = \frac{1}{2}\left(Z - iJ_oZ\right) = z^i\frac{\partial}{\partial z^i}\ , \qquad Z^{0,1} = \bar{z}^i\frac{\partial}{\partial \bar{z}^i}\ . \tag{5.1}$$

Recall also that any complex structure J of Lempert type is uniquely determined by its action on the vector fields on $\mathcal{H}$, since the action on the vector fields in $\mathcal{Z}$ is the same of the standard complex structure J_o.

Let $H^{1,0}$ and $H^{0,1} = \overline{H^{1,0}}$ be the J_o-holomorphic and J_o-anti-holomorphic subbundles of $\mathcal{H}^{\mathbb{C}}$. For any other complex structure J of Lempert type, we denote the corresponding J-holomorphic and J-antiholomorphic subbundles of $\mathcal{H}^{\mathbb{C}}$ by $H^{1,0}_J$ and $H^{0,1}_J = \overline{H^{1,0}_J}$.

Definition 5.1. Let J be a complex structure on $\widetilde{B}^n$ of Lempert type. We call *deformation tensor associated with* J any smooth section

$$\phi : \widetilde{B}^n \to \bigcup_{x \in \widetilde{B}^n} \mathrm{Hom}(H^{0,1}, H^{1,0}) = H^{0,1*} \otimes H^{1,0}$$

so that $H^{0,1}$ can be expressed as

$$H^{0,1}_J|_x = \{\ w + \phi_x(w)\ ,\ w \in H^{0,1}|_x\ \} \qquad \text{for any } x \in \widetilde{B}^n\ . \tag{5.2}$$

Notice that, a priori, not any complex structure of Lempert type has an associated deformation tensor. However if J has an associated deformation tensor, then any sufficiently small deformation J' of J, which is also of Lempert type, has an associated deformation tensor. Indeed, we will shortly see that *any complex structure J of Lempert type and L-isotopic to J_o admits an associated deformation tensor*.

We now want to exhibit some differential equations which characterize the deformation tensors. In order to do this, we first need to recall the definition of two important operators on the tensor fields in $H^{0,1*} \otimes H^{1,0}$.

We recall that $\widetilde{B}^n \subset \widetilde{\mathbb{C}}^n$ is a holomorphic bundle $\hat{\pi} : \widetilde{B}^n \to \mathbb{C}P^{n-1}$, with fibers given by the radial discs (if endowed with the standard complex structure). Since $\mathcal{H}$ is a connection in such a bundle, the holomorphic and anti-holomorphic distributions are generated by vector fields $X^{1,0} \in H^{1,0}$ and $Y^{0,1} \in \mathcal{H}^{0,1}$ that can be locally chosen so that $\hat{\pi}_*\left([X^{1,0}, Y^{0,1}]\right) = [\hat{\pi}_*(X^{1,0}), \hat{\pi}_*(Y^{0,1})] = 0$. Let us call such vector fields *holomorphic* and *anti-holomorphic* vector fields of the distribution $\mathcal{H}^{\mathbb{C}}$, respectively. It can be easily checked that if ϕ is a deformation tensor associated with a complex structure of Lempert type, then for any two anti-holomorphic vector fields $X, Y \in H^{0,1}$

$$[X, \phi(Y)] \in H^{1,0} + \mathcal{Z}^{\mathbb{C}} \ . \tag{5.3}$$

Hence, if we denote by $(\cdot)_{\mathcal{H}^{\mathbb{C}}}$ the projection onto the distribution $\mathcal{H}^{\mathbb{C}}$, we surely have that $[X, \phi(Y)]_{\mathcal{H}^{\mathbb{C}}} \in H^{1,0}$ for any pair of anti-holomorphic vector fields. Now, consider the following two operators (see [13]):

$$\bar{\partial}_b : H^{0,1*} \otimes H^{1,0} \to \Lambda^2 H^{0,1*} \otimes H^{1,0} \ ,$$

$$\bar{\partial}_b \alpha(X, Y) \overset{\text{def}}{=} [X, \alpha(Y)]_{\mathcal{H}^{\mathbb{C}}} - [Y, \alpha(X)]_{\mathcal{H}^{\mathbb{C}}} - \alpha([X, Y]) \ , \tag{5.4}$$

and

$$[\cdot, \cdot] : \left(H^{0,1*} \otimes H^{1,0}\right) \times \left(H^{0,1*} \otimes H^{1,0}\right) \longrightarrow \Lambda^2 H^{0,1*} \otimes H^{1,0} \ ,$$

$$[\alpha, \beta](X, Y) \overset{\text{def}}{=} \frac{1}{2}\left([\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\right) \tag{5.5}$$

for any pair of anti-holomorphic vector fields X, Y in $H^{0,1}$.

Proposition 5.2. *Let J be a complex structure on $\widetilde{B}^n$ of Lempert type that admits an associated deformation tensor ϕ. Then:*

i) $dd^c\tau_o(\phi(X), Y) + dd^c\tau(X, \phi(Y)) = 0$ *for anti-holomorphic* $X, Y \in H^{0,1}$*;*
ii) $\bar{\partial}_b \phi + \frac{1}{2}[\phi, \phi] = 0$*;*
iii) $\mathcal{L}_{Z^{0,1}}(\phi) = 0$.

Conversely, any tensor field $\phi \in H^{0,1*} \otimes H^{1,0}$ that satisfies (i) - (iii) is the deformation tensor of a complex structure of Lempert type.

