THE PLURICOMPLEX POISSON KERNEL FOR STRONGLY CONVEX DOMAINS

FILIPPO BRACCI, GIORGIO PATRIZIO, STEFANO TRAPANI

CONTENTS

INTRODUCTION

In the past decades the study of pluri-potential theory and of its applications played a central role in complex analysis in several variables. In particular, since the basic work of Siciak [31] and Bedford and Taylor [7], [8] a great effort was made to understand the complex Monge-Ampère operator and the associated generalized Dirichlet problems (for instance, see [15], [20] and references therein).

Let $D \subset \mathbb{C}^n$ be a bounded convex domain with $z_0 \in D$. From the work of Lempert [21], [24] and Demailly [15] it turned out that the following homogeneous Monge-Ampère equation

$$(0.1)\qquad \begin{cases} u \in \mathsf{Psh}(D) \\ (\partial\overline{\partial}u)^n = 0 \quad \text{in } D \setminus \{z_0\} \\ \lim_{z\to x} u(z) = 0 \quad \text{for all } x \in \partial D \\ u(z) - \log|z - z_0| = O(1) \quad \text{as } z \to z_0 \end{cases}$$

has a solution L_{D,z_0} which is continuous in $D \setminus \{z_0\}$ (actually it is smooth there if D is strongly convex with smooth boundary) and unique.

The function L_{D,z_0} shares many properties with the Green function for the unit disc $\mathbb{D} \subset \mathbb{C}$. For instance, from an analytic point of view it can be used to reproduce continuous plurisubharmonic functions (see [15] or Section 8) while from a geometrical point of view, its level sets are boundaries of Kobayashi balls centered at z_0 and its associated foliation is the singular pencil of complex geodesics passing through z_0 and thus it can be successfully used in questions such as classification of domains or biholomorphisms (see, *e.g.*, [28], [29], [9]). Thus, such function deserves the name of *pluricomplex Green's function*.

In [11] the first and second named authors concentrated in studying a homogeneous Monge-Ampère equation with a simple singularity at the boundary. Namely, the following result has been proved:

Theorem 0.1. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. The following Monge-Ampère equation*

$$\begin{cases} u \in \mathsf{Psh}(D) \\ (\partial\overline{\partial} u)^n = 0 \quad \textit{in } D \\ u < 0 \quad \textit{in } D \\ u(z) = 0 \quad \textit{for } z \in \partial D \setminus \{p\} \\ u(z) \approx \|z-p\|^{-1} \quad \textit{as } z \to p \textit{ non-tangentially} \end{cases} \tag{0.2}$$

has a solution $\Omega_{D,p} \in C^\infty(\overline{D} \setminus \{p\})$ such that $d(\Omega_{D,p})_z \neq 0$ and $(\partial\overline{\partial}\Omega_{D,p})^{n-1}(z) \neq 0$ for all $z \in \overline{D} \setminus \{p\}$. Moreover the level sets of $\Omega_{D,p}$ are boundaries of horospheres of D with center p.

Here $\mathsf{Psh}(D)$ denotes the real cone of plurisubharmonic functions in D and horospheres are the "limits of Kobayashi balls" introduced by Abate [1], [2] and coincide with the sub-level sets of Busemann functions of geodesics whose closure contain p (see [12]). The function $\Omega_{D,p}$ has been defined by means of the boundary spherical representation of Chang-Hu-Lee [13] (see Section 1). In [11], among other things, it has been proved that $\Omega_{D,p}$ can be used to characterize biholomorphisms and that its associated foliation is the fibration of complex geodesics of D whose closure contain p.

The aim of this paper is to study the properties of $\Omega_{D,p}$ in depth. We will show that $\Omega_{D,p}$ shares many properties with the Poisson kernel for the unit disc $\mathbb{D}$ and therefore it deserves the name of *pluricomplex Poisson kernel* of D with singularity at $p \in \partial D$.

More in detail, we show that a version of the classical Phragmen-Lindelöf theorem on the growth of subharmonic functions in $\mathbb{D}$ holds for plurisubharmonic functions in D, proving that $\Omega_{D,p}$ is the maximal element of the family

$$\begin{cases} u \in \mathsf{Psh}(D) \\ \limsup_{z\to x} u(z) \leq 0 \quad \text{for all } x \in \partial D \setminus \{p\} \\ \liminf\limits_{t\to 1} |u(\gamma(t))(1-t)| \geq 2\mathsf{Re}\,(\langle \gamma'(1), \nu_p\rangle^{-1}), \end{cases}$$

where ν_p is the unit outward normal to ∂D at p and γ is any C^1-curve in D such that $\gamma(1) = p$ and $\gamma'(1) \notin T_p\partial D$ (see Section 5). In due course we will find the exact behavior of $\Omega_{D,p}(z)$ as z goes to p along non tangential directions to ∂D at p (see Corollary 5.3).

Next, we deal with uniqueness properties of $\Omega_{D,p}$. These are essentially of two types: analytic and geometric. From an analytic point of view we show that $\Omega_{D,p}$ is the only solution of the homogeneous Monge-Ampère equation which is zero on $\partial D \setminus \{p\}$ and behaves like $\Omega_{D,p}$ as z tends to p (see Theorem 7.1). This is the analogous of the uniqueness statement for the pluricomplex Green function, except that the behavior of $\Omega_{D,p}$ near p is universal only along non tangential directions, but it might depend on the domain D itself along other directions. From a geometrical point of view we show that $\Omega_{D,p}$ is the only C^2 solution (up to multiplication by constants) of the homogeneous Monge-Ampère equation which is zero on $\partial D \setminus \{p\}$ and whose associated foliation is the fibration of D in complex geodesics whose closure contain p (see Theorem 7.3). This fact is then used to show a couple of interesting other characterizations of $\Omega_{D,p}$ both in terms of its level sets (see Proposition 7.4) and in terms of its behavior under pull-back with holomorphic self-maps of D (see Proposition 7.5).

We also show in Theorem 6.1 that L_{D,z_0} and $\Omega_{D,p}$ have the same relationship as the Green function and the Poisson kernel in $\mathbb{D}$, namely

$$\Omega_{D,p}(z_0) = -\frac{\partial L_{D,z_0}}{\partial \nu_p}(p). \tag{0.3}$$

This is used to write explicitly the “noyaux de Poisson pluricomplexes canonique” of Demailly [15] and, applying his theory, to obtain a somewhat explicit reproducing formula for continuous plurisubharmonic functions of D in terms of L_{D,z_0} and $\Omega_{D,p}$ (see Theorem 8.2). In particular, for pluriharmonic functions $F \in C^0(\overline{D})$ we obtain the following formula which is the analogous of that for harmonic functions in the disc:

$$F(z) = \int_{p\in\partial D} |\Omega_{D,p}(z)|^n F(p)\omega_{\partial D}(p),$$

where $\omega_{\partial D}$ is a positive real $(2n-1)$-form on ∂D which depends only on D.

As a spin off result, using the properties of $\Omega_{D,p}$, we also prove that horospheres are (smooth and) strongly convex away from their center (see Theorem 4.1).

The proofs of the previous properties of $\Omega_{D,p}$ are based on a mix of different techniques. In particular we will make a strong use of families of complex geodesics and their regularity properties. Thus in Section 2 we deal with regularity for such families gathering some known but disperse information and proving the precise results needed for our arguments. In particular, using a suitable “attached analytic discs” approach, we prove (Theorem 2.1) that the set of complex geodesics in D is a finite dimensional closed submanifold in the open set of the complex Banach space $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ made of non-constant holomorphic attached discs whose first k-th derivatives extend α-Hölder continuous on $\partial\mathbb{D}$. This result, interesting on its own, allows to obtain stability and regularities properties for families of complex geodesics (Section 2) and for their *Lempert's projections*, that is, the holomorphic retractions of D with affine fibers onto complex geodesics introduced by Lempert in [21] which will play a fundamental role in our discussion (see Section 3).

The plan of the paper is as follows. In the first section we recall some preliminaries about complex geodesics, the boundary spherical representation of Chang, Hu and Lee [13] and the results in [11] as needed to make this work as self-contained as possible. In section two we deal with regularity for families of complex geodesics by studying their differential properties and, as a corollary of our construction, we recover with a different proof some stability results by Huang [18], [19]. In the third section we study Lempert's projections. We first show that holomorphic retractions on a given complex geodesics are not unique but Lempert's projections can be characterize as the unique retractions with affine fibers. Then we examine the variation of Lempert's projections with respect to boundary data and prove regularity. In section four we investigate the shape of horospheres. We prove that they are strongly convex away their center (where they are $C^{1,1}$) using Jacobi vector fields. In the fifth section we state and prove the Phragmen-Lindelöf theorem for strongly convex domains and we compute the limits of $\Omega_{D,p}$ along non complex-tangential directions. In the sixth section we prove (0.3) and in section seven we deal with uniqueness. Finally, in section eight we recall Demailly's theory for reproducing plurisubharmonic functions and find the explicit reproducing formulas using $\Omega_{D,p}$.

1. Preliminaries

Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary. A *complex geodesic* is a holomorphic map $\varphi : \mathbb{D} \to D$ which is an isometry between the Poincaré metric of $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and the Kobayashi distance k_D in D.

According to Lempert (see [21] and [1]), any complex geodesic extends smoothly to the boundary of the disc and $\varphi(\partial\mathbb{D}) \subset \partial D$. Moreover, given any two points $z, w \in \overline{D}$, $z \neq w$, there exists a complex geodesic $\varphi : \mathbb{D} \to D$ such that $z, w \in \varphi(\overline{\mathbb{D}})$. Such a geodesic is unique up to pre-composition with automorphisms of $\mathbb{D}$. Also, if $z \in \overline{D}$ and $v \in \mathbb{C}^n \setminus \{O\}$ (and $v \notin T_z\partial D$ if $z \in \partial D$) there exists a unique (still, up to pre-composition with automorphisms of $\mathbb{D}$) complex geodesic $\varphi : \mathbb{D} \to D$ such that $z \in \varphi(\overline{\mathbb{D}})$ and $\varphi(\overline{\mathbb{D}})$ is parallel to v (in case $z, w \in \partial D$ this follows from Abate [3] and Chang, Hu and Lee [13]). In case $z \in D$ and $w \in \overline{D}$, $w \neq z$, (*respectively* $v \in T_zD$) one can choose uniquely a complex geodesic $\varphi : \mathbb{D} \to D$ requiring that $\varphi(0) = z$ and $\varphi(t) = w$ for some $0 < t \leq 1$, with $t = 1$ if and only if $w \in \partial D$ (*respect.* $\varphi'(0) = tv$ for some $t > 0$). With an abuse of notation, when no risk of confusion arises, we call "complex geodesic" also the image of a complex geodesic $\varphi : \mathbb{D} \to D$.

If $\varphi : \mathbb{D} \to D$ is a complex geodesic then there exists a holomorphic map $\widetilde{\varphi} : \mathbb{D} \to \mathbb{C}^n$, called the *dual map* of φ, such that $\widetilde{\varphi}$ extends smoothly to $\partial\mathbb{D}$ and $\widetilde{\varphi}(e^{i\theta}) = e^{i\theta}\mu(e^{i\theta})\partial r_{\varphi(e^{i\theta})}$, with r being a defining function of D near $\varphi(\partial\mathbb{D})$ and $\mu > 0$ normalized so that

$$\widetilde{\varphi}(\zeta) \cdot \varphi'(\zeta) \equiv 1 \tag{1.1}$$

for all $\zeta \in \mathbb{D}$ (see [21]).

Let $\varphi : \mathbb{D} \to D$ be a complex geodesic. In [22] and [23] (see also Pang [26]) Lempert defines a biholomorphic change of coordinates $G : D \to D'$ which "linearizes" φ. Namely, he proves that G extends smoothly on ∂D, that $G \circ \varphi(\zeta) = (\zeta, 0, \ldots, 0)$ and $\widetilde{G \circ \varphi}(\zeta) = (1, 0, \ldots, 0)$. The domain $D' = G(D)$ is no longer convex in general but it is strictly linearly convex near $G(\varphi(\partial\mathbb{D}))$, namely, the real Hessian of any defining function of D' is positive on the complex tangent space at any point of $\partial D'$ near $G(\varphi(\partial\mathbb{D}))$. In the rest of the paper we will refer to such a G as the Lempert biholomorphism which linearizes φ.

Considering the foliation of all complex geodesics passing through a given point $z_0 \in D$, Lempert constructed a map $\Phi_{z_0} : D \to \mathbb{B}^n$, called *spherical representation* of D at z_0, which is defined by $\Phi_{z_0}(z) = \zeta\varphi_z'(0)/\|\varphi_z'(0)\| \in \mathbb{B}^n$ where $\varphi_z : \mathbb{D} \to D$ is a complex geodesic such that $\varphi_z(0) = z_0$, $\varphi_z(\zeta) = z$ for $z \neq z_0$ and $\Phi_{z_0}(z_0) = O$. The map Φ_{z_0} which is continuous on D, extends C^∞ on $\overline{D} \setminus \{z_0\}$. In his work [21] Lempert proved that $L_{D,z_0} := \log \|\Phi_{z_0}\|$ solves (0.1).

Similarly, considering all complex geodesics whose closure contain a given boundary point $p \in \partial D$, Chang, Hu and Lee (see [13, Theorem 3]) constructed a *boundary spherical representation.* For the reader convenience and since it will be useful later, we recall here the construction of Chang, Hu and Lee as needed for our aim. Let $p \in \partial D$ and let ν_p be the unit outward normal to ∂D at p. Denote

$$L_p := \{v \in \mathbb{C}^n | \|v\| = 1,\ \langle v, \nu_p \rangle > 0,\ iv \in T_p\partial D\}$$

and let $v \in L_p$. In what follows we will say that a complex geodesic $\varphi_v : \mathbb{D} \to D$ whose closure contains the point $p \in \partial D$ is in *the Chang-Hu-Lee normal parametrization* (with respect to $v \in L_p$) if $\varphi(1) = p$ and $\varphi'(1) = \langle v, \overline{\nu_p} \rangle v$ and $\mathsf{Im}\, \langle \varphi''(1), \overline{\nu_p} \rangle = 0$. In [13] Chang, Hu and Lee proved that for all $v \in L_p$ there exists a *unique* complex geodesic in *the Chang-Hu-Lee normal parametrization* with respect to v.

Up to rigid movements of $\mathbb{C}^n$, assume that $\nu_p = e_1 = (1, 0, \ldots, 0)$ and thus L_p reduces to $L_p = \{v = (v_1, \ldots, v_n) \in \mathbb{C}^n : \|v\| = 1, v_1 > 0\}$. For any $v \in L_p$ the map $\eta_v : \mathbb{D} \ni \zeta \mapsto e_1 + (\zeta - 1)v_1 v$ is a complex geodesic of $\mathbb{B}^n$, $\eta_v(1) = e_1$ and $\eta_v'(1) = v_1 v$. Then the *boundary spherical representation* $\Phi_p : D \to \mathbb{B}^n$ is defined as follows:

$$\Phi_p(z) = e_1 + (\zeta_z - 1)v_{z,1}v_z,$$

where $\zeta_z \in \mathbb{D}$ and $v_z \in L_p$ are the unique data such that $\varphi_{v_z}(\zeta_z) = z$. The map Φ_p is a smooth diffeomorphism whose inverse is $\Phi_p^{-1}(w) = \varphi_{v_w}(\zeta_w)$, where $\zeta_w \in \mathbb{D}$ and $v_w \in L_p$ are the unique data such that $w = \eta_{v_w}(\zeta_w)$. Moreover Φ_p, Φ_p^{-1} extend continuously up to the boundary and $\Phi_p(p) = e_1$. In particular it follows that Φ_p is holomorphic on all complex geodesics in D whose closure contain p and sends such complex geodesics to complex geodesics in $\mathbb{B}^n$ whose closure contain e_1.