In addition, a complex structure J of Lempert type, associated with a deformation tensor ϕ, is so that (B^n, J, τ_o) is a manifold of circular type if and only if

iv) $dd^c\tau_o(\phi(X), \overline{\phi(X)}) < dd^c\tau_o(\bar{X}, X)$ *for any* $0 \neq X \in H^{0,1}$.

Proof. First of all, recall that by the J_o-invariance of the 2-form $dd^c\tau_o$, for any two vector fields in $H^{0,1} \oplus \mathcal{Z}^{0,1}$ or in $H^{1,0} \oplus \mathcal{Z}^{1,0}$,

$$dd^c\tau_o(W, W') = dd^c\tau_o(J_oW, J_oW') = -dd^c\tau_o(W, W') = 0 \ .$$

So, from the proof of Theorem 4.2, the reader can check that a complex structure J of Lempert type is so that (B^n, J, τ_o) is a manifold of circular type, if and only if for any $0 \neq X \in H^{0,1}$

$$dd^c\tau_o(\bar{X} + \overline{\phi(X)}, X + \phi(X)) = dd^c\tau_o(\bar{X}, X) + dd^c\tau_o(\overline{\phi(X)}, \phi(X)) > 0 \ .$$

This proves (iv). For checking the necessity and sufficiency of (i) - (iii), we only need to show that those properties are necessary and sufficient condition for the integrability of the unique almost complex structure J, which coincides with J_o on the radial discs, leaves the distribution $\mathcal{H}$ invariant and have an associated anti-holomorphic distribution $H^{0,1}_J$ which is as in (5.2). Such almost complex structure J is integrable if and only if for any anti-holomorphic vector fields $X, Y \in H^{0,1}$ one has

$$[X + \phi(X), Y + \phi(Y)] \in \mathcal{Z}^{0,1} + H^{0,1}_J \ , \qquad [Z^{0,1}, X + \phi(X)] \in \mathcal{Z}^{0,1} + H^{0,1}_J \ . \tag{5.6}$$

But conditions (5.6) are satisfied if and only if

$$[X + \phi(X), Y + \phi(Y)]_{\mathcal{H}^{\mathbb{C}}} = [X, Y] + \phi([X, Y]) \ \Leftrightarrow \ \bar{\partial}_b\phi(X, Y) + \frac{1}{2}[\phi, \phi](X, Y) = 0 \ , \tag{5.7}$$

$$[X + \phi(X), Y + \phi(Y)]_{\mathcal{Z}^{\mathbb{C}}} = 0 \ \Leftrightarrow \ dd^c\tau_o(X + \phi(X), Y + \phi(Y)) = 0 \ , \tag{5.8}$$

$$[Z^{0,1}, X + \phi(X)] = [Z^{0,1}, X] + \phi([Z^{0,1}, X]) \ \Leftrightarrow\ \mathcal{L}_{Z^{0,1}}\phi(X) = 0 \tag{5.9}$$

for any anti-holomorphic X, $Y \in H^{0,1}$, i.e. if and only if (i) - (iii) are true. □

Let J be a complex structure of Lempert type and J_t, $t \in [0,1]$, an L-isotopy between J and J_o. By the previous remark, the set of t's, for which J_t has an associated deformation tensor is open, while (iv) of the previous lemma implies that it is also closed. From this, we conclude that also $J = J_1$ has a deformation tensor and hence that *there is a natural injective map between the class of manifolds in normal form* (B^n, J, τ_o) *and the class of tensor fields* $\phi \in H^{0,1*} \otimes H^{1,0}$ *on* $\widetilde{B}^n$ *which satisfy (i) - (iv) of Proposition 5.2* .

The correspondence between normal forms and deformation tensors satisfying (i) - (iv) is a priori only injective, not surjective. However, for any deformation tensor satisfying (i) - (iv), the associated complex structure J defines a manifold of circular type and hence there exists some normalizing map $\Phi : \widetilde{B}^n \to \widetilde{B}^n$ for which $\hat{J} = \Phi_*(J)$ is in normal form and whose associated deformation tensor is $\hat{\phi} = \Phi_*(\phi)$. In other words, we may say that *any deformation tensor satisfying (i) - (iv) is, up to a diffeomorphism, the deformation tensor of some normal form.*

Proposition 5.2 (iii) has also the following consequence. Consider $n-1$ holomorphic vector fields $(e_1, \dots, e_{n-1})$, defined on some open subset of $\mathbb{C}P^{n-1} \subset \widetilde{B}^n$ and linearly independent at all points where they are defined. Extend them to $\widetilde{B}^n$ as Z and JZ invariant vector fields on taking values in $H^{1,0}$. Let also $(e^1, \dots, e^{n-1},\ Z^{1,0*})$ the holomorphic field of $(1,0)$-forms, which is dual to the frame field $(e_1, \dots, e_{n-1},\ Z^{1,0})$. Then any tensor field $\phi \in H^{0,1*} \otimes H^{1,0}$ is of the form $\phi = \sum_{a,b=1}^{n-1} \phi^a_b \overline{e^b} \otimes e_a$ and satisfies Proposition 5.2 (iii) if and only if the restrictions of the functions ϕ^a_b on the radial discs are holomorphic. In particular, using a system of coordinates $(v^1, \dots, v^{n-1}, \zeta)$ for $\widetilde{B}^n$ as in (2.5), we have that ϕ satisfies (iii) if and only if it is of the form

$$\phi = \sum_{j=0}^{\infty} \phi_j \zeta^j \ , \qquad \phi_j = \sum_{a,b=1}^{n-1} \phi^a_{bj} \overline{e^b} \otimes e_a$$