Following Abate ([1], [2]) we define a *horosphere* $E_D(p, z_0, R)$ of center $p \in \partial D$, pole $z_0 \in D$ and radius $R > 0$ as

$$E_D(p, z_0, R) := \{z \in D : \lim_{w \to p}[k_D(z, w) - k_D(z_0, w)] < \frac{1}{2}\log R\}.$$

The limit in the definition of $E_D(p, z_0, R)$ exists since D is strongly convex and any such horosphere $E_D(p, z_0, R)$ is a sub-level set of the Busemann function of any geodesic whose closure contains p (see [34]).

In [11, Corollary 6.2] it was proved that Φ_p *maps horospheres of* D *centered at* p *onto horospheres of* $\mathbb{B}^n$ *centered at* e_1, which, since horospheres of $\mathbb{B}^n$ are complex ellipsoid, implies in particular that the boundaries of horospheres are smooth away from the center p.

Let $\Omega_{\mathbb{B}^n,e_1}(z) = -\frac{1-\|z\|^2}{|1-z_1|^2}$. The sub-level sets of $\Omega_{\mathbb{B}^n,e_1}$ corresponds to horospheres of $\mathbb{B}^n$ with center e_1 and pole O (see, *e.g.*, [1], [2]). In [11] we defined

$$\Omega_{D,p} := \Omega_{\mathbb{B}^n,e_1} \circ \Phi_p$$

and proved Theorem 0.1. For further use we notice that

$$E_D(p, \Phi_p^{-1}(O), R) = \{z \in D : \Omega_{D,p}(z) < -\frac{1}{R}\}.$$

Finally, let

$$P(\zeta) := \frac{1 - |\zeta|^2}{|1 - \zeta|^2}$$

be the Poisson kernel on $\mathbb{D} = \{|\zeta| < 1\}$. Recall that P is harmonic in $\mathbb{D}$, $\lim_{\zeta\to x} P(\zeta) = 0$ for $x \in \partial\mathbb{D} \setminus \{1\}$ and $\lim_{\mathbb{R}\ni r\to 1^-} P(r)(1 - r) = 2$. From the very definition it follows that for all $v \in L_p$

$$\Omega_{D,p}(\varphi_v(\zeta)) = -P(\zeta)/v_1^2. \tag{1.2}$$

2. Regularity for families of complex geodesics

In this section we state some results about regularity of families of complex geodesics in strongly convex domains which we need later. From these we also rediscovered some facts already known or implicitly contained in other papers such as [21], [22], [18], [19]). Our presentation owes much to the works [16], [35], [33].

In all this section D will be a bounded strongly convex domain of $\mathbb{C}^n$ with smooth boundary. Given $k \geq 2$ and $\alpha \in (1/2, 1)$ we denote by $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ the set of all holomorphic maps from $\mathbb{D}$ to $\mathbb{C}^N$ which extends C^k on $\overline{\mathbb{D}}$ and such that their k-th derivatives are α-Hölder on $\overline{\mathbb{D}}$ (a map $f : \overline{\mathbb{D}} \to \mathbb{C}^n$ is α-Hölder if there exists $C > 0$ such that $\|f(\zeta_0) - f(\zeta_1)\| \leq C|\zeta_0 - \zeta_1|^\alpha$ for all $\zeta_0, \zeta_1 \in \mathbb{D}$). The set $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ is a complex Banach space when endowed with the norm

$$\|f\|_{k+\alpha} = \sum_{j=1}^{k} \sup_{\zeta\in\partial\mathbb{D}} \|f^{(j)}(\zeta)\| + \sup_{\zeta_0,\zeta_1\in\overline{\mathbb{D}},\zeta_0\neq\zeta_1} \frac{\|f^{(k)}(\zeta_0) - f^{(k)}(\zeta_1)\|}{|\zeta_0 - \zeta_1|^\alpha}.$$

Let $\mathcal{G}$ be the set of complex geodesics from $\mathbb{D}$ to D. By Lempert's theory [21] it follows that $\mathcal{G} \subset \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$. Let also denote by $M \subset \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ the set of constants with value in ∂D. It is clear that M is a closed set in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$.

Theorem 2.1. *The set* $\mathcal{G}$ *is a closed submanifold of* $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n) \setminus M$ *of real dimension* $4n-1$.

Proof. Let $\{f_n\} \subset \mathcal{G}$ and assume that $f_n \to f$ in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$. Since the domain D is strongly (pseudo)convex then either $f(\mathbb{D}) \subset D$—and from the continuity of k_D it follows easily that $f \in \mathcal{G}$ as well—or $f \in M$. Thus $\mathcal{G}$ is closed in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n) \setminus M$. Let $f_0 \in \mathcal{G}$. We want to prove that $\mathcal{G}$ is a submanifold of $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ near f_0.

Let $G : D \to D' = G(D)$ be the Lempert biholomorphisms which linearizes f_0. Then $G \circ f_0(\zeta) = (\zeta, 0, \dots, 0)$ and the dual map $\widetilde{G \circ f_0}(\zeta) \equiv (1, 0, \dots, 0)$. Notice that G extends C^∞ up to ∂D. Thus we can extend (arbitrarily) $G|_{\partial D}$ to some C^∞ map, denoted by $\tilde{G}$, from $\mathbb{C}^n$ to $\mathbb{C}^n$. We have thus a morphism $\Lambda : \mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n) \to \mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n)$ given by $\Lambda(f) = \tilde{G} \circ f$. The morphism Λ is C^∞ and maps the set of complex geodesics of D onto the set of complex geodesics of D'. Assume for the moment that we proved that $\Lambda(\mathcal{G})$ is a finite dimensional submanifold of $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ near $G \circ f_0$, and thus a finite dimensional submanifold of $\mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n)$. Repeating the argument with G^{-1}, we find a C^∞ map $\Lambda' : \mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n) \to \mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n)$ such that $\Lambda \circ \Lambda'|_{\Lambda(\mathcal{G})} = \mathsf{Id}|_{\Lambda(\mathcal{G})}$. Thus $\Lambda'|_{\Lambda(\mathcal{G})}$ is an embedding with $d\Lambda'(T\Lambda(\mathcal{G}))$ finite dimensional, thus closed and complemented in $\mathcal{C}^{k+\alpha}(\partial\mathbb{D}, \mathbb{C}^n)$. Therefore $\mathcal{G} = \Lambda'(\Lambda(\mathcal{G}))$ is a finite dimensional submanifold (see, *e.g.*, [5]). We are then left to show that $\Lambda(\mathcal{G})$ is a finite dimensional submanifold.

Thus, we can assume from the beginning that $f_0(\zeta) = (\zeta, 0, \dots, 0)$ and $\tilde{f}_0(\zeta) = (1, 0, \dots, 0)$ in D—here however the domain D is no longer strongly convex, but it is strictly linearly convex near $f_0(\partial\mathbb{D})$. By the very definition of the dual map and by (1.1) it follows that if f is a complex geodesic of D close to f_0 in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$ then $\tilde{f}$ is close to $\tilde{f}_0$ in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$, where, with some abuse of notation, we identify the one form $\tilde{f}$ with the vector of its components.

Let $\mathbb{P}^{n-1}(\mathbb{C})$ be the space of complex hyperplanes passing through the origin O. Let $\Psi : \partial D \to \mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C})$ be defined by $\Psi(p) := (p, T_p^{\mathbb{C}}\partial D)$. Let $S = \Psi(\partial D)$. By the very definition $(f_0, [\tilde{f}_0])(\partial\mathbb{D}) \subset S$. Moreover, since ∂D is strongly pseudoconvex near $f_0(\partial D)$, then S is a compact maximal totally real submanifold of $\mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C})$ near $\Psi(f_0(\partial D))$ (see [36]).

Let $(z_1, \dots, z_n)$ be coordinates in $\mathbb{C}^n$ and let $[z_1 : \dots : z_n]$ be the corresponding homogeneous coordinates in $\mathbb{P}^{n-1}(\mathbb{C})$; that is, the point $[z_1 : \dots : z_n]$ corresponds to the hyperplane $\{v = (v_1, \dots, v_n) \in \mathbb{C}^n : \sum_{j=1}^n v_j \cdot z_j = 0\}$. Let $U_1 := \{[z] \in \mathbb{P}^{n-1}(\mathbb{C}) : z_1 \neq 0\}$ be the chart obtained by identifying $\mathbb{C}^{n-1}$ with U_1 via $(w_1, \dots, w_{n-1}) \to [1 : w_1 : \dots : w_{n-1}]$ and let $R : \mathbb{C}^n \times \mathbb{C}^{n-1} \to \mathbb{R}^{2n-1}$ be a defining function for $S \cap \mathbb{C}^n \times U_1$ (such a defining function can be easily defined starting from a global defining function of D in $\mathbb{C}^n$). Let us consider $\mathcal{Q} = \{F = (f, g) \in \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n \times \mathbb{C}^{n-1}) : R(f, g)|_{\partial\mathbb{D}} \equiv 0\}$. In other words, $F = (f, g) \in \mathcal{Q}$ if and only if $(f, [1 : g])(\partial\mathbb{D}) \subset S$. In particular $(f_0, 0) \in \mathcal{Q}$. Note that if f is a complex geodesic close to f_0 with dual map $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \in \mathbb{C} \times \mathbb{C}^{n-1}$ then $\min_{\zeta \in \overline{\mathbb{D}}} |\tilde{f}_1(\zeta)| > 0$ and therefore $(f, \tilde{f}_2/\tilde{f}_1) \in \mathcal{Q}$. Conversely, if $(f, g) \in \mathcal{Q}$ and (f, g) is close to $(f_0, 0)$ in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n \times \mathbb{C}^{n-1})$ then $|f_1'(\zeta) + \sum_{j=2}^n f_j'(\zeta) g_j(\zeta)| > 0$ for all $\zeta \in \overline{\mathbb{D}}$ and then f is a stationary disc in D with dual map $(1, g)/(f_1' + \sum_{j=2}^n f_j' g_j)$ that is, a complex geodesic. It should be remarked that in this argument one cannot refer directly to Lempert's theory because D is not strongly convex in general. However, since ∂D is strongly pseudoconvex near $f_0(\partial\mathbb{D})$ then for f close to f_0 in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n)$, one can use Pang's results to relate stationarity to extremality, see [26, Section 2].

The previous discussion shows that there exists a open neighborhood $W_0 \subset \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n \times \mathbb{C}^{n-1})$ of $(f_0, 0)$ such that $\pi_1 : \mathcal{Q} \cap W_0 \to \mathcal{G} \cap \pi_1(W_0)$ is bijective, where π_1 is the projection on the first factor, namely $\pi_1(f,g) := f$. The map $\pi_1|_{\mathcal{Q}\cap W_0}$ is clearly C^∞, and its inverse is C^∞ as well, being given by $f \mapsto (f, \tilde{f}_2/\tilde{f}_1)$ with $\tilde{f} = (\tilde{f}_1, \tilde{f}_2) \in \mathbb{C} \times \mathbb{C}^{n-1}$ the dual of f. Thus $d\pi_1|_{T\mathcal{Q}}$ is injective and its image is finite dimensional and hence closed and complemented in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n)$. Therefore, if we prove that $\mathcal{Q}$ is a finite dimensional submanifold of $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n \times \mathbb{C}^{n-1})$ near $(f_0, 0)$ then the claim on $\mathcal{G}$ will follow. To prove that $\mathcal{Q}$ is a submanifold by means of the implicit function theorem in Banach spaces, it is enough to show that $dR_{f_0} : \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n \times \mathbb{C}^{n-1}) \to \mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{R}^n)$ is surjective and its kernel complemented in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n \times \mathbb{C}^{n-1})$. We have $dR_{f_0}(f(\zeta)) = 2\mathsf{Re}\, Af(\zeta)$ with A being the $(2n-1)\times(2n-1)$ matrix with entries $\frac{\partial R_j}{\partial z_k}(f_0)$. Since S is maximal totally real, arguing as in [32, Theorem 3.1, Lemma 3.2] one can prove that all the Birkhoff partial indices of the operator $f \mapsto 2\mathsf{Re}\, Af(\zeta)$ are ≥ 1 and thus, by [16] (see also [35] and [33]) dR_{f_0} is surjective. Notice that the computation of Birkhoff partial indices in [32, Lemma 3.2] was proved under the assumption that ∂D is strongly convex. It is easy to check that in fact such result holds for strictly linearly convex domains and therefore it can be used here as, in Lempert's coordinates, the domain is strictly linearly convex near $f_0(\partial\mathbb{D})$. Finally, a direct computation (or see [34]) shows that its kernel has finite (real) dimension $4n-1$ and therefore $\mathcal{Q}$ is a submanifold of dimension $4n-1$ near f_0. □

Let κ_D be the Kobayashi metric in D. According to Lempert ([21], [23]) the map $D \times (\mathbb{C}^n \setminus \{O\}) \to \mathbb{R}$ given by $(z,v) \mapsto \kappa_D(z;v)$ is C^∞. Moreover, since $\kappa_D(z,\lambda v) = \lambda\kappa_D(z,v)$ for all $(z,v) \in D\times(\mathbb{C}^n\setminus\{O\})$ and $\lambda > 0$ it follows that $d(\kappa_D)_{(z,v)} \neq 0$ for all $(z,v) \in D\times(\mathbb{C}^n\setminus\{O\})$. Therefore the set

$$\mathcal{K} = \{(z,v) \in D \times (\mathbb{C}^n \setminus \{O\}) : \kappa_D(z;v) = 1\}$$

is a $(4n-1)$-real dimensional submanifold of $D \times (\mathbb{C}^n \setminus \{O\})$.

Theorem 2.2. *The map $V : \mathcal{G} \to \mathcal{K}$ defined by $V : f \mapsto (f(0), f'(0))$ is a diffeomorphism.*

Proof. By the uniqueness of complex geodesics [21], the map V is bijective. Since V is the restriction of a linear bounded map from $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n)$ to $\mathbb{C}^{2n}$ then it is linear and C^∞. By [23, Theorem 5] the inverse V^{-1} is C^∞ as well and hence V is a diffeomorphism. □

From this result we obtain some corollaries which will be useful later on.