where $\phi^a_{bj} = \phi^a_{bj}(v^1, \dots, v^{n-1})$ are the coefficients of the series expansion in powers of ζ of the functions $\phi^a_b(v^1, \dots, v^{n-1}, \zeta)$. It can be checked that the deformation tensors $\phi^{(k)} \overset{\text{def}}{=} \phi_k \zeta^k \in H^{0,1*} \otimes H^{1,0}$ are independent on the choice of the coordinates and of the frame field $(e_1, \dots, e_{n-1})$. Moreover, ϕ satisfies (i) and (iii) of Proposition 5.2 if and only each tensor field $\phi^{(i)}$ satisfies (i) and that the following equations:

$$\bar{\partial}_b \phi^{(k)} + \frac{1}{2} \sum_{i+j=k} [\phi^{(i)}, \phi^{(j)}] = 0 \qquad \text{for any } 0 \le k < \infty\ . \tag{5.10}$$

Summarizing, we have the following theorem, which can be considered as an extension to arbitrary manifolds of circular type some of the main results in [5, 7] (see next remark).

Theorem 5.3. *Let D be a manifold of circular type in normal form, i.e. $D = (B^n, J, \tau_o)$, with J complex structure of Lempert type and L-isotopic to J_o. Then J is uniquely determined by an associated sequence of deformation tensors* $\phi^{(k)} \in$

$H^{0,1*} \otimes H^{1,0}$, $0 \leq k < \infty$, which satisfy (5.10) and (i) of Proposition 5.2 for any k, and with the series $\phi = \sum_k \phi^{(k)}$ uniformly converging on compacta.

In [5, 7], Bland and Duchamp considered small deformations of the standard CR structure of the S^{2n-1} and proved that, for $n > 1$, any such CR structure is embeddable in $\mathbb{C}^{n-1}$ as boundary of a domain which is biholomorphic to a domain in normal form (B^n, J, τ_o). In particular, they associated with any small deformation of the CR structure of S^{2n-1} a sequence of tensors, which correspond to the restrictions to S^{2n-1} of the tensors $\phi^{(k)}$ appearing in Theorem 5.3 [2]. It is therefore natural to name such sequence of deformation tensors $\phi^{(k)}$ the *Bland and Duchamp invariants of* (B^n, J, τ_o).

We conclude with the following concept, which will turn out to be quite useful in the applications of the next section.

Definition 5.4. Let (D, J, τ) be a domain of circular type. We say that D is *stable* if it admits a parabolic exhaustive function τ, whose associated circular representation $\Psi : \widetilde{I} \to D$ extends smoothly up to the boundary inducing a diffeomorphism between $\partial\widetilde{I}$ and ∂D.

Unless differently stated, for any given stable domain D, we will call *parabolic exhaustive functions of D* only those, whose associated circular representation satisfies the above condition.

We also recall that, by Lempert's and the first author's result (see e.g. [14, 23]), the class of stable domains of circular type naturally includes the smoothly bounded, strictly convex domains of $\mathbb{C}^n$ and the smoothly bounded, strongly pseudoconvex circular domains. Indeed, by Thm. 4.4 in [24], the complete circular domains may be characterized as the unique (stable) domains of circular type domains whose Monge-Ampère foliation is holomorphic.

Moreover, the "stability" property is invariant under biholomorphisms between domains of circular type. In fact:

Lemma 5.5. *Let (D, J, τ) and $(\hat{D}, \hat{J}, \hat{\tau})$ be two biholomorphic domains of circular type. Then D is stable if and only if $\hat{D}$ is stable. Furthermore they have a normal form (B^n, J, τ_o), with a complex structure J which is smoothly extendible up to the boundary and makes $\overline{B^n}$ a stable domain of circular type.*

Proof. The first claim follows from the fact that any biholomorphism $f : D \to \hat{D}$ between stable domains extends smoothly up to the boundary. This property can be checked using the local regularity results of Berteloot ([4], Prop. 3), which imply that any such f admits an Hölder continuous extension up the boundary. In fact, from Hölder boundary regularity, the standard arguments of Lempert's proof of Fefferman theorem (see [14, 30]) imply that f extends smoothly up to the boundary.

The last claim is a consequence of the proof of Theorem 4.4. In fact, if D is stable, we may construct a normalizing map which is smooth up to the boundary and induces a diffeomorphism between ∂D and $S^{2n-1} = \partial B^n$. In particular, the complex structure of $\widetilde{D}$, which extends up to ∂D, is mapped onto a complex structure on $\widetilde{B}^n$, which extends smoothly up to ∂B^n (and hence also to a small neighborhood

[2] Be aware that our deformation tensor is minus the deformation tensor considered in [5, 7].

of $\overline{B^n}$). This implies that the associated normal form (B^n, J, τ_o) is a stable domain of circular type since the circular representation of (B^n, J, τ_o) coincides with the one of (B^n, J_o, τ_o). $\square$

From the previous lemma, it is clear that the normal forms of stable domains of circular type are associated with sequences of Bland and Duchamp's invariants $\{\phi^{(k)}\}$ which converge uniformly on the closure $\overline{\widetilde{B}^n}$.

6. Miscellaneous results

6.1. **The geometrical meaning of the Bland and Duchamp invariant $\phi^{(0)}$.**

Let (B^n, J, τ_o) be a manifold in normal form, $\phi^{(k)}$ the associated Bland and Duchamp invariants, $I \subset T_0B^n \simeq \mathbb{C}^n$ the indicatrix at 0 and $\Psi : \widetilde{I} \to \widetilde{B}^n$ the circular representation. The identification $T_0B^n \simeq \mathbb{C}^n$ is done so that we may assume $J|_{T_0B^n} = J_o$. The following proposition collects a few properties which give a clear indication of the information carried by the Bland and Duchamp invariant $\phi^{(0)}$.

Proposition 6.1.

a) *The pull-backed complex structure $J' = \Psi^*(J)$ on $\widetilde{I}$ is of Lempert type.*
b) *The tensor field $\phi - \phi^{(0)} = \sum_{k\geq 1}^{\infty} \phi^{(k)}$ is identically vanishing if and only if the circular representation is a biholomorphism between I and (B^n, J, τ_o), i.e. if and only if (B^n, J) is biholomorphic to a circular domain.*
c) *The invariant $\phi^{(0)}$ is always the deformation tensor of a manifold in normal form $(B^n, J^{(0)}, \tau_o)$, more precisely, of a normal form of the indicatrix I.*

Proof. a) is a direct consequence of definitions and Proposition 3.1. For b), we remark that $\phi^{(k)} = 0$ for all $k \geq 1$ if and only if the projection $\pi : \widetilde{B}^n \to \mathbb{C}P^{n-1}$ is a J-holomorphic, i.e. if and only if the Monge-Ampère foliation of (B^n, J, τ_o) is holomorphic. Then (b) follows from [24], Prop. 3.4.

For c), notice that $\phi^{(0)}$ satisfies (i) - (iv) of Proposition 5.2 and defines a complex structure which is L-isotopic to J_o, because ϕ does it. So, by the remarks after Proposition 5.2, the first claim follows immediately. Now, consider the circular representation $\Psi : \widetilde{I} \to \widetilde{B}^n$. It is straightforward to realize that $J^{(0)}|_{\mathbb{C}P^n} = \Psi_*(J_o|_{\mathbb{C}P^n})$ and hence, by invariance along the leaves of the Monge-Ampère foliations, $J^{(0)} = \Psi_*(J_o)$ on $\widetilde{I}$. This implies that the corresponding projected structures on I and on B^n are biholomorphic. $\square$

6.2. **The parameterization of normalizing maps and the automorphism group of a manifold of circular type.**

6.2.1. *Special frames of a manifold of circular type.*

Definition 6.2. Let τ be a parabolic exhaustion function for M and x_o and $I_{x_o} \subset T_{x_o}M \simeq \mathbb{C}^n$ the corresponding center and indicatrix. We call *special frame at* x_o *associated with* τ a complex basis $(e_0, e_1, \ldots, e_{n-1})$ for $T_{x_o}M$ defined as follows:

i) $e_0 \in \partial I_{x_o}$ (i.e. $\kappa(e_0) = 1$, where κ is the Minkowski function of I_{x_o});
ii) $(e_1, \ldots, e_{n-1})$ is a unitary basis w.r.t. $dd^c\kappa^2$ for the holomorphic tangent space $\mathcal{D}^{1,0}_{e_0} \subset T_{e_0}\partial I_{x_o}$ of the CR structure of $\partial I_{x_o} \subset T_{x_o}M \simeq \mathbb{C}^n$.

Recall that if D is a *domain* of circular type, for any center x_o there is a unique parabolic exhaustion function $\tau : D \to [0, 1)$, smoothly extendible at the boundary, for which the center is exactly x_o (see Lemma 2.2). For this reason, for any such domain the following set is well defined

$$P = \bigcup_{x_o \text{is a center}} P_{x_o} \ , \qquad \text{where} \qquad P_{x_o} = \{ \text{ special frames at } x_o \ \} \ .$$

We will call it the *pseudo-bundle of special frames of* D. We will also denote by $\pi : P \to D$ the map which associates to any special frame the base point and $\mathfrak{C}(D) = \pi(P) \subset D$ will denote the set of centers of D.

It should be observed that, relatively to the biholomorphisms that are smooth up to the boundary (in case of stable domains, any biholomorphism is in such a class), the pseudo-bundle P is a biholomorphic invariant of D. In case $\mathfrak{C}(D)$ is discrete, the pseudo-bundle P is a bundle over such a set. Any fiber $P_{x_o} = \pi^{-1}(x_o)$ has a structure of U_{n-1}-principal bundle with basis equal to the boundary of the indicatrix ∂I_{x_o}. In case D is a smoothly bounded, strictly convex domain in $\mathbb{C}^n$ (and hence $D = \mathfrak{C}(D)$), P is a bundle over D. More precisely, it coincides with *unitary frame bundle of the complex Finsler metric* given by the Kobayashi metric of D ([27]).

Notice also that, via Gram-Schmidt ortonormalization, any special frame of a manifold in normal form is uniquely associated with a basis which is unitary w.r.t. the standard Poincarè-Bergman metric of the unit ball $B^n \subset \mathbb{C}^n$. Using such correspondence, it is possible to embed the pseudo-bundle P into the bundle $\mathrm{U}_n(B^n)$ of the unitary frames of B^n. Since $\mathrm{Aut}(B_n, J_o)$ acts transitively and freely on $\mathrm{U}_n(B^n)$, $\mathrm{U}_n(B^n)$ can be identified with $\mathrm{Aut}(B_n, J_o)$ and the previous immersion $P \hookrightarrow \mathrm{U}_n(B^n)$ can be considered as an immersion $P \hookrightarrow \mathrm{Aut}(B_n, J_o)$. In case $\mathfrak{C}(D) = D$, such immersion is actually a diffeomorphism (see e.g. [27, 28]).