Corollary 2.3. *Let $\{f_n\} \subset \mathcal{G}$ be such that $f_n \to f$ uniformly on compacta of $\mathbb{D}$. If f is not constant then $f \in \mathcal{G}$ and $f_n^{(j)} \to f^{(j)}$ uniformly on $\overline{\mathbb{D}}$ for all $j = 0, 1, \ldots$.*

Proof. By Theorem 2.1 if f is not constant then it belongs to $\mathcal{G}$. Thus, $f_n \to f$ uniformly on compacta of $\mathbb{D}$ implies that $f_n(0) \to f(0)$ and $f_n'(0) \to f'(0)$. By Theorem 2.2it follows that $f_n \to f$ in $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}},\mathbb{C}^n)$ for all fixed $k \in \mathbb{N}$. In particular $f_n^{(j)} \to f^{(j)}$ for all $j = 0, 1, \ldots$. □

Corollary 2.4. *It* $(0,1) \ni t \mapsto f_t \in \mathcal{G}$ *is a family of complex geodesics such that* $t \mapsto f_t(0)$ *and* $t \mapsto f_t'(0)$ *are* C^∞ *then* $t \mapsto f_t$ *is* C^∞ *in* $\mathcal{O}^{k+\alpha}(\overline{\mathbb{D}}, \mathbb{C}^n)$*. In particular the map* $\zeta \mapsto \frac{\partial^j f_t}{\partial t^j}(\zeta)$ *is smooth on* $\overline{\mathbb{D}}$ *for all* $j = 1, 2, \ldots$

Lemma 2.5. *The map* $\mathcal{G} \ni f \mapsto f(0) \in D$ *is proper.*

Proof. If $\{z_n\} \subset D$ is such that $z_n \to z \in D$ let $f_n \in \mathcal{G}$ be such that $f_n(0) = z_n$. Let $\{f_{n_k}\}$ be a converging subsequence. Since the Kobayashi distance is continuous on D it follows that the limit f of $\{f_{n_k}\}$ is not constant. Then by Corollary 2.3 it follows that $f \in \mathcal{G}$ and $f(0) = z$. Hence the map $f \mapsto f(0)$ is proper. □

As a straightforward corollary of Lemma 2.5 we have the following result, first proved with different methods by Huang [19, Proposition 1]:

Proposition 2.6. *Let* $c > 0$ *and let* $\mathcal{G}_c := \{f \in \mathcal{G} : dist(f(0), \partial D) \geq c\}$*. Then there exists* $c' > 0$ *such that* $\|f\|_{k+\alpha} \leq c'$*.*

3. Lempert's projections

Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $\varphi : \mathbb{D} \to D$ be a complex geodesic. According to Lempert ([21], [22], [23]), for all $z \in D$ the equation $\widetilde{\varphi}(\zeta) \cdot (z - \varphi(\zeta)) \equiv 0$ in the unknown $\zeta \in \mathbb{D}$ has a unique solution $\zeta := \tilde{\rho}(z)$. The map $\tilde{\rho} : D \to \mathbb{D}$ is holomorphic, extends smoothly on ∂D and it is called the *left inverse* of φ for it satisfies $\tilde{\rho} \circ \varphi = \mathsf{id}_{\mathbb{D}}$. By the very definition

$$\widetilde{\varphi}(\tilde{\rho}(z)) \cdot (z - \varphi(\tilde{\rho}(z))) \equiv 0. \tag{3.1}$$

Remark 3.1. Let $z \in \overline{D}$. If $\zeta \in \overline{\mathbb{D}}$ is such that $\widetilde{\varphi}(\zeta) \cdot (z - \varphi(\zeta)) = 0$ then $\widetilde{\rho}(z) = \zeta$. Indeed, by the strong convexity of ∂D, if $z \in \overline{D} \setminus \varphi(\partial\mathbb{D})$ then the winding number of the function $\partial\mathbb{D} \ni \zeta \mapsto \widetilde{\varphi}(\zeta) \cdot (z - \varphi(\zeta))$ is 1 (see [22], [23]) hence $\zeta = \widetilde{\rho}(z)$. On the other hand, if $z = \varphi(e^{it})$ for some $t \in \mathbb{R}$, by continuity of $\widetilde{\rho}$ it follows that $\widetilde{\rho}(\varphi(e^{it})) = e^{it}$. Suppose by contradiction that $\widetilde{\varphi}(\zeta) \cdot (\varphi(e^{it}) - \varphi(\zeta)) = 0$ for some $\zeta \in \overline{\mathbb{D}} \setminus \{e^{it}\}$. Since the domain is strongly convex the interior of the real segment ℓ joining $\varphi(e^{it})$ to $\varphi(\zeta)$ is contained in D. Then the segment ℓ belongs to the fiber of $\widetilde{\rho}$ at $\varphi(\zeta)$ and, since $\widetilde{\rho}$ is continuous on $\overline{D}$, it follows that $\widetilde{\rho}(\varphi(e^{it})) = \zeta$ which contradicts $\widetilde{\rho}(\varphi(e^{it})) = e^{it}$.

Let φ be a complex geodesic and let $\tilde{\rho}$ be its left-inverse. The map $\rho : D \to \varphi(\mathbb{D}) \subset D$ defined as $\rho := \varphi \circ \tilde{\rho}$ is a holomorphic retraction on $\varphi(\mathbb{D})$, *i.e.*, ρ is a holomorphic self-map of D such that $\rho \circ \rho = \rho$ and $\rho(z) = z$ for any $z \in \varphi(\mathbb{D})$. It extends smoothly to ∂D and it is called the *Lempert projection* associated to φ. The triple $(\varphi, \rho, \tilde{\rho})$ is the so-called *Lempert projection device*. As remarked for instance in [10, p. 145] the Lempert projection ρ depends only on the image $\varphi(\mathbb{D})$.

In this section we study regularity of Lempert's left-inverse. Before that, we make some comments on holomorphic retractions on strongly convex domains. We start with an example which shows that there exist infinitely many holomorphic retractions:

Example 3.2. Let $f_{jk} : \mathbb{B}^n \to \mathbb{D}$ be holomorphic functions, $j, k = 2, \dots, n$ and let $\epsilon < 1/2n$. The holomorphic map

$$\rho(z) := (z_1 + \epsilon \sum_{j,k=2}^{n} z_j z_k f_{jk}(z), 0, \dots, 0) \tag{3.2}$$

is a holomorphic retraction of $\mathbb{B}^n$ onto the complex geodesic $\varphi(\zeta) = (\zeta, 0, \dots, 0)$. Indeed, it is clear that $\rho(\mathbb{B}^n) \subset \mathbb{C} \times \{(0, \dots, 0)\}$, that $\rho^2 = \rho$ and that ρ is holomorphic. Moreover, if we let $r = |z_1|$ then $|z_j| \leq \sqrt{1 - r^2}$ and $|z_1 + \epsilon \sum_{j,k=2}^{n} z_j z_k f_{jk}(z)| \leq r + n\epsilon(1 - r^2)$, proving that for $\epsilon < 1/2n$ the image $\rho(\mathbb{B}^n) \subset \mathbb{B}^n$.

From (3.1) it follows that the fibers of Lempert's projection are intersections of D with complex affine hyperplanes. Lempert's projection can be characterized exactly by this property:

Proposition 3.3. *Let $\varphi : \mathbb{D} \to D$ be a complex geodesic. If $\rho : D \to \varphi(\mathbb{D})$ is a holomorphic retraction whose fibers are intersections of D with complex affine hyperplanes then ρ is the Lempert projection. In other words, the Lempert projection is the only "linear" retraction.*

Proof. Let $\rho : D \to \varphi(\mathbb{D})$ be a retraction whose fibers are intersection of D with complex affine hyperplanes. Let $E_D = E_D(\varphi(e^{it}), \varphi(\zeta_0), R)$ be a horosphere of D with radius $R > 0$. Since $\rho \circ \varphi = \varphi$, if $z \in E_D$ we have

$$\begin{aligned}
&\lim_{w \to \varphi(e^{it})} [k_D(\rho(z), w) - k_D(\varphi(\zeta_0), w)] \\
&\qquad = \lim_{r \to 1} [k_D(\rho(z), \rho(\varphi(re^{it}))) - k_D(\varphi(0), \varphi(re^{it}))] \\
&\qquad \leq \lim_{r \to 1} [k_D(z, \varphi(re^{it})) - k_D(\varphi(0), \varphi(re^{it}))] < \frac{1}{2} \log R.
\end{aligned} \tag{3.3}$$

Therefore $\rho(z) \in E_D \cap \varphi(\mathbb{D})$. Let $\eta \in \mathbb{D}$ and $\varphi(\eta) \in \partial E_D$. Let H be the affine hyperplane which contains $\rho^{-1}(\varphi(\eta))$. Then $E_D \cap H = \emptyset$, because if $z \in E_D \cap H$ then $\varphi(\eta) = \rho(z) \in E_D \cap \varphi(\mathbb{D})$, which is a contradiction. Since $\varphi(\eta) \in \overline{E_D} \cap H$ and E_D is convex, it follows that $H - \varphi(\eta) = T^{\mathbb{C}}_{\varphi(\eta)} \partial E_D$. Now, $T^{\mathbb{C}}_{\varphi(\eta)} \partial E_D = \ker(\partial \rho_L)_{\varphi(\eta)}$, where ρ_L is the Lempert projection. Thus ρ and ρ_L have the same fibers at $\varphi(\eta)$, and, by the arbitrariness of the choices it follows that $\rho = \rho_L$ as claimed. $\square$

Next we examine the variation of the left inverse of Lempert's projection with respect to boundary data.

Lemma 3.4. *Let $\{z_k\} \subset D$ be a sequence converging non-tangentially to p. Let $v_k \in L_p$ be such that $z_k \in \varphi_{v_k}(\mathbb{D})$ (where, for $v \in L_p$, $\varphi_v : \mathbb{D} \to D$ denotes the unique complex geodesic in the Chang-Hu-Lee normal parametrization with respect to v). If $v_{t_k} \to v_0$ then $v_0 \in L_p$ and $\varphi_{v_k} \to \varphi_{v_0}$, $\varphi^{(j)}_{v_k} \to \varphi^{(j)}_{v_0}$ uniformly on $\overline{\mathbb{D}}$ for all $j = 1, 2, \dots$.*

Proof. We can assume that $\nu_p = e_1$. To see that $v_0 \in L_p$ we need to show that $\langle v_0, e_1 \rangle > 0$. Assume this is not the case. Then $v_0 \in T^{\mathbb{C}}_p \partial \mathbb{D}$.

First of all, we claim that for any open neighborhood U of p it follows that $\varphi_{v_k}(\overline{\mathbb{D}}) \subset U$ eventually. Indeed, let $\Phi_p : D \to \mathbb{B}^n$ be the spherical representation of Chang-Hu-Lee and denote by $\eta_{v_k} := \Phi_p \circ \varphi_{v_k}$. By construction $\eta_{v_k}(\zeta) = e_1 + (\zeta - 1)\langle v_k, e_1\rangle v_k$ and thus $\eta_{v_k}(\overline{\mathbb{D}}) \to e_1$. Since Φ_p^{-1} is uniformly continuous on $\overline{\mathbb{B}^n}$ the claim follows.

Therefore, $\{\varphi_{v_k}(\overline{\mathbb{D}})\}$ converges to $\{p\}$ and, by [18, Theorem 2], given any $\epsilon > 0$ there exists k_0 such that, for all $k > k_0$, it follows that $\|(\varphi'_{v_k}(\zeta))_N\| \leq \epsilon \|(\varphi'_{v_k}(\zeta))_T\|$ for all $\zeta \in \mathbb{D}$ where, if $z \in D$ and $z' \in \partial D$ is the unique point of ∂D nearest to z, then, for all vectors $w \in T_p D = \mathbb{C}^n$ the vectors w_N and w_T denote the complex normal and the complex tangential components of w at z' (namely, $w_T \in T^{\mathbb{C}}_{z'}\partial D$ and $w_N = \langle w, \overline{\nu_{z'}}\rangle \nu_{z'}$ with $\nu_{z'}$ being the unit outward normal to ∂D at z').

Let $K \subset D$ be a cone with vertex p such that $\{z_k\} \subset K$. In particular, there exists $c > 0$ such that if $w \in \mathbb{C}^n$ and $(w - p) \in K$ then $\|w - p\|_N \geq c\|w - p\|_T$ (at p). Therefore, if $\gamma : [0, 1] \to D \cup \{p\}$ is a C^∞ curve such that $\gamma'(1) = p$ and $\|\gamma'(1)_N\| \leq (c/2)\|\gamma'(1)_T\|$ (at p), then $\gamma(t) \notin K$ for $t \approx 1$. Moreover, we can find a small open neighborhood U of p such that, if $\gamma([0, 1)) \subset U \cap D$ and $\|\gamma'(t)_N\| \leq (c/2)\|\gamma'(t)_T\|$ for all $t \in [0, 1]$ (here the projection is at the point of ∂D nearest to $\gamma(t)$) then $\gamma(t) \notin K$ for $t \in [0, 1)$).

Now, let k be such that $\varphi_{v_k}(\mathbb{D}) \subset U \cap D$ and $\|(\varphi'_{v_k}(\zeta))_N\| \leq (c/2)\|(\varphi'_{v_k}(\zeta))_T\|$ for all $\zeta \in \mathbb{D}$. Let $\theta_k \in \mathsf{Aut}(\mathbb{D})$ be an automorphism such that $\theta_k(1) = 1$ and $\varphi_{v_k}(\theta_k(0)) = z_k$. By the previous argument $\gamma(t) := \varphi_{v_k}(\theta_k(t))$ does not belong to K for any t, which contradicts the fact that $z_k \in K$. Thus $\langle v_0, e_1\rangle > 0$ and $v_0 \in L_p$.

We are left to show that $\varphi_{v_k} \to \varphi_{v_0}$ and $\varphi^{(j)}_{v_k} \to \varphi^{(j)}_{v_0}$ uniformly on $\overline{\mathbb{D}}$. Let $\eta_{v_k} := \Phi_p \circ \varphi_{v_k} : \mathbb{D} \to \mathbb{B}^n$. By the very definition $\eta_{v_k}(\zeta) = e_1 + (\zeta - 1)\langle v_k, e_1\rangle v_k$ and clearly $\eta_{v_k} \to \eta_{v_0}$ uniformly on $\overline{\mathbb{D}}$. Since Φ_p is a homeomorphism between $\overline{D}$ and $\overline{\mathbb{B}^n}$ it follows that $\varphi_{v_k} \to \varphi_{v_0}$ uniformly on $\overline{\mathbb{D}}$. By Corollary 2.3 then $\varphi^{(j)}_{v_k} \to \varphi^{(j)}_{v_0}$ uniformly on $\overline{\mathbb{D}}$. □

Lemma 3.5. *For any $v \in L_p$ denote by $\varphi_v : \mathbb{D} \to D$ the unique complex geodesic in the Chang-Hu-Lee normal parametrization with respect to v and let $\widetilde{\rho}_v$ be its left-inverse. Then, if $\{v_k\} \subset L_p$ is such that $v_k \to v_0 \in L_p$ it follows that $d\widetilde{\rho}_{v_k} \to d\widetilde{\rho}_{v_0}$ uniformly on $\overline{D}$.*

Proof. Differentiating (3.1) with respect to z_j we obtain for $z \in \overline{D}$

$$\frac{\partial \widetilde{\rho}_v}{\partial z_j}\widetilde{\varphi}'_v(\widetilde{\rho}_v(z)) \cdot (z - \varphi_v(\widetilde{\rho}_v(z))) + \widetilde{\varphi}_v(\widetilde{\rho}_v(z)) \cdot (e_j - \frac{\partial \widetilde{\rho}_v}{\partial z_j}\varphi'_v(\widetilde{\rho}_v(z))) \equiv 0,$$

holding for $z \in \overline{D}$. Taking into account that $\widetilde{\varphi}(\zeta) \cdot \varphi'(\zeta) \equiv 1$, we have

$$\frac{\partial \widetilde{\rho}_v}{\partial z_j}[\widetilde{\varphi}'_v(\widetilde{\rho}_v(z)) \cdot (z - \varphi_v(\widetilde{\rho}_v(z))) - 1] \equiv -\widetilde{\varphi}_v(\widetilde{\rho}_v(z)) \cdot e_j. \tag{3.4}$$

Notice that, since $\widetilde{\varphi}_v(\zeta) \neq 0$ for all $\zeta \in \overline{\mathbb{D}}$, for all $z \in \overline{D}$ there exists $j \in \{1, \ldots, n\}$ such that $\widetilde{\varphi}_v(\widetilde{\rho}_v(z)) \cdot e_j \neq 0$. In particular it follows that

$$\widetilde{\varphi}'_v(\widetilde{\rho}_v(z)) \cdot (z - \varphi_v(\widetilde{\rho}_v(z))) - 1 \neq 0$$

for all $z \in \overline{D}$. Therefore

$$\frac{\partial \widetilde{\rho}_v}{\partial z_j}(z) = \frac{-\widetilde{\varphi}_v(\widetilde{\rho}_v(z)) \cdot e_j}{\widetilde{\varphi}'_v(\widetilde{\rho}_v(z)) \cdot (z - \varphi_v(\widetilde{\rho}_v(z))) - 1}. \tag{3.5}$$

Let $\{v_k\} \subset L_p$ be such that $v_k \to v_0 \in L_p$. We claim that

$$\widetilde{\rho}_{v_k} \to \widetilde{\rho}_{v_0}, \quad \widetilde{\varphi}_{v_k} \to \widetilde{\varphi}_{v_0}, \quad \widetilde{\varphi}'_{v_k} \to \widetilde{\varphi}'_{v_0} \quad \varphi_{v_k} \to \varphi_{v_0}$$

uniformly on $\overline{D}$ and $\overline{\mathbb{D}}$ respectively. By Lemma 3.4 it follows that $\varphi_{v_k} \to \varphi_{v_0}$ uniformly on $\overline{\mathbb{D}}$.

As for $\widetilde{\varphi}_v$, if $v_k \to v_0$ in L_p then by Lemma 3.4 it follows that $\varphi_{v_k}^{(j)} \to \varphi_{v_0}^{(j)}$ uniformly on $\overline{\mathbb{D}}$ for all $j = 0, 1, 2, \ldots$. By the very definition and by (1.1), if r is a defining function for ∂D, it follows that for $\zeta \in \partial\mathbb{D}$

$$\widetilde{\varphi}_{v_k}(\zeta) = \frac{1}{\partial r_{\varphi_{v_k}(\zeta)}(\varphi'_{v_k}(\zeta))} \partial r_{\varphi_{v_k}(\zeta)} \tag{3.6}$$

and therefore, since $|\partial r_{\varphi_{v_k}(\zeta)}(\varphi'_{v_k}(\zeta))| \geq c > 0$ for all k, it follows that $\widetilde{\varphi}_{v_k} \to \widetilde{\varphi}_{v_0}$ uniformly on $\partial\mathbb{D}$. By the maximum principle then $\widetilde{\varphi}_{v_k} \to \widetilde{\varphi}_{v_0}$ uniformly on $\overline{\mathbb{D}}$. Differentiating (3.6) for $\zeta = e^{it}$ and $t \in \mathbb{R}$ by $\frac{d}{dt}$ we see that $\widetilde{\varphi}'_{v_k}$ is expressed as continuous combination of $\varphi_{v_k}, \varphi'_{v_k}, \varphi''_{v_k}$ and by Lemma 3.4 it follows then that $\widetilde{\varphi}'_{v_k} \to \widetilde{\varphi}'_{v_0}$ uniformly on $\overline{\mathbb{D}}$.

We are left to show that $\widetilde{\rho}_{v_k} \to \widetilde{\rho}_{v_0}$ uniformly on $\overline{D}$. If not, there exists a sequence $\{z_{k_m}\} \subset \overline{D}$ (which we may assume converging to some $z_0 \in \overline{D}$) such that $|\widetilde{\rho}_{v_{k_m}}(z_{k_m}) - \widetilde{\rho}_{v_0}(z_{k_m})| > \epsilon_0$ for some $\epsilon_0 > 0$ and for all k_m. By (3.1) it follows that for all k_m

$$\widetilde{\varphi}_{v_{k_m}}(\widetilde{\rho}_{v_{k_m}}(z_{k_m})) \cdot (z_{k_m} - \varphi_{v_{k_m}}(\widetilde{\rho}_{v_{k_m}}(z_{k_m}))) = 0.$$

Up to subsequences, we can assume that $\widetilde{\rho}_{v_{k_m}}(z_{k_m}) \to \zeta_0 \in \overline{\mathbb{D}}$. For what we already proved it follows then

$$\widetilde{\varphi}_{v_0}(\zeta_0) \cdot (z_0 - \varphi_{v_0}(\zeta_0)) = 0.$$

This implies that $\zeta_0 = \widetilde{\rho}_{v_0}(z_0)$, since the only zero of the function $\zeta \mapsto \widetilde{\varphi}_{v_0}(\zeta) \cdot (z_0 - \varphi_{v_0}(\zeta))$ is $\widetilde{\rho}_{v_0}(z_0)$ by Remark 3.1. But then both $\{\widetilde{\rho}_{v_{k_m}}(z_{k_m})\}$ and $\{\widetilde{\rho}_{v_0}(z_{k_m})\}$ converge to $\widetilde{\rho}_{v_0}(z_0)$, and then $|\widetilde{\rho}_{v_{k_m}}(z_{k_m}) - \widetilde{\rho}_{v_0}(z_{k_m})| \to 0$, contradiction. Thus $\widetilde{\rho}_{v_k} \to \widetilde{\rho}_{v_0}$ uniformly on $\overline{D}$ and the claim is proved.

Since, as we remarked at the beginning, the denominator of the right hand side of (3.5) for $v = v_0$ is never zero for all $z \in \overline{D}$, the previous claim implies that $d\widetilde{\rho}_{v_k} \to d\widetilde{\rho}_{v_0}$ uniformly on $\overline{D}$. □

Remark 3.6. By (3.5) it follows that $d(\widetilde{\rho}_v)_p = \widetilde{\varphi}(1)$ and by (3.6) we have (cfr. [1, Lemma 2.6.44]) for $w \in \mathbb{C}^n$

$$d(\widetilde{\rho}_v)_p(w) = \frac{\partial r_p(w)}{\partial r_p(\varphi'_v(1))} = \frac{\langle w, \nu_p\rangle}{\langle \varphi'_v(1), \nu_p\rangle}. \tag{3.7}$$

4. The shape of horospheres

Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain and let $p \in \partial D$. As we recalled in Section 1, for any $R > 0$ and $z_0 \in D$, the set $\partial E_D(p, z_0, R)$ is smooth away from its center $p \in \partial D$. It should be noted that smoothness of horospheres away from the center was known after [34, Section 4], but we do not know any previous reference for this fact.

In [1] (see also [4]) it is proved that horospheres are convex domains (since they are increasing union of Kobayashi balls of D). In [11, Remark 4.2], referring to [1, Corollary 2.6.49] it was claimed that (boundaries of) horospheres are strongly convex at their center. Unfortunately the proof of [1, Corollary 2.6.49] does not seem to show smoothness at the center and thus one can only infer that horospheres are geometrically strictly convex (*i.e.*, the intersection of their closure with the supporting hyperplane at the center is just the center). However from [2, p. 231-232] it follows that if $E_D(p, z_0, R) \subset D$ is a horosphere of center $p \in \partial D$ and radius $R > 0$ and $\mathbb{B} \subset D$ is a ball tangent to ∂D at p then there exists a horosphere $E_{\mathbb{B}}(p, R') \subset \mathbb{B}$ for some $R' > 0$ such that $E_{\mathbb{B}}(p, R') \subset E_D(p, z_0, R)$. In particular, since horospheres of the ball $\mathbb{B}$ are smooth complex ellipsoids, it follows that there exists a ball $\mathbb{B}' \subset E_D(p, z_0, R)$ tangent to $\partial E_D(p, z_0, R)$ at p. Namely, horospheres have the inner-ball property at the center. Therefore $\partial E_D(p, z_0, R)$ is $C^{1,1}$ at p (see, *e.g.*, [17, Proposition 2.4.3]).

We prove here that the boundaries of horospheres are strongly convex away from the center:

Theorem 4.1. *Let $D \subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $p \in \partial D$. Let $E_D(p, R)$ be a horosphere in D with center p and radius $R > 0$. The boundary $\partial E_D(p, R) \setminus \{p\}$ is smooth and strongly convex.*

Proof. Let $\Omega_{D,p}$ be the function defined in Theorem 0.1. Its level sets are boundaries of horospheres of D with center p. Thus, to show that such boundaries are strongly convex we need to prove that the (real) Hessian of $\Omega_{D,p}$ is positive definite on the tangent space of $\partial E_D(p, R)$ (for all $R > 0$). It is known (see, [1]) that $\partial E_D(p, R)$ are convex for all $R > 0$, (and strongly pseudoconvex for all $R > 0$ and strongly convex for big radii, see [11, Remark 7]). Thus the real Hessian of $\Omega_{D,p}$ is non-negative definite on the (real) tangent space of $\partial E_D(p, R)$ for all $R > 0$.

Let $q \in D$ and let $\varphi : \mathbb{D} \to D$ be a complex geodesic such that $\varphi(0) = q$ and $\varphi(1) = p$. Up to post-composing with automorphisms of $\mathbb{B}^n$ fixing e_1, we can suppose that $\Phi_p(q) = O$. Thus $\Phi_p(\varphi(\zeta)) = (\zeta, O)$. Let $F : D \to \mathbb{H}^n := \{(\zeta, w) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathsf{Im}\, \zeta > \|w\|^2\}$ be given by $F = C \circ \Phi_p$ where $C : \mathbb{B}^n \to \mathbb{H}^n$ is the Cayley transform defined as

$$C(\zeta, w) = (i\frac{1+\zeta}{1-\zeta}, \frac{w}{1-\zeta}), \quad (\zeta, w) \in \mathbb{C} \times \mathbb{C}^{n-1}.$$

We write $F(z) = (F_0(z), \tilde{F}(z)) \in \mathbb{C} \times \mathbb{C}^{n-1}$. By definition,

$$F_0(\varphi(\zeta)) = i\frac{1+\zeta}{1-\zeta}, \quad \tilde{F}(\varphi(\zeta)) \equiv O. \tag{4.1}$$

By the very definition of $\Omega_{D,p}$ (see Remark 1.2) it follows that $\Omega_{D,p}(F^{-1}(\zeta,w)) = \|w\|^2 - \mathsf{Im}\,\zeta$ for $(\zeta,w)\in\mathbb{C}\times\mathbb{C}^n$, $(\zeta,w)\in\mathbb{H}^n$. Therefore

$$\Omega_{D,p}(z) = \Omega_{D,p}(F^{-1}(F(z))) = \|\tilde F(z)\|^2 - \mathsf{Im}\,F_0(z). \tag{4.2}$$

Thus, from (4.1) and (4.2) we have for $v\in\mathbb{C}^n$

$$\mathsf{Hess}(\Omega_{D,p})_{\varphi(\zeta)}(v,v) = 2\|d\tilde F_{\varphi(\zeta)}(v)\|^2 - \mathsf{Hess}(\mathsf{Im}\,F_0)_{\varphi(\zeta)}(v,v) \tag{4.3}$$

where, for a real function f, $\mathsf{Hess}(f)_x$ denotes the real Hessian of f at x.

Now, let $r\in\mathbb{D}\cap\mathbb{R}$ and let $\theta_r\in\mathsf{Aut}(\mathbb{D})$ be such that $\theta_r(0)=r$ and $\theta_r(1)=1$ (notice that necessarily $\theta_r'(0)\in\mathbb{R}$). Let $\widetilde{\varphi\circ\theta_r}:\mathbb{D}\to D$ be the dual map of $\varphi\circ\theta_r$. from the very definition, a direct computation shows that

$$\widetilde{\varphi\circ\theta_r}(\zeta) = \frac{\widetilde\varphi(\theta_r(\zeta))}{\theta_r'(\zeta)}.$$

By [34, Lemma 4.1] (and since $\theta_r'(0)\in\mathbb{R}$) it follows that if $\varphi(r)\in\partial E_D(p,R(r))$ then

$$\begin{aligned} T_{\varphi(r)}(\partial E_D(p,R(r))) &= \{v\in\mathbb{C}^n : \mathsf{Re}\,(\widetilde{\varphi\circ\theta_r}(0)\cdot v)=0\} \\ &= \{v\in\mathbb{C}^n : \mathsf{Re}\,(\widetilde\varphi(r)\cdot v)=0\}. \end{aligned} \tag{4.4}$$

On the other hand, by (4.2) and (4.1) it follows that

$$T_{\varphi(r)}(\partial E_D(p,R(r))) = \ker d(\Omega_{D,p})_{\varphi(r)} = \ker d(\mathsf{Im}\,F_0)_{\varphi(r)}.$$

Thus, since they have the same kernel, the two (real) forms $v\mapsto\mathsf{Re}\,(\widetilde\varphi(r)\cdot v)$ and $v\mapsto\mathsf{Im}\,d(F_0)_{\varphi(r)}(v)$ are multiple each other. Since $\mathsf{Re}\,(\widetilde\varphi(r)\cdot\varphi'(r))=1$ by (1.1), and by (4.1)

$$\mathsf{Im}\,d(F_0)_{\varphi(r)}(\varphi'(r)) = \mathsf{Im}\,\frac{d}{d\xi}F_0(\varphi(\xi))|_{\xi=r} = \mathsf{Re}\,\frac{d}{d\xi}\frac{1+\xi}{1-\xi}|_{\xi=r} = \frac{2}{(1-r)^2},$$

it follows that for all $v\in\mathbb{C}^n$

$$d(\mathsf{Im}\,F_0)_{\varphi(r)}(v) = \frac{2\mathsf{Re}\,(\widetilde\varphi(r)\cdot v)}{(1-r)^2}. \tag{4.5}$$

Now, let $R>0$ be such that $q\in\partial E_D(p,R)$ and assume that $v\in T_q\partial E_D(p,R)$ verifies $\mathsf{Hess}(\Omega_{D,p})_q(v,v)=0$. We want to show that $v=0$.

Write $(\lambda,U)=(d(F_0)_q(v),d(\tilde F)_q(v))$. Since the map Φ_p transforms boundaries of horospheres onto boundaries of horospheres, it follows that the vector (λ,U) is tangent to the boundary of the horosphere $\{(\zeta,w)\in\mathbb{C}\times\mathbb{C}^{n-1} : \mathsf{Im}\,\zeta-\|w\|^2>1\}$ whose closure contains $(i,O)\in\mathbb{H}^n$. Thus $\lambda\in\mathbb{R}$.