6.2.2. *Parameterization of normalizing maps by means of special frames.*

Consider a manifold of circular type (M, J, τ) of dimension n and denote by $\widetilde{\mathcal{N}(M)}$ the class of all normalizing maps $\Phi : \widetilde{M} \to \widetilde{B}^n$ between a blow up of M at a center x_o and the blow up of B^n at the origin. A priori, $\widetilde{\mathcal{N}(M)}$ is a very large class of maps: In fact, for any $\Phi \in \widetilde{\mathcal{N}(M)}$, if $\psi : \widetilde{B}^n \to \widetilde{B}^n$ is a diffeomorphism, which is a fiber bundle automorphism of $\pi : \widetilde{B}^n \to \mathbb{C}P^{n-1}$ and preserves the complex structure on the fibers, the composition $\psi \circ \Phi$ is still in $\widetilde{\mathcal{N}(M)}$. This means that, for any given τ and x_o, the class $\widetilde{\mathcal{N}(M)}$ contains a family of maps of cardinality larger or equal to the cardinality of $\mathit{Diff}(\mathbb{C}P^{n-1})$.

On the other hand, any normalizing map $\Phi : \widetilde{M} \to \widetilde{B}^n$ induces a complex structure on $\widetilde{B}^n$, which in turn projects down onto a complex manifold structure on B^n (see Definition 4.1). In general, the charts of two distinct complex manifold structures on B^n of this kind do not smoothly overlap, i.e. *belong to two distinct real manifolds structures*, even if they are surely diffeomorphic and they coincide with the standard manifold structure of $\mathbb{C}^n$ when restricted to $B^n \setminus \{0\}$.

For this reason, in the following, given a manifold M of circular type, we fix *once and for all* one of such real manifold structures and we will denote by $\mathcal{N}(M) \subset \widetilde{\mathcal{N}}(M)$ the class of normalizing maps which induce on B^n that real manifold structure. In other words, a normalizing map $\Phi : \widetilde{M} \to \widetilde{B}^n$ belongs to $\mathcal{N}(M)$ if and only if the corresponding projected map $\phi : M \to B^n$ is a diffeomorphism w.r.t. to the real manifold structure of M and the fixed manifold structure on B^n. Finally, for any parabolic exhaustive function τ on M and corresponding center x_o, we denote by $\mathcal{N}(M, \tau, x_o)$ the subclass of $\mathcal{N}(M)$ consisting of normalizing maps which are associated with τ and x_o.

Lemma 6.3. *Let $\Phi, \Phi' \in \mathcal{N}(M, \tau, x_o)$ and denote by $\phi, \phi' : M \to B^n$ the corresponding projected maps. Then $\Phi = \Phi'$ if and only if $\left(\phi' \circ \phi^{-1}\right)_*\big|_{T_0 B^n} = Id$.*

Proof. The necessity is immediate. For the sufficiency, notice that, from definitions, the hypothesis implies that the lifted maps $\Phi, \Phi' : \widetilde{M} \to \widetilde{B}^n$ are so that $\Phi' \circ \Phi^{-1}$ is the identity when restricted to the exceptional divisor $\pi^{-1}(0) = \mathbb{C}P^{n-1}$ and $\left(\Phi' \circ \Phi^{-1}\right)_*$ induces the identity map on each tangent space of a standard radial disc at the intersection with $\mathbb{C}P^{n-1} = \pi^{-1}(0)$. So, $Id_{\widetilde{B}^n}$ and $\Phi' \circ \Phi^{-1}$ are normalizing maps for B^n that satisfy the hypothesis of Theorem 4.4. Hence $\Phi' \circ \Phi^{-1} = Id_{\widetilde{B}^n}$.□

With the help of the previous lemma, we can prove the following.

Proposition 6.4. *Fix a normalizing map $\Phi_o \in \mathcal{N}(M, \tau, x_o)$ and a special frame $(e^o_0, e^o_1, \dots, e^o_{n-1})$ at $x_o \in M$, associated with τ. Also, for any $\Phi \in \mathcal{N}(M, \tau, x_o)$ denote by $\phi : M \to B^n$ the corresponding projected map and let (e^ϕ_i) be the frame at $T_{x_o}M$ defined by $e^\phi_i = \left(\phi^{-1} \circ \phi_o\right)_* (e^o_i)$.*

Then the frames of the form (e^ϕ_i) are special frames, relatively to x_o and τ, and the correspondence $\Phi \longrightarrow (e^\phi_i)$ is a one to one map between $\mathcal{N}(M, \tau, x_o)$ and the class of special frames at x_o.

Proof. By definition of normalizing maps, the map $f = \phi \circ \phi_o^{-1}$ is a biholomorphism of (M, J) into itself, fixing x_o and so that $\tau \circ f = \tau$. This implies that $f_*|_{T_{x_o}M}$ maps the class of special frames at x_o into itself. Moreover, if Φ and Φ' are so that $(e^\phi_i) = (e^{\phi'}_i)$, it follows from definitions that $\left(\phi' \circ \phi^{-1}\right)_*\big|_{T_0 B^n} = Id$ and hence that $\Phi = \Phi'$ by the previous lemma. It remains to show that the correspondence $\Phi \longrightarrow (e^\phi_i)$ is surjective.