Let us now consider the smooth one-parameter family of complex geodesics $g_t:\mathbb{D}\to\mathbb{H}^n$ depending smoothly on t given by

$$g_t(\zeta) := (i\frac{1+\zeta}{1-\zeta} + t\lambda + it^2\|U\|^2, tU)$$

and we denote

$$\dot{g}\,(\zeta) := \frac{\partial g_t(\zeta)}{\partial t}|_{t=0}, \quad \ddot{g}\,(\zeta) := \frac{\partial^2 g_t(\zeta)}{\partial^2 t}|_{t=0}.$$

Notice that $\dot{g}\,(\zeta) = (\lambda, U)$ and $\ddot{g}\,(\zeta) = (2i\|U\|^2, O)$ are independent of ζ. Let $f_t := F^{-1}(g_t)$. By construction $f_t : \mathbb{D} \to D$ is a smooth one-parameter family of complex geodesics, $f_t(1) = p$ and $f_0(0) = q$. Therefore $f_0 = \varphi$. Again, denoting by $\dot{f}\,(\zeta), \ddot{f}\,(\zeta)$ the derivative of $f_t(\zeta)$ with respect to t at $t = 0$, it follows that $\dot{f}\,(1) = 0$ and $\ddot{f}\,(1) = 0$ because $f_t(1) = p$ for all t (see Corollary 2.4).

Let us denote by $J(\zeta)$ the *Jacobi vector field* $J(\zeta) := \dot{f}\,(\zeta)$ *along* φ. We can write $J(\zeta) = \lambda(\zeta)\varphi'(\zeta) + J^\perp(\zeta)$ for some holomorphic function λ and vector field $J^\perp$ such that $\widetilde{\varphi}(\zeta) \cdot J^\perp(\zeta) \equiv 0$ and $\lambda(\zeta) = \alpha + i\beta\zeta - \overline{\alpha}\zeta^2$ for some $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$ (see [34, Theorem 3.1.c]).

Since $\dot{f}\,(1) = 0$ then $J(1) = 0$. Since the map Φ_p transforms boundaries of horospheres onto boundaries of horospheres, it follows that $J(0) \in T_{\varphi(0)}(\partial E_D(p, R))$. In other words, by (4.4), $\mathsf{Re}\,(\widetilde{\varphi}(0) \cdot J(0)) = 0$ which implies that $\mathsf{Re}\,\alpha = 0$ for $\widetilde{\varphi}(0) \cdot \varphi'(0) = 1$. Therefore $J(\zeta) = i\gamma(1-\zeta)^2\varphi'(\zeta) + J^\perp(\zeta)$ with $\gamma \in \mathbb{R}$ and $J^\perp(1) = 0$. This implies that $\mathsf{Re}\,(\widetilde{\varphi}(r) \cdot J(r)) = 0$ for $r \in (-1, 1)$. Hence by (4.4) it follows that $J(r) \in T_{\varphi(r)}(\partial E_D(p, R(r)))$, where $R(r) > 0$ is such that $\varphi(r) \in \partial E_D(p, R(r))$. Since boundaries of horospheres are convex, we have

$$\mathsf{Hess}(\Omega_{D,p})_{\varphi(r)}(J(r), J(r)) \geq 0, \quad r \in (-1, 1). \tag{4.6}$$

Now differentiating with respect to t the identity $F \circ f_t = g_t$ and setting $t = 0$ we obtain

$$\begin{aligned} &d\tilde{F}_{\varphi(\zeta)}(J(\zeta)) = U \\ &\mathsf{Hess}(\mathsf{Im}\,F_0)_{\varphi(\zeta)}(J(\zeta), J(\zeta)) + d(\mathsf{Im}\,F_0)_{\varphi(\zeta)}(\ddot{f}\,(\zeta)) = \mathsf{Im}\,\ddot{g}\,(\zeta) = 2\|U\|^2. \end{aligned} \tag{4.7}$$

Putting together (4.3), (4.5) and (4.7), we obtain for $r \in (-1, 1)$

$$\mathsf{Hess}(\Omega_{D,p})_{\varphi(r)}(J(r), J(r)) = \frac{\mathsf{Re}\,\widetilde{\varphi}(r) \cdot \ddot{f}\,(r)}{(1-r)^2}. \tag{4.8}$$

Our next aim is to compute $\widetilde{\varphi}(r) \cdot \ddot{f}\,(r)$. In order to do this, we choose a suitable defining function: according to Pang [26, Proposition 2.36] there exists a C^∞ defining function ρ for D near $\varphi(\overline{D})$ such that for all $\theta \in \mathbb{R}$ it follows that $\widetilde{\varphi}(e^{i\theta}) = e^{i\theta}\partial\rho_{\varphi(e^{i\theta})}$. For all t and for all $\theta \in \mathbb{R}$ it follows that $\rho(f_t(e^{i\theta})) \equiv 0$, thus differentiating such an identity (as we can, by Corollary 2.4) with respect to t at $t = 0$ we obtain $2\mathsf{Re}\,(\partial\rho \cdot \ddot{f}\,(e^{i\theta}) + \mathsf{Hess}(\rho)_{e^{i\theta}}(J(e^{i\theta}), J(e^{i\theta})) \equiv 0$, namely,

$$\mathsf{Re}\,(\overline{\zeta}\widetilde{\varphi}(\zeta) \cdot \ddot{f}\,(\zeta)) = -\frac{1}{2}\mathsf{Hess}(\rho)_\zeta(J(\zeta), J(\zeta)), \quad |\zeta| = 1. \tag{4.9}$$

Now, the function $\zeta \mapsto \widetilde{\varphi}(\zeta)\cdot \ddot{f}(\zeta)$ is holomorphic in $\mathbb{D}$. Write $\widetilde{\varphi}(\zeta)\cdot \ddot{f}(\zeta) = A + \zeta B + \zeta^2 C(\zeta)$ for some $A, B \in \mathbb{C}$ and some holomorphic function C. Then

$$\mathsf{Re}\,(\bar{\zeta}\widetilde{\varphi}(\zeta)\cdot \ddot{f}(\zeta)) = \mathsf{Re}\,(\overline{A}\zeta + B + \zeta C(\zeta)), \quad |\zeta| = 1. \tag{4.10}$$

Let T_1 denote the Hilbert transform which associates to any harmonic function u in $\mathbb{D}$, Hölder continuous on $\partial\mathbb{D}$, its harmonic conjugated $T_1(u)$, still Hölder continuous on $\partial\mathbb{D}$, normalized so that $T_1(u)(1) = 0$. Let h denote the holomorphic function in $\mathbb{D}$ whose trace on $\partial\mathbb{D}$ is $-1/2(\mathsf{id} + iT_1)(\mathsf{Hess}(\rho)(J, J))$. Notice that $\mathsf{Re}\, h \leq 0$ on $\partial\mathbb{D}$ since ∂D is convex. Moreover, since $J(1) = 0$ and by the normalization chosen for T_1 it follows that $h(1) = 0$.

By (4.9) and (4.10) we obtain that $h(\zeta) = \overline{A}\zeta + B + \zeta C(\zeta) + i\alpha$ for some $\alpha \in \mathbb{R}$. Hence $\widetilde{\varphi}(\zeta)\cdot \ddot{f}(\zeta) - \zeta h(\zeta) = -\overline{A}\zeta^2 - i\alpha\zeta + A$ and, since $h(1) = \ddot{f}(1) = 0$, we get $-\overline{A} - i\alpha + A = 0$. Writing $A = a + ib$ with $a, b \in \mathbb{R}$, we obtain

$$\widetilde{\varphi}(\zeta)\cdot \ddot{f}(\zeta) = \zeta h(\zeta) + a(1 - \zeta^2) + ib(1 - \zeta)^2.$$

Substituting this expression in (4.8) for $\zeta = r \in (-1, 1)$, we find

$$\mathsf{Hess}(\Omega_{D,p})_{\varphi(r)}(J(r), J(r)) = a\frac{1 + r}{1 - r} + \frac{r\mathsf{Re}\,(h(r))}{(1 - r)^2}. \tag{4.11}$$

By construction then $a = \mathsf{Hess}(\Omega_{D,p})_{\varphi(0)}(J(0), J(0)) = \mathsf{Hess}(\Omega_{D,p})_q(v, v) = 0$. By (4.6) and (4.11) it follows then that $\mathsf{Re}\,(h(r)) \geq 0$ for $r \in (0, 1)$. However $\mathsf{Re}\, h(\zeta)$ is harmonic on $\mathbb{D}$ and non-positive on $\partial\mathbb{D}$ and thus by the maximum principle $\mathsf{Re}\, h(\zeta) \equiv 0$. Thus

$$\mathsf{Hess}(\rho)_\zeta(J(\zeta), J(\zeta)) = 0, \quad \zeta \in \partial\mathbb{D}$$

and, since ∂D is strongly convex, it follows that $J = 0$ on $\partial\mathbb{D}$ and thus $J \equiv 0$ on $\mathbb{D}$ proving that $v = 0$. □

5. EXTREMALITY

Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary. We let Γ_p be the set of all C^∞ curves $\gamma : [0, 1] \to D \cup \{p\}$ such that $\gamma(1) = p$ and $\gamma'(1) \notin T_p\partial D$ (notice that, if ν_p is the unit outward normal to ∂D at p then $\gamma'(1) \notin T_p\partial D$ if and only if $\mathsf{Re}\,\langle\gamma'(1), \nu_p\rangle > 0$).

Theorem 5.1. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let ν_p be the unit outward normal to ∂D at p. Consider the following family $\mathcal{S}_p(D)$:*

$$\begin{cases} u \in \mathsf{Psh}(D) \\ \limsup_{z \to x} u(z) \leq 0 \quad \textit{for all } x \in \partial D \setminus \{p\} \\ \liminf\limits_{t \to 1} |u(\gamma(t))(1 - t)| \geq 2\mathsf{Re}\,(\langle\gamma'(1), \nu_p\rangle^{-1}) \quad \textit{for all } \gamma \in \Gamma_p, \end{cases} \tag{5.1}$$

Then $\Omega_{D,p} \in \mathcal{S}_p(D)$ (where $\Omega_{D,p}$ is the function defined in Theorem 0.1) and $u \leq \Omega_{D,p}$ for all $u \in \mathcal{S}_p(D)$.

To prove the theorem we need some preliminary results. Let $\mathsf{subh}(\mathbb{D})$ denote the real cone of subharmonic functions in the unit disc $\mathbb{D}$.

Lemma 5.2 (Phragmen-Lindelöf)**.** *Let $c > 0$. Consider the following family $\mathcal{S}_c(\mathbb{D})$ in the unit disc:*

$$\begin{cases} u \in \mathsf{subh}(\mathbb{D}) \\ u < 0 \quad \textit{in } \mathbb{D} \\ \liminf_{\mathbb{R} \ni r \to 1^-} |u(r)(1-r)| \geq 2c \end{cases} \tag{5.2}$$

Then $-cP(\zeta) \in \mathcal{S}_c(\mathbb{D})$ and for all $u \in \mathcal{S}_c(\mathbb{D})$ it follows $u \leq -cP(\zeta)$.

For the sake of completeness we give a short proof of Lemma 5.2, based on some notes of Prof. P. Poggi-Corradini. We thank him for letting us to use such notes.

Proof. It is clear that $-cP(\zeta) \in \mathcal{S}_c(\mathbb{D})$. We have to show that $-cP$ is the maximal element of the family.

First of all, let $C(\zeta) = (1+\zeta) \cdot (1-\zeta)^{-1}$ be the Cayley transformation from $\mathbb{D}$ to $\mathbb{H} = \{w \in \mathbb{C} : \mathsf{Re}\, w > 0\}$. Then we consider the family $C^*(\mathcal{S}_c(\mathbb{D}))) = \{\tilde{u} : \tilde{u} = u \circ C^{-1}$ for some $u \in \mathcal{S}_c(\mathbb{D})\}$. Then, if $\tilde{u} \in C^*(\mathcal{S}_c(\mathbb{D}))$ it follows that $\tilde{u} \in \mathsf{subh}(\mathbb{H})$, $\tilde{u} < 0$ in $\mathbb{H}$ and

$$\limsup_{\mathbb{R} \ni R \to +\infty} \frac{\tilde{u}(R)}{R} \leq -c.$$

Notice that $P \circ C^{-1}(w) = \mathsf{Re}\, w$ is the Poisson kernel in $\mathbb{H}$. Let $\tilde{u} = u \circ C^{-1} \in C^*(\mathcal{S}_c(\mathbb{D}))$ and let $L = \limsup_{\mathbb{R} \ni R \to +\infty} \tilde{u}(R)/R \leq -c$. We are going to show that $\tilde{u} \leq L \mathsf{Re}\, w$, from which it follows that $u \leq -cP$.

Let $\epsilon > 0$ be such that $\epsilon < -L$. Let $v(w) = \tilde{u}(w) - (L+\epsilon)\mathsf{Re}\, w$. Now, $v \in \mathsf{subh}(\mathbb{H})$, $\limsup_{w \to iy} v(w) \leq 0$ for all $y \in \mathbb{R}$ and

$$\limsup_{\mathbb{R} \ni R \to +\infty} v(R) = \limsup_{\mathbb{R} \ni R \to +\infty} R \left(\frac{\tilde{u}(R)}{R} - (L+\epsilon) \right) \leq 0.$$

Therefore there exists $\delta > 0$ such that $v(R) \leq 1$ for $R \leq \delta$ and $R \geq \frac{1}{\delta}$. Moreover, since v is semicontinuous, there exists $K > 0$ such that $v(R) \leq K$ for $\delta < R < \frac{1}{\delta}$. We consider now $V(w) = v(\sqrt{iw}) - K$. Again, $V \in \mathsf{subh}(\mathbb{H})$ and $\limsup_{w \to iy} V(w) \leq 0$ for all $y \in \mathbb{R}$. Moreover,

$$\begin{aligned} \sup_{-\pi/2 < \theta < \pi/2} V(re^{i\theta}) &= \sup_{0 < \theta < \pi/2} v(\sqrt{r} e^{i\theta}) - K = \sup_{0 < \theta < \pi/2} [\tilde{u}(\sqrt{r} e^{i\theta}) - (L+\epsilon)\sqrt{r} \cos\theta - K] \\ &\leq \sup_{0 < \theta < \pi/2} (-(L+\epsilon)\sqrt{r}\cos\theta - K) = -(L+\epsilon)\sqrt{r} - K. \end{aligned}$$

By the classical estimates on sub-linear growth of subharmonic functions (see, *e.g.* [30]), it follows that $V(w) \leq 0$ for all $w \in \mathbb{H}$ and therefore, $v \leq K$ in the first quadrant. A similar argument shows that $v \leq K$ on the fourth quadrant and as before, $v \leq 0$ on $\mathbb{H}$ which implies $\tilde{u}(w) \leq (L+\epsilon)\mathsf{Re}\, w$ for $w \in \mathbb{H}$. By the arbitrariness of ϵ we have the statement. $\square$

Proof of Theorem 5.1. Up to rigid movements, we can suppose that $\nu_p = e_1$.

First of all, notice that the function identically 0 does not belong to $\mathcal{S}_p(D)$ because of the estimates at p.

We claim that if $u \in \mathcal{S}_p(D)$ then $u < 0$ in D. Indeed, let $\varphi : \mathbb{D} \to D$ be a complex geodesic not containing p in its closure (in fact, any attached analytic disc not containing p would be enough). Then $\tilde{u} = u \circ \varphi : \mathbb{D} \to \mathbb{R}$ is subharmonic and $\limsup_{\zeta \to x} \tilde{u}(\zeta) \leq 0$ for all $x \in \partial\mathbb{D}$. Thus by the maximum principle for subharmonic functions, $\tilde{u} \leq 0$ in $\mathbb{D}$ and hence $u \leq 0$ in D as φ was an arbitrary complex geodesic. Again, the maximum principle for plurisubharmonic functions implies that $u < 0$ in D or $u \equiv 0$, and the latter cannot be the case. Thus $u < 0$ in D as wanted.

Now, let $v \in L_p$ and let $\varphi_v : \mathbb{D} \to D$ be a complex geodesic parameterized as in [13]. Let $\widetilde{\rho}_v : D \to \mathbb{D}$ be its left inverse. We show that the function $u_v : D \to \mathbb{R}^-$ defined by

$$u_v(z) = -\frac{P(\widetilde{\rho}_v(z))}{v_1^2} \tag{5.3}$$

belongs to $\mathcal{S}_p(D)$. It is clear that $u_v \in \mathsf{Psh}(D)$, $\limsup_{z\to x} u_v(z) \leq 0$ for all $x \in \partial D \setminus \{p\}$ since $\tilde{\rho}_v(\overline{D} \setminus \varphi_v(\partial\mathbb{D})) \subset \mathbb{D}$. We claim that for all smooth curves $\gamma : [0,1] \to D \cup \{p\}$ such that $\gamma(1) = p$ and $\langle \gamma'(1), e_1\rangle \neq 0$ (that is $\gamma'(1)$ is not complex tangential to ∂D) it follows

$$\lim_{t\to 1} |u_v(\gamma(t))(1-t)| = \frac{2\mathsf{Re}\,\langle \gamma'(1), e_1\rangle}{|\langle \gamma'(1), e_1\rangle|^2}. \tag{5.4}$$

Indeed,

$$|u_v(\gamma(t))(1-t)| = \frac{1}{v_1^2}\frac{1-|\widetilde{\rho}_v(\gamma(t))|^2}{1-t}\frac{(1-t)^2}{|1-\widetilde{\rho}_v(\gamma(t))|^2},$$

now

$$\lim_{t\to 1}\frac{1-\widetilde{\rho}_v(\gamma(t))}{1-t} = \frac{d}{dt}(\widetilde{\rho}_v(\gamma(t)))|_{t=1} = d(\widetilde{\rho}_v)_p(\gamma'(1)) = \frac{\langle \gamma'(1), e_1\rangle}{\langle \varphi_v'(1), e_1\rangle},$$

where the last equality follows from (3.7) and since $\varphi_v'(1) = v_1 v$. Moreover

$$\begin{aligned}\lim_{t\to 1}\frac{1-|\widetilde{\rho}_v(\gamma(t))|^2}{1-t} &= \frac{d}{dt}(|\widetilde{\rho}_v(\gamma(t))|^2)|_{t=1}\\ &= \widetilde{\rho}_v(\gamma(1))\frac{d}{dt}(\overline{\widetilde{\rho}_v(\gamma(t))})|_{t=1} + \overline{\widetilde{\rho}_v(\gamma(1))}\frac{d}{dt}(\widetilde{\rho}_v(\gamma(t)))|_{t=1}\\ &= 2\mathsf{Re}\,\frac{d}{dt}(\widetilde{\rho}_v(\gamma(t)))|_{t=1} = 2\mathsf{Re}\,\frac{\langle \gamma'(1), e_1\rangle}{\langle \varphi_v'(1), e_1\rangle}.\end{aligned}$$

Therefore

$$\lim_{t\to 1}|u_v(\gamma(t))(1-t)| = \frac{2}{v_1^2}\mathsf{Re}\,\frac{\langle \gamma'(1), e_1\rangle}{\langle \varphi_v'(1), e_1\rangle}\cdot\frac{|\langle \varphi_v'(1), e_1\rangle|^2}{|\langle \gamma'(1), e_1\rangle|^2}.$$

Taking into account that $\langle \varphi_v'(1), e_1\rangle = v_1^2$, we have the claim. In particular equation (5.4) holds if $\gamma \in \Gamma_p$, showing that u_v belongs to $\mathcal{S}_p(D)$.

Notice that $\Omega_{D,p}(\varphi_v(\zeta)) = u_v(\varphi_v(\zeta))$ for all $\zeta \in \mathbb{D}$. Moreover, if $u \in \mathcal{S}_p(\mathbb{D})$ then $\tilde{u} : \zeta \mapsto u(\varphi_v(\zeta))$ is in the family $\mathcal{S}_{1/v_1^2}(\mathbb{D})$ given by (5.2) (with $c = 1/v_1^2$). Indeed, $\tilde{u} \in \mathsf{subh}(\mathbb{D})$, $u < 0$ in $\mathbb{D}$ and

$$\liminf_{\mathbb{R}\ni r\to 1} |\tilde{u}(r)|(1-r) \geq \frac{2\mathsf{Re}\,\langle \varphi_v'(1), e_1\rangle}{|\langle \varphi_v'(1), e_1\rangle|^2} = \frac{2}{\langle \varphi_v'(1), e_1\rangle} = \frac{2}{v_1^2},$$

since $\varphi_v'(1) = v_1^2 e_1 + e_1^\perp$. Thus, by Lemma 5.2 it follows that for all $\zeta \in \mathbb{D}$

$$u(\varphi_v(\zeta)) = \tilde{u}(\zeta) \leq \frac{-1}{v_1^2} P(\zeta) = \frac{-1}{v_1^2} u_v(\varphi_v(\zeta)) = \Omega_{D,p}(\varphi_v(\zeta)).$$

Thus for all $u \in \mathcal{S}_p(D)$ we have $u \leq \Omega_{D,p}$. It remains only to show that $\Omega_{D,p} \in \mathcal{S}_p(D)$. To this aim, we let $\varphi_{v_t} : \mathbb{D} \to D$ be the complex geodesic in Chang-Hu-Lee normal parametrization such that $\gamma(t) \in \varphi_{v_t}(\mathbb{D})$. Moreover, we denote by $v_t = \varphi_{v_t}'(1) \in L_p$. Thus

$$\Omega_{D,p}(\gamma(t)) = u_{v_t}(\gamma(t)) = \frac{-1}{\langle v_t, e_1\rangle^2} P(\widetilde{\rho}_{v_t}(\gamma(t))).$$

Hence

$$|\Omega_{D,p}(\gamma(t))|(1-t) = \frac{1}{\langle v_t, e_1\rangle^2} \frac{1 - |\widetilde{\rho}_{v_t}(\gamma(t))|^2}{1-t} \frac{(1-t)^2}{|1 - \widetilde{\rho}_{v_t}(\gamma(t))|^2}. \tag{5.5}$$

Fix $v = v_t$. By the mean value theorem it follows that

$$\frac{1 - \mathsf{Re}\,\widetilde{\rho}_v(\gamma(t))}{1-t} = \frac{d}{dt}\mathsf{Re}\,\widetilde{\rho}_v(\gamma(t))|_{t=s} = \mathsf{Re}\, d(\widetilde{\rho}_v)_{\gamma(s)}(\gamma'(s)),$$

for some $t < s < 1$, and similarly for the imaginary part and for the modulus $|\widetilde{\rho}_{v_t}(\gamma(t))|^2$. Notice that s depends on v but clearly, $s \to 1$ as $t \to 1$.

Now let $\{v_{t_k}\}$ be a converging subsequence of $\{v_t\}$. By Lemma 3.4 if $v_{t_k} \to v_0$ then $v_0 \in L_p$ (and in particular $\langle v_{t_k}, e_1\rangle^2 \to \langle v_0, e_1\rangle^2 > 0$). Therefore, by Lemma 3.5 we have

$$\lim_{t\to 1} d(\widetilde{\rho}_{v_t})_{\gamma(s)}(\gamma'(s)) = d(\widetilde{\rho}_{v_0})_p(\gamma'(1)).$$

Thus by (5.5) and (3.7) it follows

$$\lim_{t_k\to 1} |\Omega_{D,p}(\gamma(t_k))|(1-t_k) = \frac{1}{\langle v_0, e_1\rangle^2} \frac{2\mathsf{Re}\, d(\widetilde{\rho}_{v_0})_p(\gamma'(1))}{|d(\widetilde{\rho}_{v_0})_p(\gamma'(1))|^2} = \frac{2\mathsf{Re}\,\langle \gamma'(1), e_1\rangle}{|\langle \gamma'(1), e_1\rangle|^2}.$$

Since this holds for any converging subsequence of $\{v_t\}$ then we have that

$$\lim_{t\to 1} |\Omega_{D,p}(\gamma(t))|(1-t) = \frac{2\mathsf{Re}\,\langle \gamma'(1), e_1\rangle}{|\langle \gamma'(1), e_1\rangle|^2}.$$

□

Corollary 5.3. *Let $\Omega_{D,p}$ be the function given by Theorem 0.1. Then for all smooth curves $\gamma : [0,1] \to D \cup \{p\}$ such that $\gamma(1) = p$ and $\gamma'(1) \notin T_p^{\mathbb{C}}\partial D$ it follows*

$$\lim_{t\to 1} |\Omega_{D,p}(\gamma(t))|(1-t) = \mathsf{Re}\,\frac{2}{\langle \gamma'(1), \nu_p\rangle}.$$

Proof. If $\gamma'(1) \notin T_p\partial D$ then the claim follows from the proof of Theorem 5.1. In case $\gamma'(1) \in T_p\partial D \setminus T_p^{\mathbb{C}}\partial D$—that is $\mathsf{Re}\,\langle \gamma'(1), \nu_p\rangle = 0$ but $\langle \gamma'(1), \nu_p\rangle \neq 0$—let $v \in L_p$ and let u_v be given by (5.3). By Theorem 5.1 it follows that for all $z \in D$

$$0 \leq |\Omega_{D,p}(z)| \leq |u_v(z)|.$$

By (5.4) it follows that $|u_v(\gamma(t))|(1-t) \to 0$ and then $|\Omega_{D,p}(\gamma(t))|(1-t) \to 0$, proving the statement. □

6. Green's versus Poisson's pluricomplex functions

Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary and let $z_0 \in D$. Consider the problem in (0.1). In his outstanding work [21], [24], Lempert proved that there exists a unique solution L_{D,z_0}, given by $L_{D,z_0} = \log \|\Phi_{z_0}\|$, where $\Phi_{z_0} : D \to \mathbb{B}^n$ is the Lempert spherical representation with center z_0 introduced in Section 1. Rephrasing the very definition of Φ_{z_0}, it follows that for $z \in D$

$$L_{D,z_0}(z) = \log(\tanh k_D(z_0, z)). \tag{6.1}$$

We have the following relations between the pluricomplex Green function L_{D,z_0} and the pluricomplex Poisson kernel $\Omega_{D,p}$ solution of the problem (0.2) which generalizes the corresponding relation in $\mathbb{D}$ between the classical Green function and the classical Poisson kernel (see for instance [20, Proposition 2.2.2]):

Theorem 6.1. *Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary. Let $z_0 \in D$ and $p \in \partial D$. Let ν_p be the outer normal of ∂D at p. Then*

$$\Omega_{D,p}(z_0) = -\frac{\partial L_{D,z_0}}{\partial \nu_p}(p) \tag{6.2}$$

Proof. Let $K_{z_0} := \exp(L_{D,z_0})$. Let $\varphi : \mathbb{D} \to D$ be a complex geodesic such that $\varphi(0) = z_0$ and $\varphi(1) = p$. Since $\frac{\partial K_z}{\partial \nu_p}(p) > 0$ for all $z \in D$, by [1, Theorem 2.6.47] (see also [4]) it follows that

$$\lim_{\mathbb{R}\ni t\to 1}[k_D(z, \varphi(t)) - k_D(z_0, \varphi(t))] = \frac{1}{2}[\log \frac{\partial K_{z_0}}{\partial \nu_p}(p) - \log \frac{\partial K_z}{\partial \nu_p}(p)].$$

On the other hand by [11, Proposition 7.1]

$$\lim_{\mathbb{R}\ni t\to 1}[k_D(z, \varphi(t)) - k_D(z_0, \varphi(t))] = \frac{1}{2}[\log |\Omega_{D,p}(z_0)| - \log |\Omega_{D,p}(z)|],$$

which implies that there exists $C > 0$ such that for all $z \in D$

$$\Omega_{D,p}(z) = -C\frac{\partial K_z}{\partial \nu_p}(p).$$

We want to show that $C = 1$. Let $\varphi : \mathbb{D} \to D$ be the unique complex geodesic in Chang-Hu-Lee normal parametrization such that $\varphi(1) = p$ and $\varphi'(1) = \nu_p$. By the very definition

$\Omega_{D,p}(\varphi(\zeta)) = -P(\zeta)$, where P is the Poisson kernel of $\mathbb{D}$ and $K_{\varphi(0)}(\varphi(\zeta)) = |\zeta|$ for all $\zeta \in \mathbb{D}$. Since

$$\frac{\partial K_{\varphi(0)}}{\partial \nu_p}(p) = \frac{d}{dr}(K_{\varphi(0)} \circ \varphi(r))|_{r=0} = \frac{d}{dr} r = 1,$$

and $P(0) = 1$ it follows that $C = 1$, as wanted. Finally, since $\frac{\partial K_z}{\partial \nu_p}(p) = K_z(p)\frac{\partial L_{D,z}}{\partial \nu_p}$ and $K(p) = 1$ for $p \in \partial D$, we get (6.2). $\square$

7. Uniqueness properties

In this section we study some analytical and geometrical properties which characterize the pluricomplex Poisson kernel introduced before.

Before start, recall that, according to Bedford and Taylor [8] (see also [20, Section 3.5], the complex Monge-Ampère operator $(dd^c)^n$ (here $d^c = i(\overline{\partial} - \partial)$) can be defined for all $u \in \mathsf{Psh}(D) \cap L^\infty_{loc}(D)$ for any bounded domain $D \subset \mathbb{C}^n$. Moreover, if $u \in \mathsf{Psh}(D) \cap L^\infty_{loc}(D)$ then $(dd^c u)^n = (\partial\overline{\partial} u)^n = 0$ if and only if u is *maximal* in D; namely, for all relatively compact open subsets $E \subset D$ and all plurisubharmonic functions v in E such that $\limsup_{E \ni z \to x} v(z) \le u(x)$ for all $x \in \partial E$ it follows that $v \le u$ in E.

Now we can state and prove the first uniqueness result, which is the analogous in our setting of the uniqueness statement for the Monge-Ampère equation with one concentrated logarithmic singularity in the domain D (see [24]).

Theorem 7.1. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let $u \in \mathsf{Psh}(D) \cap L^\infty_{loc}(D)$ be such that $(\partial\overline{\partial} u)^n = 0$, $\lim_{z \to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$ and*

$$\lim_{z \to p} \frac{u(z)}{\Omega_{D,p}(z)} = 1. \tag{7.1}$$

Then $u \equiv \Omega_{D,p}$.