Let $J^{(o)}$ be the complex structure on B^n obtained by pushing forward the complex structure of M and denote by (f^o_i) and (f_i) the special frames of $(B^n, \tau_o, J^{(o)})$ obtained as images of (e^o_i) and of another special frame (e_i), respectively. Let also $([f^o_i])$ and $([f_i])$ the corresponding points in $\mathbb{C}P^{n-1} \subset \widetilde{B}^n$. Observe that the restriction $J^{(o)}\big|_{\mathbb{C}P^{n-1}}$ coincides with the complex structure $J^{(o,0)}$ on $\widetilde{B}^n$ defined by the Bland and Duchamp invariant $\phi^{(0)}$ (see §6.1) and it is diffeomorphic to the standard complex structure of $\mathbb{C}P^{n-1}$ (to see this, simply use the biholomorphism between $(\widetilde{B}^n, J^{(o,0)})$ and the blow up $\widetilde{I}$ of its indicatrix - see Proposition 6.1). Hence $\mathrm{Aut}(\mathbb{C}P^{n-1}, J^{(o)})$ is isomorphic to $\mathrm{Aut}(\mathbb{C}P^{n-1}) = \mathrm{PGL}_n(\mathbb{C})$ and there exists a unique $J^{(o)}$-biholomorphism $\Psi : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ mapping $[f_i]$ into $[f_i^o]$ for any $0 \leq i \leq n-1$. Let us extend such a map to a diffeomorphisms $\Psi : \widetilde{B}^n \to \widetilde{B}^n$ in such a way that $\zeta \cdot \Psi(w) = \Psi(\zeta w)$ for any $\zeta \in \Delta$. We stress the fact that even if $\Psi|_{\mathbb{C}P^{n-1}}$ is by construction a biholomorphism of $(\mathbb{C}P^{n-1}, J^{(o)}|_{\mathbb{C}P^{n-1}})$, Ψ is not in general a biholomorphism of $(\widetilde{B}^n, J^{(o)})$.

Now, observe that $\Psi : \widetilde{B}^n \to B^n$ is a normalizing map which maps the $[f_i]$'s into the $[f_i^o]$'s. Assume for the moment that Ψ projects onto a map $\psi : B^n \to B^n$ which preserves the real manifold structure of B^n. Then, $\psi_*(f_j) = e^{i\theta_j} f_j^o$ for some suitable complex numbers $e^{i\theta_j}$ and, by a suitable adjustment of the definition of Ψ, we may always assume that $e^{i\theta_j} = 1$ for any $0 \leq j \leq n-1$. From this and its construction, it follows that $\Phi = \Psi \circ \Phi_o$ is a normalizing map in $\mathcal{N}(M, \tau, x_o)$ so that $(e_i^\phi) = (e_i)$. This implies the surjectivity of the map $\Phi \longrightarrow (e_i^\phi)$.

So, in order to conclude, we only need to show that Ψ projects onto a map $\psi : B^n \to B^n$ which preserves the real manifold structure of B^n. This is equivalent to check that there is a chart on B^n on a neighborhood of the origin, which belongs to the real manifold structure of $(B^n, J^{(o)})$ and in which ψ is smooth. This fact can be done using the circular representation. In fact, we may use it in order to identify the differentiable manifolds $(B^n, J^{(o)})$ and $(\widetilde{B}^n, J^{(o)})$ with the indicatrix I and its blow up $\widetilde{I} \subset \widetilde{\mathbb{C}}^n$, respectively, both endowed with a suitable complex structure, say $\hat{J}^{(o)}$. Since the circular representation is a diffeomorphism between a domain in $\mathbb{C}^n$ and the manifold of circular type, we have that the real manifold structure on I determined by projection from the manifold structure of $\widetilde{I}$ is the same of the standard manifold structure of I, considered as open subset of $\mathbb{C}^n$. Writing the explicit expression Ψ as a map $\Psi : \widetilde{I} \to \widetilde{I}$ it can be checked directly that the projected map $\psi : I \to I$ is smooth in any system of standard coordinates of $\mathbb{C}^n$. $\square$

The main result of the section can be now immediately inferred.

Theorem 6.5. *Let (D, J, τ) be a stable domain of circular type and $\mathcal{N}(D)$ the class of all normalizing maps, which induce on B^n the same real manifold structure. Fix also a frame $(e_i^o) \subset T_xD$ belonging to the pseudo-bundle of special frames $\pi : P \to D$ and a normalizing map $\Phi_o \in \mathcal{N}(D)$. Then, the correspondence $\Phi \longrightarrow (e_i^\phi)$ of the previous proposition gives a continuous one to one map between $\mathcal{N}(M)$ and the pseudo-bundle of special frames P.*

We remark that if P is identified with a subset of $\mathrm{U}_n(B^n) \simeq \mathrm{Aut}(B^n)$, the previous result can be stated saying that: *$\mathcal{N}(D)$ is parameterized by a special subset of* $\mathrm{Aut}(B^n)$. In particular, we have that if the set of centers $\mathfrak{C}(D)$ is a singleton, then $\mathcal{N}(D)$ is parameterized by $\mathrm{U}_n = \mathrm{Aut}(B^n)_0$, while if $\mathfrak{C}(D) = D$, $\mathcal{N}(D)$ is parameterized by $\mathrm{Aut}(B^n)$. It is interesting to observe the analogy (and the difference) between this class of normalizing maps and the class of Chern-Moser normalizing maps for Levi non-degenerate hypersurfaces of $\mathbb{C}^n$.