Proof. First of all we notice that (7.1) implies that u belongs to the family (5.1) because for all $\gamma \in \Gamma_p$ (here Γ_p is the set of curves defined in section 5) it follows that

$$\lim_{t \to 1} u(\gamma(t))(1-t) = \lim_{t \to 1} \frac{u(\gamma(t))}{\Omega_{D,p}(\gamma(t))} \Omega_{D,p}(\gamma(t))(1-t) = -\mathsf{Re}\, 2(\langle \gamma'(1), \nu_p \rangle)^{-1}.$$

Therefore, by Theorem 5.1 it follows that $u(z) \le \Omega_{D,p}(z)$ for all $z \in D$. Suppose that $u(z_0) < \Omega_{D,p}(z_0)$ for some $z_0 \in D$. Then there exist $0 < c < 1$ and $\delta > 0$ such that the set

$$E_{\delta,c} := \{z \in D : \Omega_{D,p}(z) > cu(z) + \delta\}$$

is non-empty. Since u is upper semi-continuous the set $E_{\delta,c}$ is open. If we prove that $E_{\delta,c}$ is relatively compact in D, since $(\partial\overline{\partial}(cu + \delta))^n = 0$ and $\Omega_{D,p}(z) \le cu(z) + \delta$ on $\partial E_{\delta,c}$, by maximality it follows that $\Omega_{D,p}(z) \le cu(z) + \delta$ in $E_{\delta,c}$, contradicting the definition of $E_{\delta,c}$.

Thus we are left to show that $E_{\delta,c}$ is relatively compact in D. First of all, since $u(x) = \Omega_{D,p}(x) = 0$ for all $x \in \partial D \setminus \{p\}$, then $\overline{E_{\delta,c}} \subset D \cup \{p\}$. Seeking a contradiction, we assume that $p \in \overline{E_{\delta,c}}$. Thus there exists $\{z_k\} \subset E_{\delta,c}$ such that $z_k \to p$. Therefore for all $k \in \mathbb{N}$

$$\Omega_{D,p}(z_k) - cu(z_k) - \delta > 0. \tag{7.2}$$

Up to subsequences, we can assume that $\Omega_{D,p}(z_k) \to L$ for some $L \in [-\infty, 0]$. If $L < 0$ then dividing (7.2) by $\Omega_{D,p}(z_k) < 0$ and passing to the limit, taking into account (7.1), we would find $1 - c \leq 0$, a contradiction since $c < 1$. If $L = 0$, (7.1) implies that $u(z_k) \to 0$ as $k \to \infty$ and therefore we reach a contradiction by passing to the limit for $k \to \infty$ in (7.2). Hence p is not in the closure of $E_{\delta,c}$ which is thus relatively compact in D. $\square$

Remark 7.2. As pointed out in the introduction, Theorem 7.1 is the analogous of the uniqueness statement for the problem (0.1), where uniqueness is established in the class of plurisubharmonic functions $u \in \mathsf{Psh}(D)$ such that $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D$ and $u(z)$ goes like the pluricomplex Green function L_{D,z_0} for $z \to z_0$. Since for any convex domain the function L_{D,z_0} goes like $\log \|z - z_0\|$ at z_0 then in the case of a inner singularity, there is a "universal" behavior. When the singularity is at $p \in \partial D$, it turns out that, thanks to Corollary 5.3, we know that the behavior of $\Omega_{D,p}$ along all non tangential directions is independent of D, but we do not have any hint on the behavior of $\Omega_{D,p}$ along the tangential directions, which might depend on D near p.

Next we characterize the pluricomplex Poisson kernel in terms of its associated Monge-Ampère foliation, with no hypotheses on the behavior near the boundary singularity.

Theorem 7.3. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let $u \in \mathsf{Psh}(D) \cap C^2(D)$ be such $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$. Then the restriction of u to each complex geodesic whose closure contains p is harmonic if and only if there exists $c \geq 0$ such that $u = c\Omega_{D,p}$.*

Proof. One direction is obvious. Assume then that $u \in \mathsf{Psh}(D) \cap C^2(D)$ is harmonic on each complex geodesic whose closure contains p and $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$. Arguing as in the proof of Theorem 5.1 we see that $u < 0$ in D or $u \equiv 0$. In the latter case $c = 0$ and the theorem is proved. Thus we can assume that $u < 0$ in D.

First of all, it is a well known result that if $v \geq 0$ is a harmonic function in $\mathbb{D}$ such that $\lim_{\zeta\to x} v(\zeta) = 0$ for all $x \in \partial\mathbb{D} \setminus \{1\}$ then $v = cP$ for some $c \geq 0$ (here, as usual, P denotes the Poisson kernel).

Therefore $u = \lambda\Omega_{D,p}$ for some C^2 function $\lambda : D \to (0, \infty)$ which is constant on each complex geodesic whose closure contains p. We need to show that λ is constant.

To this aim, we argue as in the proof of Theorem 4.1 and retain the notations introduced there. Let $q \in D$. Up to post-composing with automorphisms of $\mathbb{B}^n$ and with the Cayley transform, we let $F : D \to \mathbb{H}^n$ be the diffeomorphism defined by means of the boundary spherical representation Φ_p, such that $F(q) = (i, O)$. We let $U = u \circ F^{-1}$. Then U is a C^2 negative function on $\mathbb{H}^n$ and by the very definition of $\Omega_{D,p}$ and [11, Theorem 7.3], it follows

$U(\xi, w) = \tilde{\lambda}(w)(\|w\|^2 - \operatorname{Im}\xi)$. We are going to prove that $w = O$ is a critical point for $\tilde{\lambda}$; from this, since F is a diffeomorphism from D to $\mathbb{H}^n$, it will follow that λ has a critical point at $q = F^{-1}(i, O)$ and, by the arbitrariness of q, it will follow that all points of D are critical for λ which turns out to be constant.

Since $\tilde{\lambda}$ is a real function, it is enough to prove that the vector $V := (\frac{\partial\tilde{\lambda}}{\partial w_1}(0), \ldots, \frac{\partial\tilde{\lambda}}{\partial w_{n-1}}(0))$ is zero. Let $\varphi : \mathbb{D} \to D$ be the complex geodesic such that $\varphi(0) = q$ and $\varphi(1) = p$. According to [26, Section 2.39] we can assume to be working with a system of holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighborhood of $\varphi(\overline{\mathbb{D}})$ for which (among other conditions on the defining function of D which we only use implicitly when referring to the paper [32] in the course of the proof) $\varphi(\zeta) = (\zeta, 0, \ldots, 0)$ for $\zeta \in \mathbb{D}$.

By construction it follows that if we write $G := F^{-1} = (G_1, \ldots, G_n)$ then $G_1(\xi, O) = (\xi - i)/(\xi + i)$ and $G_j(\xi, O) = 0$ for $j > 1$ and $\operatorname{Im}\xi > 0$.

Now let $t \mapsto w(t)$ be a smooth curve in $\mathbb{C}^{n-1}$ such that $w(0) = O$. Let $g_t(\zeta) := (i(1+\zeta)/(1-\zeta) + i\|w(t)\|^2, w(t))$ for $\zeta \in \mathbb{D}$ and t close to 0. By definition, $\{g_t\}$ is a family of complex geodesics in $\mathbb{H}^n$, and thus $\varphi_t := G(g_t(\zeta))$ is a smooth real one-parameter family $\{\varphi_t\}$ of complex geodesics in D such that $\varphi_0(\zeta) = (\zeta, O)$. The associated Jacobi vector field $J(\zeta) = \frac{\partial\varphi_t}{\partial t}(\zeta)$ can be written in the form

$$J(\zeta) = J_1(\zeta)\frac{\partial}{\partial z_1} + J^\perp(\zeta),$$

where $J^\perp(\zeta) = \sum_{k=2}^n J_k(\zeta)\frac{\partial}{\partial z_k}$ and, since $\varphi_t(1) = p$ for all t, by Corollary 2.4 it follows that $J(1) = O$. Therefore, from [32, Section 3] it follows that there exist $a \in \mathbb{C}$, $X, Y \in \mathbb{C}^{n-1}$ (depending on J) and a unique continuous map $M : \overline{\mathbb{D}} \to \mathsf{GL}(2n-2, \mathbb{C})$ holomorphic in $\mathbb{D}$ which depends only on D and φ with the following properties. If $M(\zeta) = \begin{pmatrix} M_1(\zeta) & M_2(\zeta) \\ M_3(\zeta) & M_4(\zeta) \end{pmatrix}$ where the M_j's are suitable $(n-1)\times(n-1)$-matrices with $M_1(1) = \frac{1}{2}\mathsf{Id}$, $M_2(1) = \frac{-i}{2}\mathsf{Id}$ (and $M_3(1), M_4(1)$ satisfy suitable conditions that we do not need here), then

$$\begin{aligned} J_1(\zeta) &= (1-\zeta)(a + \overline{a}\zeta), \\ J^\perp(\zeta) &= i(1-\zeta)(M_1(\zeta)X + M_2(\zeta)Y). \end{aligned} \tag{7.3}$$

By the very definition of G and by (7.3), taking into account that G maps complex tangent spaces to the boundary of horospheres in $\mathbb{H}^n$ to complex tangent spaces to the boundary of horospheres in D (see the proof of *Theorem 6.3* in [11]) it follows that for $\mathsf{Im}\,\xi > 0$ and $\zeta \in \mathbb{D}$

$$
\begin{aligned}
&\frac{\partial G_j}{\partial \overline{\xi}}(\xi, 0) = 0 \quad \text{for } j = 1, \dots, n \\
&\frac{\partial G_1}{\partial \xi}(\xi, 0) = \frac{\partial}{\partial \xi}(\xi - i)(\xi + i)^{-1} \\
&\frac{\partial G_j}{\partial \xi}(\xi, 0) = 0 \quad \text{for } j = 2, \dots, n \\
&\frac{\partial G_1}{\partial w_j}(\xi, 0) = \frac{\partial G_1}{\partial \overline{w_j}}(\xi, 0) = 0 \quad \text{for } j = 1, \dots, n-1. \\
&\frac{\partial G_k}{\partial w_j}(i\frac{1+\zeta}{1-\zeta}, 0) = i(1-\zeta)(M_1(\zeta)S_j + M_2(\zeta)T_j)_k \quad \text{for } j, k = 2, \dots, n \\
&\frac{\partial G_k}{\partial \overline{w_j}}(i\frac{1+\zeta}{1-\zeta}, 0) = i(1-\zeta)(M_1(\zeta)\overline{S}_j + M_2(\zeta)\overline{T}_j)_k \quad \text{for } j, k = 2, \dots, n,
\end{aligned}
\tag{7.4}
$$

for some vectors $(S_2, \dots, S_n), (T_2, \dots, T_n) \in \mathbb{C}^{n-1}$. Let S (respectively T) be the matrix whose columns are $S_2, \dots, S_{n-1}$ (respect. $T_2, \dots, T_{n-1}$) and set

$$
N = \begin{pmatrix} S & \overline{S} \\ T & \overline{T} \end{pmatrix}.
$$

We claim that N is invertible. Indeed, since dG is invertible at (i, O), equations (7.4) imply that the only vector v satisfying $(M_1(0)\ M_2(0))(2\mathsf{Re}\,(\begin{smallmatrix} S \\ T \end{smallmatrix})\,v) = 0$ is the zero vector $v = O$. Therefore $S_2, \dots, S_n, T_2, \dots, T_n$ form a real basis of $\mathbb{C}^{n-1}$. From this it follows easily that if the vector $(\begin{smallmatrix} v \\ w \end{smallmatrix})$ belongs to the kernel of N^t then $v = w = 0$ and thus N is invertible.

Now we are in the good shape to compute $\frac{\partial U}{\partial w_j}(\xi, O)$. Since $U = u \circ G = \tilde{\lambda}(w)(\|w\|^2 - \mathsf{Im}\,\xi)$, from (7.4) we have for $j = 1, \dots, n-1$ and $\mathsf{Im}\,\xi > 0$

$$
-\frac{\partial \tilde{\lambda}}{\partial w_j}(O)\mathsf{Im}\,\xi = \sum_{k=2}^{n} [\frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial w_j}(\xi, O) + \overline{\frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial \overline{w_j}}(\xi, O)}]. \tag{7.5}
$$

Notice that since u is plurisubharmonic in D and harmonic on the complex geodesics whose closure contains p, it follows that the functions $\frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)$ are holomorphic for $\mathsf{Im}\,\xi > 0$. Moreover, by (7.4) both $\frac{\partial G}{\partial w_j}(\xi, O)$ and $\frac{\partial G}{\partial \overline{w}_j}(\xi, O)$ are holomorphic for $\mathsf{Im}\,\xi > 0$. Taking the real and imaginary part in (7.5) and writing $V(= \frac{\partial \tilde{\lambda}}{\partial w}(i, O)) = C + iD$ with $C, D \in \mathbb{R}^{n-1}$, we find that there exist two vectors $C', D' \in \mathbb{R}^{n-1}$ such that for all $\mathsf{Im}\,\xi > 0$

$$
iC_j\xi + iC'_j = \sum_{k=2}^{n} [\frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial w_j}(\xi, O) + \frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial \overline{w_j}}(\xi, O)], \tag{7.6}
$$

$$
-D_j\xi + D'_j = \sum_{k=2}^{n} [\frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial w_j}(\xi, O) - \frac{\partial u}{\partial z_k}(\frac{\xi - i}{\xi + i}, O)\frac{\partial G_k}{\partial \overline{w_j}}(\xi, O)]. \tag{7.7}
$$

Let $V' = iC' + D'$, let $f_k(\zeta) = -2i(1-\zeta)^2 \frac{\partial u}{\partial z_k}(\zeta, O)$ and let $f = (f_1, \ldots, f_n)$ for $\zeta \in \mathbb{D}$. Summing (respectively subtracting) (7.6) with (7.7), composing with $\zeta \mapsto i\frac{1+\zeta}{1-\zeta}$, multiplying by $(1-\zeta)$ and using (7.4) we obtain for $\zeta \in \overline{\mathbb{D}} \setminus \{1\}$

$$\begin{pmatrix} \zeta(V+V') + (V-V') \\ \zeta(\overline{V} - \overline{V}') + (\overline{V} + \overline{V}') \end{pmatrix} = N^t \begin{pmatrix} M_1(\zeta)^t \\ M_2(\zeta)^t \end{pmatrix} \cdot f(\zeta).$$

From this, since N is invertible and also $M_1(\zeta), M_2(\zeta)$ are invertible for ζ close to 1 (since by the very definition $M_1(1) = \frac{1}{2}\mathsf{Id}$ and $M_2(1) = \frac{-i}{2}\mathsf{Id}$) it follows that $f(\zeta)$ has a limit L at $\zeta = 1$ and

$$\begin{pmatrix} 4V \\ 4\overline{V} \end{pmatrix} = N^t \begin{pmatrix} \mathsf{Id} \\ -i\mathsf{Id} \end{pmatrix} L. \tag{7.8}$$

Therefore $(S^t - iT^t)L - (S^t + iT^t)\overline{L} = O$. Writing $L = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}^{n-1}$, we have $S^t\beta - T^t\alpha = O$ and, since α, β are real, this is equivalent to

$$N^t \begin{pmatrix} \beta \\ -\alpha \end{pmatrix} = O.$$

But N is invertible and therefore $\alpha = \beta = O$ which means $L = O$. Finally, from (7.8) it follows that $V = O$. $\square$

The pluricomplex Poisson kernel can be also characterized in terms of its level sets:

Proposition 7.4. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let $u \in \mathsf{Psh}(D) \cap L^\infty_{loc}(D)$ be such $(\partial\overline{\partial}u)^n = 0$ in D and $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$. If u has the same level sets of $\Omega_{D,p}$ then there exists $c > 0$ such that $u = c\Omega_{D,p}$.*

Proof. By hypothesis there exists a function $Y : \mathbb{R}^- \to \mathbb{R}^-$ such that $u(z) = Y(\Omega_{D,p}(z))$ for all $z \in D$. We need to show that there exists $c > 0$ such that $Y(t) = ct$ for all $t \in \mathbb{R}^-$. To this aim, since each complex geodesic whose closure contains p intersects every horosphere, it is enough to prove that $u(z) = c\Omega_{D,p}(z)$ for z belonging to any complex geodesic whose closure contains p.

Let S be a complex geodesic in D such that $p \in \overline{S}$ and $\rho : D \to S$ the associated Lempert's projection. We can assume that D is linearizated along S in Lempert's coordinates. Let $\tilde{B}$ be a open disc relatively compact in S. Let

$$\mathcal{P} := \begin{cases} \tilde{v} \in \mathsf{subh}(\tilde{B}) \\ \limsup_{\zeta\to x} \tilde{v}(\zeta) \le u(x) \quad \forall x \in \partial\tilde{B} \end{cases}$$

If we prove that $u|_{\tilde{B}}$ is the maximum of $\mathcal{P}$ then, by the arbitrariness of $\tilde{B}$ it follows that u is harmonic on S. Therefore $u \circ \varphi$ is harmonic and negative in $\mathbb{D}$ and it is zero on $\partial\mathbb{D} \setminus \{1\}$, hence it is a constant multiple of the Poisson kernel of $\mathbb{D}$. That is, there exists $c > 0$ such that $u(\varphi(\zeta)) = c\Omega_{D,p}(\varphi(\zeta))$ for all $\zeta \in \mathbb{D}$, as wanted.

In order to prove that $u|_{\tilde{B}}$ is the maximum of $\mathcal{P}$, let $\epsilon > 0$ small. Let $T = \rho^{-1}(\tilde{B}) \cap D$ and let $B = \{z \in T : \text{dist}(z, \partial D) > \epsilon\}$ (a cylinder in D). The boundary of the set B is made of two parts: R_1 which has the property that $\rho(R_1) = \partial\tilde{B}$ and R_2 (the bottom and top of the cylinder) such that $\rho(R_2) \subset \tilde{B}$; $\partial B = R_1 \cup R_2$. Since $u = 0$ on ∂D and $p \notin T$, then we can choose ϵ so small that $\inf_{x\in R_2} u(x) > \max_{x\in\partial\tilde{B}} u(x)$.

Let $\tilde{v} \in \mathcal{P}$. Let $v := \tilde{v} \circ \rho|_B$. Then v is plurisubharmonic in B and $\sup_{x\in B} v(x) = \sup_{x\in\partial\tilde{B}}(\limsup_{z\to x} v(z))$. In particular by construction $\limsup_{z\to x} v(z) \leq u(x)$ for all $x \in R_2$. Also, we have that $\limsup_{B\ni z\to x} v(z) = \limsup_{B\ni z\to x} \tilde{v}(\rho(z)) \leq u(\rho(x))$ for all $x \in R_1$. Now u has the same level sets of $\Omega_{D,p}$ and thus by (3.3) we have that $u(x) \geq u(\rho(x))$ for all $x \in D$ and hence $\limsup_{B\ni z\to x} v(z) \leq u(x)$ for all $x \in R_1$. Therefore $\limsup_{B\ni z\to x} v(z) \leq u(x)$ for all $x \in \partial B$ and by the maximality of Monge-Ampere solutions, it follows that $v \leq u$ in B and in particular $\tilde{v} \leq u|_{\tilde{B}}$ and the arbitrariness of $\tilde{v}$ implies that $u|_{\tilde{B}}$ is maximal in $\mathcal{P}$. □

The previous argument, together with Theorem 7.3, shows that if $u \in \mathsf{Psh}(D) \cap C^2(D)$ is such that $(\partial\overline{\partial}u)^n = 0$ on D and $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$ then $u = c\Omega_{D,p}$ for some $c > 0$ if and only if $u(\rho(z)) \leq u(z)$ for all $z \in D$ and for all Lempert's projections ρ.

More generally, if $f : D \to D$ is holomorphic and $f(p) = p$ as non-tangential limit we can define the *boundary dilatation coefficient* $\alpha_f(p)$ of f at p by means of

$$\frac{1}{2}\log\alpha_f(p) := \liminf_{z\to p}[k_D(z_0, z) - k_D(z_0, f(z))].$$

It turns out that $\alpha_f(p) > 0$ and, if $\alpha_f(p) < \infty$, we can rephrase Abate's generalization of the classical Julia Lemma (see [1], [2]) saying that $\alpha_f(p)f^*(\Omega_{D,p}) \leq \Omega_{D,p}$. In [11, Theorem 7.3], with a slightly more regularity assumption required on f at p, it is proved that f is an automorphism of D if and only if $f^*(\Omega_{D,p}) = \alpha_f(p)\Omega_{D,p}$.

Using Abate's version of the Julia-Wolff-Caratheodory theorem for strongly convex domains (see [1], [4]) it is easy to see that $\alpha_\rho(p) = 1$ for all Lempert's projections ρ. Therefore, the above discussion shows that the property $f^*(\Omega_{D,p}) \leq \alpha_f(p)\Omega_{D,p}$ characterizes $\Omega_{D,p}$. In other words:

Proposition 7.5. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let $u \in \mathsf{Psh}(D) \cap C^2(D)$ be such that $(\partial\overline{\partial}u)^n = 0$ in D and $\lim_{z\to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$. Then there exists $c \geq 0$ such that $u = c\Omega_{D,p}$ if and only if for all $f : D \to D$ holomorphic such that $f(p) = p$ as non-tangential limit and $\alpha_f(p) < \infty$ it follows that*

$$\alpha_f(p)f^*(u) \leq u.$$

Some remarks about uniqueness properties are in order. First, it would be interesting to see whether Theorem 7.3 (and thus its corollaries) holds without any regularity hypothesis on u. A direct argument using the sub-media property of plurisubharmonic functions shows that Theorem 7.3 holds in the unit ball $\mathbb{B}^n$ with no regularity hypothesis on u. Such an argument seems however to fail in general.

Another (maybe more) interesting open question is the following:

Question 7.6. Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let $u \in \mathsf{Psh}(D) \cap L^\infty_{loc}(D)$ be such that $(\partial\overline{\partial} u)^n = 0$ in D and $\lim_{z \to x} u(z) = 0$ for all $x \in \partial D \setminus \{p\}$. Is it true that $u = c\Omega_{D,p}$ for some constant $c \geq 0$?

As we already recalled, the answer to such a question is "yes" in case $D = \mathbb{D}$ the unit disc, $u < 0$ in $\mathbb{D}$ and $\Omega_{D,p}$ is the (negative) Poisson kernel.

8. Reproducing formulas

Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary. As usual, let $d^c := i(\overline{\partial} - \partial)$. Let r be a defining function of D and let ω_D be the real $(2n-1)$-form defined as

$$\omega_{\partial D} := \frac{(dd^c r)^{n-1} \wedge d^c r}{\|dr\|^n}|_{\partial D}.$$

such a form $\omega_{\partial D}$ is positive and it is easily seen to be independent of the defining function r chosen to define it.

Let L_{D,z_0} denote the Lempert solution of (0.1) and denote by $\Omega_{D,p}$ the solution of (0.2) with singularity at $p \in \partial D$ given by Theorem 0.1. From the very definition of $\Omega_{D,p}$ and since the boundary spherical representation Φ_p of Chang-Hu-Lee is smooth out of the diagonal of $\partial D \times \partial D$ as the vertex p varies on ∂D (see [13, Theorem 3]) it follows that the map $\overline{D} \times \partial D \ni (z, p) \mapsto \Omega_{D,p}(z) \in \mathbb{R}$ is C^∞ on $(\overline{D} \times \partial D) \setminus \{(p,p) \in \partial D \times \partial D\}$.

We briefly recall Demailly's theory [14], [15]. Let $\varphi \in \mathsf{Psh}(D)$ be such that $\exp(\varphi) \in C^0(\overline{D})$, that $\varphi < 0$ on D and that $\varphi = 0$ on ∂D. Let $R < 0$ and let $B_R = \{z \in D : \varphi(z) < R\}$. Moreover let $S_R = \partial B_R$ and $\varphi_R(z) = \max\{\varphi(z), R\}$. By [15, (1.4)] we can write

$$(dd^c \varphi_R)^n = \mathbf{1}_{\mathbb{C}^n \setminus B_R} (dd^c \varphi)^n + \mu_{\varphi,R}$$

where $\mathbf{1}_{\mathbb{C}^n \setminus B_R}$ is the characteristic function of $\mathbb{C}^n \setminus B_R$ and $\mu_{\varphi,R}$ is a positive measure supported on S_R. By [15, Théorème 3.1] if the total Monge-Ampère mass of φ is finite, *i.e.*, if $\int_D (dd^c\varphi)^n < +\infty$, then as $R \to 0$ the measures $\mu_{\varphi,R}$ converge weakly on $\mathbb{C}^n$ to a positive measure μ_φ supported on ∂D, with total mass $\int_D (dd^c\varphi)^n$. We denote by μ_z the limit measure of $L_{D,z}$. By [15, Théorème 5.1] it follows that for all $F \in \mathsf{Psh}(D) \cap C^0(\overline{D})$ we have the following representation formula:

$$F(z) = \mu_z(F) - \frac{1}{2\pi^n} \int_{w \in D} |L_{D,z}(w)| \, dd^c F(w) \wedge (dd^c L_{D,z})^{n-1}(w). \tag{8.1}$$

We can prove the following result:

Theorem 8.1. *Let D be a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary. Then*

$$d\mu_z(p) = |\Omega_{D,p}(z)|^n \omega_{\partial D}(p).$$

Proof. First of all, since $L_{D,z}$ is C^∞ on $\overline{D} \setminus \{z\}$ and $dL_{D,z}|_{\partial D} \neq 0$, arguing as in [15] we see that

$$d\mu_z = (dd^c L_{D,z})^{n-1} \wedge d^c L_{D,z}|_{\partial D}.$$

From (6.2) we have

$$|\Omega_{D,p}(z)| = \|\frac{\partial L_{D,z}}{\partial \nu_p}(p)\| = \|d(L_{D,z})_p\|,$$

where the last equality follows from $L_{D,z}|_{\partial D} = 0$ which implies that $d(L_{D,z})_p$ is a positive multiple of ν_p, the unit normal to ∂D at $p \in \partial D$ (here, as usual and with an abuse of notation, we identify the gradient of a function with its differential). Thus

$$d\mu_z = |\Omega_{D,p}(z)|^n \frac{(dd^c L_{D,z})^{n-1} \wedge d^c L_{D,z}}{\|dL_{D,z}\|^n}|_{\partial D}.$$

To end the proof we only need to check that

$$\omega_{\partial D} = \frac{(dd^c L_{D,z})^{n-1} \wedge d^c L_{D,z}}{\|dL_{D,z}\|^n}|_{\partial D}.$$

To this aim, it is enough to show that if r is a (local) defining function for D on a neighborhood U_p of $p \in \partial D$, then $L_{D,z} = h \cdot r$ on $U_p \cap \overline{D}$ for some positive $h \in C^\infty(U_p \cap \overline{D})$ for then a direct computation gives the result. Up to changes of coordinates we can assume that $U_p \cap D = \{(x,y) \in \mathbb{C} \times \mathbb{C}^{n-1} : x < 0\}$. Thus $L_{D,z}(x,y)/x$ is defined and positive on $U_p \cap D$. If we let $h(x,y) = \int_0^1 \frac{\partial L_{D,z}}{\partial x}(tx,y)dt$ then h is $C^\infty(U_p \cap \overline{D})$ and coincides with $L_{D,z}(x,y)/x$ in $U_p \cap D$. Moreover, since $dL_{D,z} \neq 0$ on ∂D it follows that $h > 0$ on $U_p \cap \overline{D}$. $\square$

From (8.1) and Theorem 8.1 we obtain:

Theorem 8.2. *Let $F \in \mathsf{Psh}(D) \cap C^0(\overline{D})$. Then for all $z \in D$*

$$\begin{aligned} F(z) = &\int_{p \in \partial D} |\Omega_{D,p}(z)|^n F(p) \omega_{\partial D}(p) \\ &- \frac{1}{2\pi^n} \int_{w \in D} |L_{D,z}(w)| \, dd^c F(w) \wedge (dd^c L_{D,z})^{n-1}(w). \end{aligned}$$

In particular if F is pluriharmonic then

$$F(z) = \int_{p \in \partial D} |\Omega_{D,p}(z)|^n F(p) \omega_{\partial D}(p).$$

Remark 8.3. If $F \in C^2(\overline{D})$ (but *not* plurisubharmonic in D) then there exists $C > 0$ such that $F(z) + C\|z\|^2 \in \mathsf{Psh}(D) \cap C^0(\overline{D})$. Thus Theorem 8.2 applies and one gets

$$\begin{aligned}
F(z) + C\|z\|^2 &= \int_{p\in\partial D} |\Omega_{D,p}(z)|^n F(p)\omega_{\partial D}(p) + C\int_{p\in\partial D} |\Omega_{D,p}(z)|^n \|p\|^2 \omega_{\partial D}(p)\\
&- \frac{1}{2\pi^n}\int_{w\in D} |L_{D,z}(w)|\, dd^cF(w) \wedge (dd^cL_{D,z})^{n-1}(w)\\
&- C\frac{1}{2\pi^n}\int_{w\in D} |L_{D,z}(w)|\, dd^c\|w\|^2 \wedge (dd^cL_{D,z})^{n-1}(w)\\
&= \int_{p\in\partial D} |\Omega_{D,p}(z)|^n F(p)\omega_{\partial D}(p)\\
&- \frac{1}{2\pi^n}\int_{w\in D} |L_{D,z}(w)|\, dd^cF(w) \wedge (dd^cL_{D,z})^{n-1}(w) + C\|z\|^2.
\end{aligned}$$

Therefore Theorem 8.2 applies to any $F \in C^2(\overline{D})$ (not necessarily plurisubharmonic). As a consequence it follows that the kernel $|\Omega_{D,p}(z)|^n\omega_{\partial D}(p)$ is the *unique* reproducing kernel associated to $L_{D,z}$, namely, (8.1) cannot hold with any other measure T_z in place of μ_z.

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F. BRACCI: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY.

E-mail address: `fbracci@mat.uniroma2.it`

G. PATRIZIO: DIPARTIMENTO DI MATEMATICA "ULISSE DINI", UNIVERSITÀ DI FIRENZE, VIALE MORGAGNI 67-A, 50134 FIRENZE, ITALY.

E-mail address: `patrizio@math.unifi.it`

S. TRAPANI: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY.
E-mail address: trapani@mat.uniroma2.it