6.2.3. *The automorphism group of a manifold of circular type.*

Let M be a circular domain of circular type. By means of a normalizing map, there is no loss of generality if we assume that M is (B^n, J, τ_o), where J is a complex structure of Lempert type, L-isotopic to the standard one. In this case, any automorphism $\Phi \in \mathrm{Aut}(M) = \mathrm{Aut}(B^n, J)$ is also a normalizing map and, if we set $\Phi_o = Id_{B^n}$, Theorem 6.5 implies that the action of $G = \mathrm{Aut}(M)$ on special frames determines a one to one map between G and the points of any orbits $G \cdot (e^o_i)$. In particular, if M is a stable domain of circular type and G is a Lie group, then G is diffeomorphic to any of its orbits $G \cdot u$ in the pseudo-bundle P. In such a case, if $\mathfrak{C}(M) \subset M$ is contained in a real submanifold of dimension a ($\leq 2n = \dim_{\mathbb{R}} M$), we get that

$$\dim G \leq a + n^2 \leq 2n + n^2 \ .$$

By the classical results of H. Cartan (see e.g. [1]), the property that G is a Lie group is granted whenever M is biholomorphic to a bounded complex domain in $\mathbb{C}^n$. The previous remark gives therefore a refinement of the well known upper bound $2n + n^2$ for $\dim G$.

We expect that $G = \mathrm{Aut}(M)$ is a Lie group for *any* stable domain (not necessarily embedded in $\mathbb{C}^n$) and hence that the above estimate is true in full generality: We plan to attack such a problem in a future paper.

6.3. **Characterizations of circular domains and of the unit ball.**

Definition 6.6. Let (D, J, τ) be a stable domain of circular type. An element $g \in G = \mathrm{Aut}(D, J)$ is called *rotational at the center* x_o if $g_*|_{x_o} = \lambda Id_{T_{x_o}D}$ for some $\lambda \in \mathbb{C}^n$ so that $\lambda^n \neq 1$ for any $n \in \mathbb{Z}$. We say that M *is rotational at* x_o if G contains a rotational element at x_o.

The following two theorems are extensions to the case of stable domains of a couple of results in [23].

Theorem 6.7. *A stable domain of circular type (D, J, τ) is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if it is rotational at some point $x_o \in D$.*

Proof. There is no loss of generality if we assume that D is in normal form, i.e. that $(D, J, \tau) = (B^n, J, \tau_o)$, and that $x_o = 0$. If g is rotational, the lifted map $\widetilde{g} : \widetilde{B}^n \to \widetilde{B}^n$ is so that $\widetilde{g}|_{\mathbb{C}P^{n-1}} = Id_{\mathbb{C}P^{n-1}}$. Moreover, by Lemma 2.2, $\tau_o \circ \widetilde{g} = \tau_o$ and hence $\widetilde{g}$ induces on any standard radial disc a biholomorphism that fixes the origin and with derivative at 0 equal to λ. This implies that $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and, in a system of coordinates as described before (2.5), $\widetilde{g}$ is of the form $\widetilde{g}(v, \zeta) = (v, e^{i\theta}\zeta)$. Moreover, since $\widetilde{g}$ is a J-biholomorphism and induces on $\mathbb{C}P^{n-1}$ the identity map, all Bland and Duchamp invariants do not change under the action of $\widetilde{g}$. In particular any invariant $\phi^{(k)}$, $k \geq 1$, is so that $\phi^{(k)} - e^{ik\theta}\phi^{(k)} = 0$. Since $\lambda^k = e^{ik\theta} \neq 1$ for all k, we get that $\phi^{(k)} = 0$ for any $k \geq 1$ and the conclusion follows from Proposition 6.1 b). □

Theorem 6.8. *A stable domain (D, J, τ) is biholomorphic to the unit ball B^n if and only if it is rotational at two distinct centers $x_o, x'_o \in D$.*

Proof. The necessity is trivial. For the sufficiency notice that, by the previous theorem, D is biholomorphic to a circular domain $I \subset \mathbb{C}^n$, which is rotational w.r.t. to two distinct points. The conclusion follows automatically from [23], Thm. 9.4. $\square$

In the following statement, for a given manifold of circular type (M, J, τ) with $\tau : M \to [0, r^2)$ and center x_o, for any $c \in (0, r^2)$, we denote by $D_{<c}$ the domain contained in M defined by $M = \{ \ x \in M \ , \ \tau(x) < c \ \}$. We remark that the indicatrix at x_o of (M, J, τ) coincides (up to rescaling) with the indicatrix at that point of $(D_{<c}, J, \tau)$. From this and Proposition 3.1, it can be directly checked that $(D_{<c}, J, \tau)$ is always a stable domain if $D_{<c} \subsetneq M$.

Theorem 6.9. *Let (M, J, τ) be a manifold of circular type of dimension n. Then M is biholomorphic to a circular domain in $\mathbb{C}^n$ if and only if for some given parabolic exhaustive function τ there exist two domains $D_{<c}$, $D_{<c'}$, $0 < c < c' < r^2$, that are biholomorphic one to the other by means of a map fixing the center of τ.*

Proof. The necessity is direct: If $M \subset \mathbb{C}^n$ is a circular domain with Minkowski function μ and we set $\tau = \mu^2$, then the map $f(z) = \frac{c'}{c} z$ determines a biholomorphism between $D_{<c}$ and $D_{c'}$ for any two c, c'.

To prove the sufficiency, we may assume that M is in normal form, i.e. $M = (B^n, J, \tau_o)$ and that the parabolic exhaustive function which defines the two biholomorphic stable domains $D_{<c}$ and $D_{<c'}$ is the function $\tau = \tau_o$. Notice also that the restriction on $D_{<c'}$ of the map

$$\psi : \widetilde{\mathbb{C}}^n \to \widetilde{\mathbb{C}}^n \qquad \psi([z], z) = ([z], \frac{1}{c'} z)$$

is a normalizing map for $D_{<c'}$ (i.e. maps $(D_{<c'}, J, \tau_o)$ into $(B^n, \tilde{J}, \tau_o)$) and maps $D_{<c}$ into D_k, $k = c/c'$.

It follows directly from definitions that the indicatrix I at $x_o = 0$ of $(B^n, \tilde{J}, \tau_o)$ is the same of the $(D_k, \tilde{J}, \tau_o)$ (up to rescaling) and hence all special frames of the first domain at 0 coincide with the special frames of the second domain up to multiplication by k. On the other hand, by hypothesis and Lemma 2.2, we have a biholomorphism of domains of circular type $f : (D_{<k}, \tilde{J}, \tau_o) \to (B^n, \tilde{J}, \tau_o)$ so that $f(0) = 0$. The differential $f_*|_0 : T_0 D_{<k} = T_0 B^n \to T_0 B^n$ is a $\mathbb{C}$-linear map mapping $k \cdot I$ into I and hence mapping a fixed special frame (e_i) into another special frame (e'_i) rescaled by the factor $1/k$. Let us denote by $\hat{f} : B^n \to B^n$ the unique normalizing map of $(B^n, \tilde{J}, \tau_o)$ which transforms (e_i) into (e'_i) (see Theorem 6.5). The lift at the blow up level of $f^{-1} \circ \hat{f}$ is a diffeomorphism between $\widetilde{B}^n$ and $\widetilde{D}_{<k} \subset \widetilde{B}^n$ which induces the identity map on $\mathbb{C}P^{n-1}$ and so that, when restricted to any radial disc, it is a holomorphic map (w.r.t. the standard complex structure) fixing the center, mapping the unit disc into the disc of radius k and with derivative equal to k at the origin. Therefore $f^{-1} \circ \hat{f}$ is of the form

$$(f^{-1} \circ \hat{f})([z], z) = \psi_k([z], z) \ , \quad \text{where} \quad \psi_k([z], z) \stackrel{\text{def}}{=} ([z], kz) \ .$$

The same argument can be repeated for any iterated map $f^n = f \circ \cdots \circ f$ and we obtain that, for any $n \in \mathbb{N}$, the map

$$\hat{f}^{(n)} = f^n \circ \psi_k^n$$

is a normalizing map of $(B^n, \tilde{J}, \tau_o)$, fixing the origin. Since the normalizing maps fixing the origin are continuously parameterized by the compact set of special frames at the origin ($\simeq \mathrm{U}_n$), we may consider a subsequence n_j so that the sequence of normalizing maps $\hat{f}^{(n_j)}$ converges uniformly on compacta to a normalizing map $\hat{f}^{(\infty)}$. In particular, the sequence of complex structures $\tilde{J}^{(n_j)} \overset{\text{def}}{=} \hat{f}^{(n_j)*}(\tilde{J})$ converges to a complex structure $\tilde{J}^{(\infty)} = \hat{f}^{(\infty)*}(\tilde{J})$. On the other hand, since f is a $(\tilde{J}, \tilde{J})$-biholomorphism, we have that

$$\tilde{J}^{(n_j)} = \hat{f}^{(n_j)*}(\tilde{J}) = \psi_k^{n_j *}(\tilde{J}) \ .$$

A direct computation shows that, for any point $([z], z) \in \widetilde{B}^n$, the deformation tensor $\phi^{(n_j)}|_{([z],z)}$ of $\psi_k^{n_j *}(\tilde{J})$ coincides with the deformation tensor ϕ of $\tilde{J}$, but evaluated at the point $([z], k^{n_j} z)$. It follows that

$$\phi^{(\infty)}|_{([z],z)} = \lim_{n_j \to \infty} \phi^{(n_j)}|_{([z],z)} = \lim_{n_j \to \infty} \phi|_{([z],k^{n_j} z)} = \phi|_{([z],0)} \ ,$$

i.e. the deformation tensor $\phi^{(\infty)}$ of $\widetilde{J}^{(\infty)}$ is independent on the coordinate of the radial discs. By Proposition 6.1, this means that the domains

$$(B^n, \widetilde{J}^{(\infty)}, \tau_o) \simeq (B^n, \widetilde{J}, \tau_o) \simeq (D_{<c'}, J, \tau_o)$$

are all circular. In particular, the restriction to $D_{<c'}$ of the deformation tensor of the manifold from which started, i.e. of (B^n, J, τ_o), is independent on the coordinates of the radial discs. By Proposition 5.2 (iii) (i.e. analyticity along the radial discs), we get independence on the coordinates of the radial discs over the entire (B^n, J, τ_o). Using once again Proposition 6.1, we get the claim, i.e. (B^n, J, τ_o) is circular. □

Remark 6.10. Notice that the proof of previous theorem implies that, if we assume that (M, J, τ) is a stable domain (in particular if it is a strictly convex domain in $\mathbb{C}^n$), the claim is true also when $c' = r^2$ and $D_{<c'} = M$.

By the previous theorem and remark and by Theorem 6.8, we have the following direct corollary.

Corollary 6.11. *A stable domain D is biholomorphic to the unit ball if and only it has at least two centers $x_o \neq x'_o$ and it is biholomorphic, by means of two maps fixing x_o and x'_o, respectively, to two proper subdomains $D_{<c} = \{ \tau < c \}$, $D'_{<c'} = \{ \tau' < c' \}$, with τ, τ' parabolic exhaustive functions associated with x_o and x'_o, respectively.*

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